

Recall: • For an $(n \times m)$ -matrix

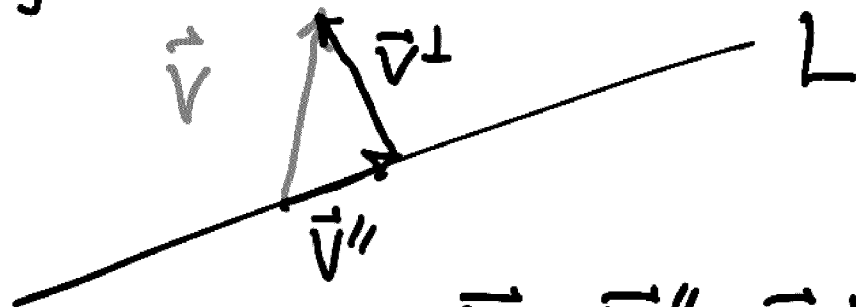
$A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_m \\ | & & | \end{bmatrix}$, its transpose A^T is the $(m \times n)$ -matrix $A^T = \begin{bmatrix} \text{---} \vec{v}_1 \text{---} \\ \vdots \\ \text{---} \vec{v}_m \text{---} \end{bmatrix}$.

• An $(n \times n)$ -matrix A is orthogonal if $A^T = A^{-1}$, or equivalently, if its columns form an ON-basis.

• An $(n \times n)$ -matrix A is symmetric if $A = A^T$.

Orthogonal projections:

We briefly discussed orthogonal projections:



$$\vec{v} = \vec{v}'' + \vec{v}^\perp \quad \text{where}$$

$\vec{v}''$ lies in L , and $\vec{v}^\perp$ is orthogonal to L , and

$$\text{proj}_L(\vec{v}) = \vec{v}'' = \vec{v} - \vec{v}^\perp.$$

If L is a line spanned by the unit vector $\vec{u}_1$, then

$$\vec{v}'' = (\vec{u}_1 \cdot \vec{v}) \vec{u}_1, \text{ so}$$

$$\text{proj}_L(\vec{v}) = (\vec{u}_1 \cdot \vec{v}) \vec{u}_1.$$

In more generality, if V is a subspace in $\mathbb{R}^n$ with ON-basis $(\vec{u}_1, \dots, \vec{u}_m)$, then

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$$\text{proj}_V(\vec{v}) = (\vec{u}_1 \cdot \vec{v}) \vec{u}_1 + \dots + (\vec{u}_m \cdot \vec{v}) \vec{u}_m.$$

EX: Consider the subspace V in $\mathbb{R}^4$, spanned by $\vec{u}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$.
Let's compute the orthogonal

projection of $\vec{v} = \begin{bmatrix} 1 \\ 4 \\ 2 \\ 3 \end{bmatrix}$ onto V .

First we have to check that $(\vec{u}_1, \vec{u}_2)$ is an ON-basis.

$$\begin{cases} \vec{u}_1 \cdot \vec{u}_2 = \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{4} (1 \cdot 1 + 1 \cdot (-1) + 1 \cdot (-1) + 1 \cdot 1) = 0 \\ \|\vec{u}_1\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{4}{4}} = 1 \\ \|\vec{u}_2\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{4}{4}} = 1 \end{cases} \quad \text{ok}$$

By the projection formula we now get:

$$\text{Proj}_V(\vec{v}) = (\vec{u}_1 \cdot \vec{v})\vec{u}_1 + (\vec{u}_2 \cdot \vec{v})\vec{u}_2$$

$$\begin{aligned} &= \left(\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ 2 \\ 3 \end{bmatrix} \right) \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \left(\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ 2 \\ 3 \end{bmatrix} \right) \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \\ &= \frac{1}{4} (1+4+2+3) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{4} (1-4-2+3) \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \\ &= \frac{5}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix}. \end{aligned}$$

