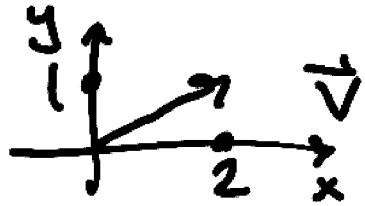
MAT 211 L2 Wed Aug 26

<u>Recall:</u>

- Matrices: $\begin{bmatrix} 3 & 1 & 3 & 4 \\ 5 & 2 & 3 & 2 \end{bmatrix}, \dots$
- Vectors $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



- Gauss-Jordan elimination to solve systems of linear eqns

<u>Ex</u>: Let's use Gauss-Jordan elimination to find the row-reduced echelon form (rref) of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{matrix} -4 \\ -7 \end{matrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} -1$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \cdot(-1/3) \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} -2$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{rref}(A).$$

Def: The rank of a matrix A is the number of leading elements in $\mathrm{rref}(A)$.

Ex. We found
$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ so}$$
$$\mathrm{rank}(A) = 2$$

Theorem: Let A be an $(\underbrace{n}_{\text{rows}} \times \underbrace{m}_{\text{columns}})$-matrix that is the coefficient matrix of a system of linear equs.

[Note $n = \#\text{equations}$
$m = \#\text{ variables}$.]

1. $\mathrm{rank}(A) \leq n$ and $\mathrm{rank}(A) \leq m$

② If the system is inconsistent, $\operatorname{rank}(A) < n$

③ If the system has a unique solution $\operatorname{rank}(A) = m$

④ If the system has infinitely many solutions, $\operatorname{rank}(A) < m$.

⑤ If $n < m$ (#eqns < #variables), the system is either inconsistent or has infinitely many solutions.

⑥ If $n = m$ (#eqns = #variables), the system has a unique solution if and only if $\operatorname{rank}(A) = n$.

This happens if and only if

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{bmatrix}$$

(1's on the diagonal & 0's everywhere else)

Ex: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ Let's find rref(A) & rank(A).

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{rref}(A)$$

rank(A) = 1.

Matrix algebra:

Addition: We can add two matrices if they have the same size:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \text{not defined.}$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1m}+b_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & \cdots & a_{nm}+b_{nm} \end{bmatrix}$$

<u>Scalar multiplication</u>:

If k is a scalar (= real number),

$$k \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} ka_{11} & \cdots & ka_{1m} \\ \vdots & \ddots & \vdots \\ ka_{n1} & \cdots & ka_{nm} \end{bmatrix}$$

<u>Ex</u>: $3\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ -3 & 9 \end{bmatrix}$

Now let $\vec{v}$ and $\vec{w}$ be two vectors. Both $\vec{v}$ & $\vec{w}$ can be either rows or column vectors.

Let $v_1, \dots, v_n$ & $w_1, \dots, w_n$ be the components of $\vec{v}$ & $\vec{w}$. Note they have the same # of components.

<u>Dot product:</u>

$$\vec{v} \cdot \vec{w} = \underbrace{v_1 w_1 + \dots + v_n w_n}_{\text{real number!}}$$

vectors

<u>Ex:</u> $[1 \ \ 2 \ \ 3] \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = 3 \cdot 1 + 2 \cdot 1 + 3 \cdot 2 = 11$

Now let A be an $(n \times m)$-matrix, and let $\vec{x}$ be a vector w/ m components.

Can write

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} - & \vec{w}_1 & - \\ & \vdots & \\ - & \vec{w}_n & - \end{bmatrix}$$

$\vec{w}_i = [a_{i1} \cdots a_{im}]$ is the i-th row of A. Then we define

$$A\vec{x} = \begin{bmatrix} — \vec{w}_1 — \\ \vdots \\ — \vec{w}_n — \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}$$

Note each $\vec{w}_i \cdot \vec{x}$ is a scalar, so $A\vec{x}$ is a vector w/ n components

$$\underbrace{\overbrace{A}^{n \times m} \overbrace{\vec{x}}^{m \times 1}}_{n \times 1}$$

<u>Ex</u>: $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} [1\ 2\ 3] \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \\ [1\ 0\ -1] \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$

