MAT 310 L3 Tue Feb 4

§ Span and linear independence

Let V be a vector field over $\mathbb{F}$.

Def: A linear combination of vectors $v_1, \dots, v_m$ is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m = \sum_{i=1}^{m} a_i v_i \quad \begin{array}{l} a_i \in \mathbb{F} \\ v_i \in V \end{array}$$

Ex:
- $(17,-4,2)$ is a lin. comb. of $(2,1,-3)$ and $(1,-2,4)$ because

$$(17,-4,2) = 6(2,1,-2) + 5(1,-2,4).$$

- $(17,-4,5)$ is not a lin comb of $(2,1,-3)$ and $(1,-2,4)$ because the equation

$$(17,-4,5) = a_1(2,1,-3) + a_2(1,-2,4)$$

has no solutions. It's equivalent to the system of linear eqns

$$\begin{cases} 17 = 2a_1 + a_2 \\ -4 = a_1 - 2a_2 \\ 5 = -3a_1 + 4a_2 \end{cases}$$ which has no sol as you can check.

Def: The *span* of $v_1, \dots, v_m \in V$ is the set of all linear combinations

$$\text{span}(v_1, \dots, v_m) = \left\{ \sum_{i=1}^{m} a_i v_i \;\middle|\; a_i \in \mathbb{F} \ \forall i \right\}.$$

Ex From previous ex

$v_1 = (2, 1, -3)$, $v_2 = (1, -2, 4)$,

$w_1 = (17, -4, 2)$, $w_2 = (17, -4, 5)$

and we have $w_1 \in \text{span}(v_1, v_2)$ but $w_2 \notin \text{span}(v_1, v_2)$.

Def: A vector space V is *finite dimensional* if V is the span of a finite list of vectors, i.e.,

$V = \text{span}(v_1, \dots, v_m)$ for some $v_1, \dots, v_m \in V$.

Def: If V is not finite dim we say that it's infinite dim.

Ex: • $\mathbb{F}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{F} \ \forall i\}$ is finite dim since

$\mathbb{F}^n = \operatorname{span}((1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1))$

• $\mathbb{F}^\infty = \{(x_1, x_2, \ldots) \mid x_i \in \mathbb{F} \ \forall i\}$ is infinite dimensional.

Def: • Let $\mathcal{P}(\mathbb{F})$ denote the set of all polynomials w/ coefficients in $\mathbb{F}$.

• Let $\mathcal{P}_m(\mathbb{F})$ be the set of all polynomials of degree $\leq m$.

Prop. $\mathcal{P}(\mathbb{F})$ is infinitely dimensional.

Proof: Going for a contradiction, assume it's not & write

$$\mathcal{P}(\mathbb{F}) = \operatorname{span}(p_1, \ldots, p_m).$$

Let $D = \max_{1 \le i \le m} \deg p_i$. This exists because each (non-zero) polynomial has a non-negative degree. But then $x^{D+1} \notin \operatorname{span}(p_1, \dots, p_m)$ which yields a contradiction. $\square$

<u>Def</u> A set of vectors $v_1, \dots, v_m$ is said to be <u>linearly independent</u> iff

$$\sum_{i=1}^{m} a_i v_i = 0 \Rightarrow a_i = 0 \ \forall i.$$

If they are not linearly independent, they are called linearly dependent.

<u>Rmk</u>: A set of vectors $v_1, \dots, v_m$ is linearly independent if $w \in \operatorname{span}(v_1, \dots, v_m)$ can only be written as a linear combination in <u>one</u> way.

(Compare this to sum/direct sum of subspaces.)

Ex: (a) $(1,0,0)$ and $(0,1,0)$ are linearly independent:

$$a_1(1,0,0)+a_2(0,1,0)=0$$

$$\Leftrightarrow (a_1,a_2,0)=(0,0,0)$$

$$\Leftrightarrow a_1=a_2=0.$$

(b) Let m be a non-negative integer. Then $1,z,\dots,z^m$ are linearly independent in $\mathcal{P}(\mathbb{F})$. Then

$a_0+a_1z+\dots+a_mz^m=0$ means

an equality in $\mathcal{P}(\mathbb{F})$, so this equality should be true for all $z\in\mathbb{F}$. The only option is

$$a_0=a_1=\dots=a_m=0.$$

(c) Any non-zero $v\in V$ is linearly independent since $av=0 \Rightarrow a=0$.

(d) The zero vector $0 \in V$ is linearly dependent since eg
$1 \cdot 0 = 0$.

(e) V_1 and V_2 are linearly dependent iff $V_1 = \lambda V_2$ for some $\lambda \in \mathbb{F}$.

(f) $(2,3,1)$, $(1,-1,2)$, $(7,3,8)$ is lin dep in $\mathbb{F}^3$ since
$2(2,3,1) + 3(1,-1,2) + (-1)(7,3,8) = 0$
i.e. we can write 0 as a non-trivial linear combination.

<u>Prop</u>: If $v_1, \dots, v_m$ is lin dep. Then $\exists k \in \{1, \dots, m\}$ s.th.
$$v_k \in \operatorname{span}(v_1, \dots, v_{k-1}).$$
Furthermore if v_k is such a vector then $\operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(v_1, \dots, \hat{v}_k, \dots, v_m)$

($v_1, \dots, \hat{v}_k, \dots, v_m$ is the list $v_1, \dots, v_m$ with v_k removed.)