V subspace of $\mathbb{R}^n$, with ON-basis $(\vec{u}_1, \dots, \vec{u}_m)$, then we can view

orthogonal projection as a linear transformation

$\text{proj}_V: \mathbb{R}^n \longrightarrow \mathbb{R}^n$, and we can also find the matrix by the projection formula

$$\text{proj}_V(\vec{v}) = (\vec{u}_1 \cdot \vec{v})\vec{u}_1 + \dots + (\vec{u}_m \cdot \vec{v})\vec{u}_m$$

The matrix for proj_V is

$$A = \begin{bmatrix} \text{proj}_V(\vec{e}_1) & \dots & \text{proj}_V(\vec{e}_n) \end{bmatrix}$$

where $(\vec{e}_1, \dots, \vec{e}_n)$ is the standard basis for $\mathbb{R}^n$.

Another way to view this comes from the following observation:

Let $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, then

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = \begin{matrix} 1 \times n & n \times 1 \\ [u_1 \dots u_n] \end{matrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

dot product matrix multiplication

$$(\vec{u} \cdot \vec{v}) \vec{u} = \underbrace{\vec{u}}_{n \times 1} (\underbrace{\vec{u} \cdot \vec{v}}_{1 \times n}) = \underbrace{\vec{u}}_{n \times 1} (\underbrace{\vec{u}^T \vec{v}}_{n \times 1})$$

$$= \underbrace{\vec{u} \vec{u}^T}_{n \times n} \vec{v} = M \vec{v},$$

$$M = \vec{u} \vec{u}^T = \begin{bmatrix} u_1 u_1 & \dots & u_1 u_n \\ \vdots & \ddots & \vdots \\ u_n u_1 & \dots & u_n u_n \end{bmatrix}$$

The projection formula can now be written as follows:

$$\begin{aligned} \text{proj}_V(\vec{v}) &= (\vec{u}_1 \cdot \vec{v}) \vec{u}_1 + \dots + (\vec{u}_m \cdot \vec{v}) \vec{u}_m \\ &= \vec{u}_1 \vec{u}_1^T \vec{v} + \dots + \vec{u}_m \vec{u}_m^T \vec{v} \\ &= (\vec{u}_1 \vec{u}_1^T + \dots + \vec{u}_m \vec{u}_m^T) \vec{v} \end{aligned}$$

Now if we set

$$Q = \underbrace{\begin{bmatrix} | & & | \\ \vec{u}_1 & \dots & \vec{u}_m \\ | & & | \end{bmatrix}}_{n \times m}, \text{ then } Q^T = \underbrace{\begin{bmatrix} \text{---} \vec{u}_1^T \text{---} \\ \vdots \\ \text{---} \vec{u}_m^T \text{---} \end{bmatrix}}_{m \times n}$$

$$Q Q^T = \underbrace{\begin{bmatrix} | & & | \\ \vec{u}_1 & \dots & \vec{u}_m \\ | & & | \end{bmatrix}}_{n \times m} \underbrace{\begin{bmatrix} \text{---} \vec{u}_1^T \text{---} \\ \vdots \\ \text{---} \vec{u}_m^T \text{---} \end{bmatrix}}_{m \times n}$$

$$\begin{aligned}
QQ^T \vec{x} &= \begin{bmatrix} | & & | \\ \vec{u}_1 & \dots & \vec{u}_m \\ | & & | \end{bmatrix} \begin{bmatrix} -\vec{u}_1^T- \\ \vdots \\ -\vec{u}_m^T- \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= \begin{bmatrix} | & & | \\ \vec{u}_1 & \dots & \vec{u}_m \\ | & & | \end{bmatrix} \begin{bmatrix} \vec{u}_1^T \vec{x} \\ \vdots \\ \vec{u}_m^T \vec{x} \end{bmatrix} \\
&= \begin{bmatrix} | & & | \\ \vec{u}_1 & \dots & \vec{u}_m \\ | & & | \end{bmatrix} \begin{bmatrix} \vec{u}_1 \cdot \vec{x} \\ \vdots \\ \vec{u}_m \cdot \vec{x} \end{bmatrix} \\
&= (\vec{u}_1 \cdot \vec{x}) \vec{u}_1 + \dots + (\vec{u}_m \cdot \vec{x}) \vec{u}_m \\
&= \text{proj}_V(\vec{x}),
\end{aligned}$$

