MAT 310 L2 Thu Jan 30

<u>Recall</u>: A vector space (over $\mathbb{F}$) V is a set w/ addition and scalar mult such that

(1) $u+v = v+u \quad \forall u,v \in V$

(2) $(u+v)+w = u+(v+w) \quad \forall u,v,w \in V$

(3) $(ab)v = a(bv) \quad \forall a,b \in \mathbb{F}, v \in V$

(4) There's $0 \in V$ such that

$$v + 0 = v \quad \forall v \in V$$

(5) For every $v \in V$ there's $-v \in V$ such that $v+(-v) = 0$

(6) $1v = v \quad \forall v \in V$

(7) $a(v+w) = av+aw \quad \forall a \in \mathbb{F}, v,w \in V$

(8) $(a+b)v = av+bv \quad \forall a,b \in \mathbb{F}, v \in V$

---

§ <u>Subspaces</u> Let V be a vector space over $\mathbb{F}$.

<u>Def</u>: A <u>subspace</u> of V is a subset $U \subset V$ that is a vector

space itself, with the same addition, scalar mult. and additive identity as in V.

The list of axioms defining a vector space is a little long, but thankfully there's a faster way to check if a subset is a subspace.

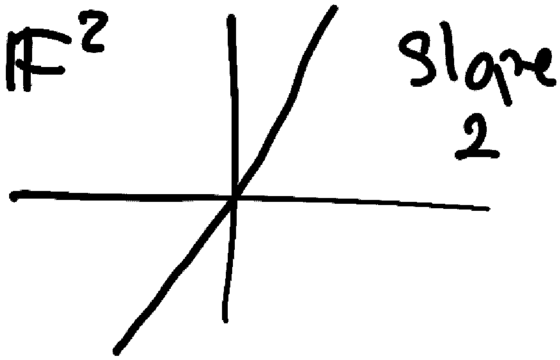
Theorem: A subset $U \subset V$ is a subspace if and only if U satisfies the following 3 conditions:

1. (additive identity) $0 \in U$
2. (closed under addition) $u, w \in U$ implies $u + w \in U$
3. (closed under scalar mult.) $a \in \mathbb{F}$ and $u \in U$ implies $au \in U$.

Ex (1) $U = \{(x,y) \in \mathbb{F}^2 \mid y = 2x\}$

$= \{(x, 2x) \mid x \in \mathbb{F}\} \subset \mathbb{F}^2$

is a subspace.



1. $0 \in U$ by picking $x = 0$.

2. $(x_1, 2x_1) + (x_2, 2x_2) = (x_1 + x_2, 2(x_1 + x_2)) \in U$

3. $\lambda(x, 2x) = (\lambda x, 2(\lambda x)) \in U$

so $U \subset \mathbb{F}^2$ is a subspace.

(2) Let $b \in \mathbb{F}$. For what values of b is $\underbrace{\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \mid x_3 = 5x_4 + b\}}_{U} \subset \mathbb{F}^4$ a subspace?

1. We need $0 \in U$, and a general element in U is

$$(x_1, x_2, 5x_4 + b, x_4) \in U$$

Need to pick $x_1 = x_2 = x_4 = 0$, which gives $(0, 0, b, 0)$, which forces us to have $\boxed{b=0}$ (else 0 is not in U).

For $b=0$ we can also check that U is closed under addition & scalar mult.

(3) $\{f:[0,1]\to\mathbb{R} \text{ continuous}\}\subset\mathbb{R}^{[0,1]}$ is a subspace:

1. $0(x)=0$ is continuous.
2. If $f,g:[0,1]\to\mathbb{R}$ are continuous we know from calculus that $f+g:[0,1]\to\mathbb{R}$ is continuous
3. Similarly, if $\lambda\in\mathbb{R}$ and $f:[0,1]\to\mathbb{R}$ is continuous, then so is $\lambda f:[0,1]\to\mathbb{R}$.

It is very useful to be able to "combine" subspaces. The union of two subspaces is rarely a subspace (see problem in HW#1), so it's not the kind of operation that is very useful to us. A better operation is the <u>sum</u> of subspaces.