<u>Proof</u>: By assumption $v_1, \dots, v_m$ is lin dep so $\exists a_1, \dots, a_m$ not all $=0$ such that $a_1 v_1 + \dots + a_m v_m = 0$. Let $k = \max\{i \mid a_i \neq 0\}$. Then

$$a_1 v_1 + \dots + a_m v_m = a_1 v_1 + \dots + a_k v_k = 0$$

with $a_k \neq 0$. So:

$$v_k = -\frac{a_1}{a_k} v_1 - \dots - \frac{a_{k-1}}{a_k} v_{k-1}$$

$\Leftrightarrow$ $v_k \in \mathrm{span}(v_1, \dots, v_{k-1})$. For the second part, it's obvious that

$\mathrm{span}(v_1, \dots, \hat{v}_k, \dots, v_m) \subset \mathrm{span}(v_1, \dots, v_m)$

so it suffices to prove

$\mathrm{span}(v_1, \dots, \hat{v}_k, \dots, v_m) \supset \mathrm{span}(v_1, \dots, v_m)$

Let $v = a_1 v_1 + \dots + a_m v_m \in \mathrm{span}(v_1, \dots, v_m)$.

Since we now assume $v_k \in \operatorname{span}(v_1, \dots, v_{k-1})$

So $v_k = b_1 v_1 + \cdots + b_{k-1} v_{k-1}$, and

$$v = a_1 v_1 + \cdots + a_k v_k + \cdots + a_m v_m$$

$$= a_1 v_1 + \cdots + a_k (b_1 v_1 + \cdots + b_{k-1} v_{k-1}) + \cdots + a_m v_m$$

$$= (a_1 + a_k b_1) v_1 + \cdots + (a_{k-1} + a_k b_{k-1}) v_{k-1}$$

$$+ a_{k+1} v_{k+1} + \cdots + a_m v_m$$

$\in \operatorname{Span}(v_1, \dots, \hat{v}_k, \dots, v_m)$ since this is now a lin comb not involving v_k anymore. This shows

$$\operatorname{span}(v_1, \dots, \hat{v}_k, \dots, v_m) \supset \operatorname{Span}(v_1, \dots, v_m)$$

and hence

$$\operatorname{span}(v_1, \dots, \hat{v}_k, \dots, v_m) = \operatorname{Span}(v_1, \dots, v_m)$$

$\square$

<u>Ex</u>: Consider $v_1 = (1,2,3)$, $v_2 = (6,5,4)$, $v_3 = (15,16,17)$, $v_4 = (8,9,7)$. Let's find the smallest k s.th. $v_k \in \operatorname{span}\{v_1, \dots, v_{k-1}\}$.

$\underline{k=1}$: $v_1 \neq 0$ so v_1 is lin indep.

$\underline{k=2}$: We see that v_2 is not a scalar multiple of v_1:

$$(6,5,4) = c(1,2,3) \text{ means } \begin{cases} c = 6 \\ 2c = 5 \\ 3c = 4 \end{cases}$$

which obviously has no sol.

$\underline{k=3}$: Now, we do have

$v_3 = av_1 + bv_2$ for $a=3, b=2$, so $v_3 \in \mathrm{span}(v_1, v_2)$ which means that v_1, v_2, v_3, v_4 is lin dep &

$$\mathrm{Span}(v_1, v_2, v_3, v_4) = \mathrm{Span}(v_1, v_2, v_4)$$

by the previous proposition.

the "linear dependence lemma"

Prop: In a fin. dim vector space, the length of every lin independent list is less than or equal to the length of every spanning list.

Proof: Supp. $u_1, \dots, u_m$ is lin indep and $w_1, \dots, w_n$ spans V. We will show $m \leq n$.

Step 1: Because $u_1 = a_1 w_1 + \dots + a_n w_n$ for some $a_j \in \mathbb{F}$, the list

$u_1, w_1, \dots, w_n$ is lin dep. Since $u_1 \neq 0$, the lin dep lemma implies that there is some w_k such that $w_k \in \operatorname{span}(u_1, w_1, \dots, w_{k-1})$ and

$$\operatorname{span}(u_1, w_1, \dots, w_n) = \operatorname{span}(u_1, w_1, \dots, \hat{w}_k, \dots, w_n)$$
$$= V$$

Repeat this process. At step k ($k \geq 2$) the list from step $k-1$ is

$u_1, \dots, u_{k-1}, w_{i_k}, \dots, w_{i_n}$ (n vectors in total)

& its span $= V$. So

$$u_k = a_1 u_1 + \dots + a_{k-1} u_{k-1} + a_k w_{i_k} + \dots + a_n w_{i_n}$$

So the new list

$u_1, \dots, u_{k-1}, u_k, w_{i_k}, \dots, w_{i_n}$ is lin dep.

becuse $u_1, \dots, u_k$ is lin indep, $\exists$ some $w_j \in \operatorname{span}(u_1, \dots, u_k, w_{i_k}, \dots, w_{j-1})$

Now at the final step when we have the list

$u_1, \dots, u_m, w_{j_m}, \dots, w_{j_n}$ it must

still be lin dep by the above reasoning. But since $(u_1, \dots, u_m)$ is lin indep, the list of w's <u>can not</u> exhausted yet, meaning $m \geq n$.

$\square$

<u>Ex</u>: • Since $(1,0)$ and $(0,1)$ spans $\mathbb{F}^2$, no list of ≥ 3 vectors in $\mathbb{F}^2$ can be linearly dependent.

e.g. $(1,2), (-1,5), (7,0)$ must be lin dep.

- Since $(1,0,0), (0,1,0), (0,0,1)$ are linearly indep in $\mathbb{F}^3$, no list of ≤ 2 vectors in $\mathbb{F}^3$ can span it!

<u>Prop</u>: Every subspace of a fin dim vector space is finite dimensional