

Recall: • A basis $(\vec{u}_1, \dots, \vec{u}_m)$ of a subspace V is orthonormal (ON) if $\|\vec{u}_i\| = 1$ and $\vec{u}_i \cdot \vec{u}_j = 0$ if $i \neq j$.

§5.3 Orthogonal transformations

Def: A linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is orthogonal if

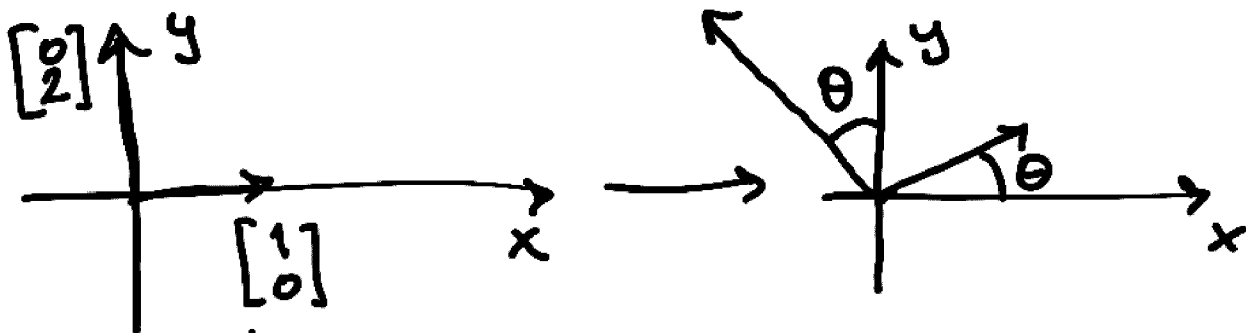
$$\|T(\vec{x})\| = \|\vec{x}\| \text{ for all } \vec{x} \text{ in } \mathbb{R}^n.$$

In other words, T preserves the length of vectors.

Ex: Last time we discussed reflections and saw if V is a subspace in $\mathbb{R}^n$ then

$$\|\text{ref}_V(\vec{x})\| = \|\vec{x}\|.$$

Ex: $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ rotation by θ radians



Rotation preserves lengths of vectors, so it's an orthogonal transformation

$$T(\vec{x}) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \vec{x}$$

Orthogonal transformations do not only preserve lengths, but they also preserve angles.

Theorem: Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be orthogonal.

If $\vec{x}$ and $\vec{y}$ are two orthogonal vectors in $\mathbb{R}^n$ (meaning $\vec{x} \cdot \vec{y} = 0$), then $T(\vec{x})$ and $T(\vec{y})$ are orthogonal.

Proof: We have seen previously that $\vec{x}$ and $\vec{y}$ are orthogonal if and only if $\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$. So to prove

that $T(\vec{x})$ and $T(\vec{y})$ are orthogonal
it suffices to prove

$$\|T(\vec{x}) + T(\vec{y})\|^2 = \|T(\vec{x})\|^2 + \|T(\vec{y})\|^2.$$

We have:

$$\begin{aligned}\|T(\vec{x}) + T(\vec{y})\|^2 &= \|T(\vec{x} + \vec{y})\|^2 && (T \text{ linear}) \\ &= \|\vec{x} + \vec{y}\|^2 && (T \text{ orthogonal}) \\ &= \|\vec{x}\|^2 + \|\vec{y}\|^2 && (\vec{x}, \vec{y} \text{ orthogonal}) \\ &= \|T(\vec{x})\|^2 + \|T(\vec{y})\|^2 && (T \text{ orthogonal})\end{aligned}$$

□

For an orthogonal transformation

$T: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$ we also
say that the matrix A is orthogonal.

Thm: (a) A linear transformation

$T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is orthogonal if and
only if the vectors

$T(\vec{e}_1), \dots, T(\vec{e}_n)$ form an ON-basis

(b) An $(n \times n)$ -matrix A is
orthogonal if and only if

the columns of A form an ON-basis for $\mathbb{R}^n$.

Ex: $A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$ is not orthogonal because $\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix}\right)$ is not an ON-basis. The two vectors are orthogonal, since $\begin{bmatrix} 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 4 \end{bmatrix} = 4(-3) + 3 \cdot 4 = 0$, but the vectors are not unit vectors

$$\left\{ \begin{array}{l} \left\| \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5 \\ \left\| \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\| = 5 \end{array} \right.$$

The matrix $\frac{1}{5} \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$ is orthogonal, though.

Ex:

Is the matrix $A = \frac{1}{2} \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$ orthogonal?