<u>Def</u> (Sum of subspaces)
Let $V_1, \dots, V_m \subset V$ be subspaces of V. The <u>sum</u> of $V_1, \dots, V_m$ is defined as

$$V_1 + \cdots + V_m = \{v_1 + \cdots + v_m \mid v_1 \in V_1, \dots, v_m \in V_m\}.$$

Sometimes we write it as $\sum_{i=1}^{m} V_i$.

<u>Ex</u>: $U = \{(x,0,0) \in \mathbb{F}^3\} \subset \mathbb{F}^3$ and

$W = \{(0,y,0) \in \mathbb{F}^3\} \subset \mathbb{F}^3$ are

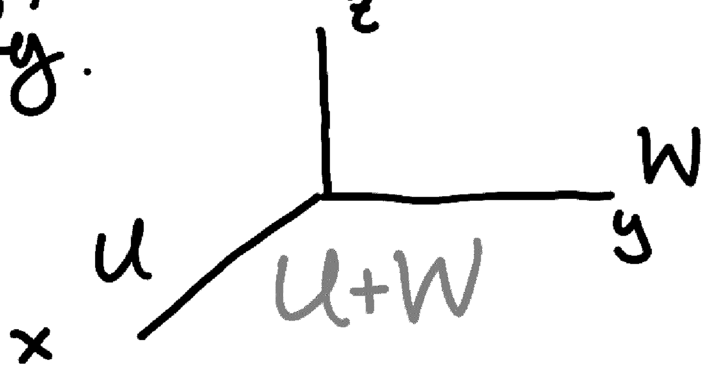
two subspaces of $\mathbb{F}^3$. Then

$$U + W = \{u + w \mid u \in U, w \in W\}$$

$$= \{(x,0,0) + (0,y,0) \mid (x,0,0) \in U, (0,y,0) \in W\}$$

$$= \{(x,y,0) \in \mathbb{F}^3\} \subset \mathbb{F}^3$$

Note that $U+W$ is again a subspace of $\mathbb{F}^3$ as you can verify.



Ex: Let $U = \{(x,x,y,y) \in \mathbb{F}^4\} \subset \mathbb{F}^4$

$W = \{(x,x,x,y) \in \mathbb{F}^4\} \subset \mathbb{F}^4$.

and let's find $U+W$. It's the set of all possible sums of elements from U and W. Let

$$(x,x,y,y) \in U \quad (z,z,z,w) \in W$$

then their sum is

$$(x+z, x+z, y+z, y+w) \in U+W.$$

Only the first two components are equal, so we have

$U + W \subset \{(a,a,b,c) \in \mathbb{F}^4\}$ (subset).

We now show that we in fact have equality. To show it, we must show

$$U + W \supset \{(a,a,b,c) \in \mathbb{F}^4\}. \quad (*)$$

To prove this we must show that any element $(a,a,b,c) \in \mathbb{F}^4$ can be written as a sum

$$(x,x,y,y) + (z,z,z,w) \text{ for some } x,y,z,w \in \mathbb{F}.$$

Namely

$$(a,a,b,c) = (a,a,b,b) + (0,0,0,c-b).$$

This shows $(*)$ and hence

$$U+W = \{(x,x,y,z) \in \mathbb{F}^4\}.$$

Can again show $U+W$ is a subspace.

That $U+W$ was a subspace in the two previous examples is not a coincidence.

<u>Theorem</u>: If $V_1,\dots,V_m \subset V$ are subspaces then $V_1+\dots+V_m \subset V$ is a subspace. In fact, it's the smallest subspace containing $V_1,\dots,V_m$.

<u>Proof</u>: We first check that $\sum_{i=1}^{m} V_i$ is a subspace:

1. $0 = 0+\dots+0 \in \sum_{i=1}^{m} V_i$
2. If $v_1+\dots+v_m,\ w_1+\dots+w_m \in \sum_{i=1}^{m} V_i$ then

$$(v_1+\dots+v_m)+(w_1+\dots+w_m)$$
$$=(v_1+w_1)+\dots+(v_m+w_m) \in \sum_{i=1}^{m} V_i$$

Since $v_i+w_i \in V_i$ for all $i=1,\dots,m$.

3. If $\lambda \in \mathbb{F}$ and $v_1+\dots+v_m \in \sum_{i=1}^{m} V_i$

then

$$\lambda(v_1+\dots+v_m)=(\lambda v_1)+\dots+(\lambda v_m)\in\sum_{i=1}^{m}V_i$$

since $\lambda v_i \in V_i$ for all $i=1,\dots,m$.