So to summarize:

V subspace in $\mathbb{R}^n$ w/ ON-basis $(u_1, \dots, u_m)$, then the matrix for

$\text{proj}_V: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is

$$P = [\text{proj}_V(\vec{e}_1) \ \dots \ \text{proj}_V(\vec{e}_n)] = QQ^T,$$

where $Q = \begin{bmatrix} | & & | \\ \vec{u}_1 & \dots & \vec{u}_m \\ | & & | \end{bmatrix}.$

Ex: Consider the subspace V in $\mathbb{R}^4$, spanned by $\bar{u}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\bar{u}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$.

We already checked that $(\bar{u}_1, \bar{u}_2)$ is an ON-basis. The matrix for

$\text{Proj}_V: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is

$$P = QQ^T = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$
$$= \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Theorem: The matrix of an orthogonal projection is symmetric.

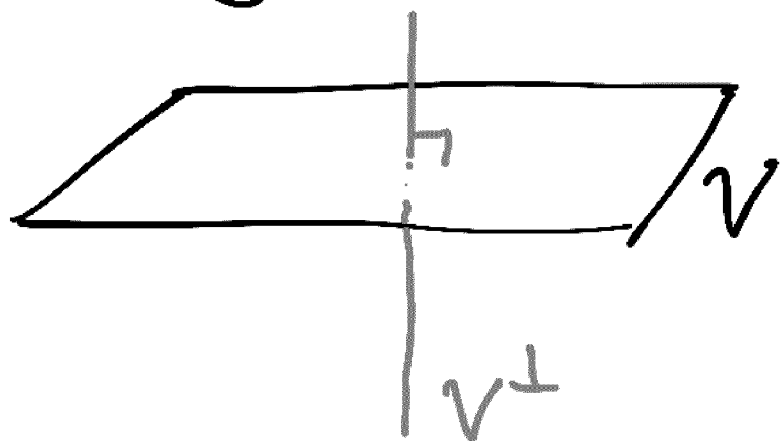
Proof: The matrix is of the form

$P = QQ^T$. We compute its transpose:

$$P^T = (QQ^T)^T = (Q^T)^T Q^T = QQ^T = P. \quad \square$$

Orthogonal complements

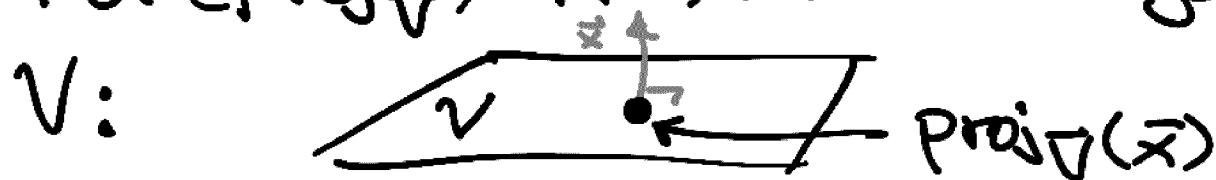
If V is a subspace of $\mathbb{R}^n$, then its orthogonal complement is the set of all vectors that are orthogonal to V :



More formally

$$V^\perp = \{ \vec{x} \text{ in } \mathbb{R}^n \mid \vec{v} \cdot \vec{x} = 0 \text{ for all } \vec{v} \text{ in } V \}$$

If $\text{proj}_V : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is orthogonal projection, then geometrically we realize that a vector $\vec{x}$ is in $\ker(\text{proj}_V)$ if $\vec{x}$ is orthogonal to V :



$$\text{So } \ker(\text{proj}_V) = V^\perp.$$

Some properties:

Let V be a subspace of $\mathbb{R}^n$.

(a) $V^\perp$ is a subspace of $\mathbb{R}^n$

(b) The only vector in both V and $V^\perp$ is $\vec{0}$.

(c) $\dim V + \dim V^\perp = n$

(d) $(V^\perp)^\perp = V$.