<u>Ex</u>: $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - z \\ y + 2z \\ 0 \end{bmatrix}$

Can also compute $A\vec{x}$ in a different way:

If we let $\vec{v}_1, \dots, \vec{v}_m$ be the columns of A, and $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$, then

$$A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \overset{\text{scalar}}{x_1}\,\overset{\text{vector}}{\vec{v}_1} + \cdots + x_m\vec{v}_m$$

<u>Ex</u>: $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} -z \\ 2z \\ 0 \end{bmatrix} = \begin{bmatrix} x - z \\ y + 2z \\ 0 \end{bmatrix}.$$

Such an expression $x_1\vec{v}_1 + \cdots x_m\vec{v}_m$ is called a <u>linear combination</u> of the vectors $\vec{v}_1, \ldots, \vec{v}_m$.

<u>Def:</u> If $\vec{b}$ and $\vec{v}_1, \ldots, \vec{v}_m$ are $(n\times 1)$-vectors, we say that $\vec{b}$ is a linear combination of the vectors $\vec{v}_1, \ldots, \vec{v}_m$ if there exist scalars $x_1, \ldots, x_m$ such that

$$\vec{b} = x_1\vec{v}_1 + \cdots + x_m\vec{v}_m.$$

Finding out whether a given vector is a linear combination of other vectors is related to systems of linear equations!

<u>Ex.</u> Is $\vec{b} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ a linear combination of $\vec{v}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}4\\5\\6\end{bmatrix}$?

Sol: Want to find scalars x_1, x_2 such that

$$\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 = x_1\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{bmatrix}4\\5\\6\end{bmatrix}$$

$$= \begin{bmatrix}x_1+4x_2\\2x_1+5x_2\\3x_1+6x_2\end{bmatrix}$$

So $\begin{cases} x_1+4x_2 = 1 \\ 2x_1+5x_2 = 1 \\ 3x_1+6x_2 = 1 \end{cases}$. Solve by Gauss-Jordan:

$$\begin{array}{c} -2 \\ \\ \\ \end{array}\left[\begin{array}{cc:c} 1 & 4 & 1 \\ 2 & 5 & 1 \\ 3 & 6 & 1 \end{array}\right]\begin{array}{c} -3 \\ \\ \\ \end{array} \longrightarrow \left[\begin{array}{cc:c} 1 & 4 & 1 \\ 0 & -3 & -1 \\ 0 & -6 & -2 \end{array}\right]\begin{array}{c} \\ -2 \\ \\ \end{array}$$

$$\longrightarrow \left[\begin{array}{cc:c} 1 & 4 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{array}\right]\begin{array}{c} \\ -1/3 \\ \\ \end{array} \longrightarrow \left[\begin{array}{cc:c} 1 & 4 & 1 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{array}\right]\begin{array}{c} -4 \\ \\ \\ \end{array}$$

$$\longrightarrow \left[\begin{array}{cc:c} 1 & 0 & -1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{array}\right] \rightsquigarrow \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-1/3\\1/3\end{bmatrix}$$

So $\vec{b} = -\frac{1}{3}\vec{v}_1 + \frac{1}{3}\vec{v}_2$ & we can also verify this:

$$-\frac{1}{3}\begin{bmatrix}1\\2\\3\end{bmatrix} + \frac{1}{3}\begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}-1/3\\-2/3\\-1\end{bmatrix} + \begin{bmatrix}4/3\\5/3\\2\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} = \vec{b}.$$

Algebraic rules: • A : $(n\times m)$-matrix

- $\vec{x}$ and $\vec{y}$ are $(m\times 1)$-matrices (vectors w/ m components)
- k is a scalar

① $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$

② $A(k\vec{x}) = k(A\vec{x})$

$$\begin{cases} a_{11}x_1 + \cdots + a_{1m}x_m = b_1 \\ \quad\vdots \qquad\qquad\qquad \vdots \\ a_{n1}x_1 + \cdots + a_{nm}x_m = b_m \end{cases} \longleftrightarrow A\vec{x} = \vec{b}$$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

The augmented matrix is $[A \,\vdots\, \vec{b}]$