Therefore $V_1+\dots+V_m \subset V$ is a subspace. Next we show that it's the smallest subspace containing $V_1,\dots,V_m$. This mean that if

$U \subset V$ is any subspace

such that $V_1,\dots,V_m \subset U \subset V$, then $V_1+\dots+V_m \subset U$.

Since $V_1,\dots,V_m \subset U$, and U is closed under addition, it follows $v_1+\dots+v_m \in U$ for any $v_1 \in V_1$, $\dots, v_m \in V_m$.

Therefore $V_1+\dots+V_m \subset U$.

$\square$

The sum operation may be further specialized:

Definition (Direct Sum).

Let $V_1, \dots, V_m \subset V$ be subspaces.

The sum $V_1 + \cdots + V_m \subset V$ is called a direct sum if each element $w \in \sum_{i=1}^{m} V_i$ can be written as

$$w = v_1 + \cdots + v_m \quad , v_i \in V_i \qquad i = 1, \dots, m$$

in only one way. In this case we use the notation

$$V_1 \oplus \cdots \oplus V_m \text{ or } \bigoplus_{i=1}^{m} V_i$$

Ex: $U = \{(x,0,0) \in \mathbb{F}^3\} \subset \mathbb{F}^3$

$W = \{(0,y,0) \in \mathbb{F}^3\} \subset \mathbb{F}^3$

then we saw previously $U + W = \{(x,y,0) \in \mathbb{F}^3\}$. This is in fact a direct sum since

$(x,y,0) = (x,0,0) + (0,y,0)$ can not be written as such a sum in any other way.

$$U \oplus W = \{(x,y,0) \in \mathbb{F}^3\}.$$

Ex: Let $V_i = \{(0,\dots,0,x_i,0,\dots,0) \in \mathbb{F}^n\}$

i-th component

for $i = 1,\dots,m$. Then

$$\mathbb{F}^n = V_1 \oplus \cdots \oplus V_m = \bigoplus_{i=1}^{m} V_i.$$

Ex: $U = \{(x,x) \in \mathbb{F}^2\}$

$W = \{(x,0) \in \mathbb{F}^2\}$

$Z = \{(x,2x) \in \mathbb{F}^2\}$

The sum $U+W+Z$ is not a direct sum. For example the element $(2,2) \in \mathbb{F}^2$ can be written as

$$(2,2) = \underbrace{(2,2)}_{U} + \underbrace{(0,0)}_{W} + \underbrace{(0,0)}_{Z}$$

but also as

$$(2,2) = (0,0) + (1,0) + (1,2)$$

So the sum decomposition of $(2,2)$ isn't unique.

<u>Proposition</u>: Let $V_1, \dots, V_m \subset V$ be subspaces. The sum $\sum_{i=1}^{m} V_i$ is a direct sum iff

$$0 = v_1 + \dots + v_m \Rightarrow v_i = 0 \quad i = 1, \dots, m.$$

i.e. the only way of writing 0 as a sum is the sum of 0's.

<u>Ex</u>: In our prev example we could've showed $U+W+Z$ isn't a direct sum by writing $(0,0)$ as

$$(0,0) = \underbrace{(2,2)}_{U} + \underbrace{(-1,0)}_{W} + \underbrace{(-1,-2)}_{Z}.$$

<u>Proposition</u>. Let $U, W \subset V$ be subspaces. $U+W$ is a direct sum $\Leftrightarrow U \cap W = \{0\}$

<u>Proof:</u> $\Longrightarrow$. Assume $U+W$ is a direct sum. Let $v \in U \cap W$. Then we can write 0 as the sum

$$0 = v + (-v)$$

which implies $v = -v = 0$ so $U \cap W = \{0\}$, since our choice of $v \in U \cap W$ was arbitrory.

$\Longleftarrow$: Assume $U \cap W = \{0\}$. To show $U+W$ is a direct sum, we need to show

$$0 = u + w \implies u = w = 0.$$

Namely, the equation $0 = u + w$ implies $w = -u$ (i.e. the additive inverse of u). Since $u \in U$, we have $-u \in U$, so $w = -u \in U$ and thus $w \in U \cap W$, which means $w = 0$, and so $u = 0$. $\square$

Remark/warning: The previous result does not hold for sums of 3 or more vector spaces.