In other words we need to check if

the columns form an ON-basis. Call the columns $\vec{v}_1, \dots, \vec{v}_4$.

$$\|\vec{v}_1\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{4}{4}} = 1$$

$$\vec{v}_1 \cdot \vec{v}_2 = \frac{1}{4}(1(-1) + 1(-1) + 1 \cdot 1 + 1 \cdot 1) = 0$$

$$\vec{v}_1 \cdot \vec{v}_3 = \frac{1}{4}(1(-1) + 1 \cdot 1 + 1(-1) + 1 \cdot 1) = 0$$

$$\vec{v}_1 \cdot \vec{v}_4 = \frac{1}{4}(1(-1) + 1 \cdot 1 + 1 \cdot 1 + 1(-1)) = 0$$

$$\|\vec{v}_2\| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{4}{4}} = 1$$

$$\vec{v}_2 \cdot \vec{v}_3 = \frac{1}{4}((-1)(-1) + (-1)1 + 1(-1) + 1 \cdot 1) = 0$$

$$\vec{v}_2 \cdot \vec{v}_4 = \frac{1}{4}((-1)(-1) + (-1) \cdot 1 + 1 \cdot 1 + 1(-1)) = 0$$

$$\|\vec{v}_3\| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{4}{4}} = 1$$

$$\vec{v}_3 \cdot \vec{v}_4 = \frac{1}{4}((-1)(-1) + 1 \cdot 1 + (-1) \cdot 1 + 1 \cdot (-1)) = 0$$

$$\|\vec{v}_4\| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{4}{4}} = 1$$

So $(\vec{v}_1, \dots, \vec{v}_4)$ is an ON-basis, and A is therefore orthogonal.

Thm: Let A and B be orthogonal $(n \times n)$ -matrices.

(a) AB is orthogonal

(b) A^{-1} exists and is orthogonal.

Proof: A matrix A is orthogonal by definition if $\|A\vec{x}\| = \|\vec{x}\|$.

$$(a) \|A(B(\vec{x}))\| \underset{\substack{\uparrow \\ A \text{ orthogonal}}}{=} \|B(\vec{x})\| \underset{\substack{\uparrow \\ B \text{ orthogonal}}}{=} \|\vec{x}\|$$

(b) First, since the columns form an ON-basis, they are in particular linearly independent. Therefore $\text{rank}(A) = n$, so A^{-1} exists. Then

$$\|A^{-1}(\vec{x})\| \underset{\substack{\uparrow \\ A \text{ orthogonal}}}{=} \|A(A^{-1}(\vec{x}))\| = \|\mathbf{I}_n \vec{x}\| = \|\vec{x}\|.$$

□

Ex: Consider $A = \frac{1}{7} \begin{bmatrix} 2 & 6 & 3 \\ 3 & 2 & -6 \\ 6 & -3 & 2 \end{bmatrix}$. One

can check that it's orthogonal.

Now construct a new matrix B so that entry ij in B is entry ji in A . This is the same thing as

making the rows in A the columns of B .

$$B = \frac{1}{7} \begin{bmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ 3 & -6 & 2 \end{bmatrix}. \text{ Then let's compute } BA.$$

$\uparrow \quad \uparrow \quad \uparrow$
rows in A

$$BA = \frac{1}{49} \begin{bmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ 3 & -6 & 2 \end{bmatrix} \begin{bmatrix} 2 & 6 & 3 \\ 3 & 2 & -6 \\ 6 & -3 & 2 \end{bmatrix}$$
$$= \frac{1}{49} \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Conclusion: B is the inverse to A !

Note that if $A = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \\ 1 & & 1 \end{bmatrix}$ then

$$B = \begin{bmatrix} -\vec{v}_1 & - \\ \vdots & \\ -\vec{v}_n & - \end{bmatrix}, \text{ and}$$

$$BA = \begin{bmatrix} -\vec{v}_1 & - \\ \vdots & \\ -\vec{v}_n & - \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \\ 1 & & 1 \end{bmatrix} = \begin{bmatrix} \vec{v}_1 \cdot \vec{v}_1 & \dots & \vec{v}_1 \cdot \vec{v}_n \\ \vdots & \ddots & \vdots \\ \vec{v}_n \cdot \vec{v}_1 & \dots & \vec{v}_n \cdot \vec{v}_n \end{bmatrix}$$

and since $(\vec{v}_1, \dots, \vec{v}_n)$ is an ON-basis

we have $\vec{v}_i \cdot \vec{v}_j = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$. This explains why $BA = I_n$.

The matrix B in the previous example has a special name: It's called the transpose of A , and is denoted A^T .

The example & the explanation proves the following:

Thm: If A is an orthogonal $(n \times n)$ -matrix, then $A^{-1} = A^T$.

In fact $A^{-1} = A^T$ also implies that A is orthogonal.

Let's summarize what we now know about orthogonal matrices:

The following are equivalent for an

$(n \times n)$ -matrix A :

- (a) A is orthogonal
 - (b) The linear transformation $T(\vec{x}) = A\vec{x}$ preserves lengths, that is $\|T(\vec{x})\| = \|\vec{x}\|$.
 - (c) The columns of A form an ON-basis of $\mathbb{R}^n$.
 - (d) $A^{-1} = A^T$
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A closely related property of matrices is the following:

Def: An $(n \times n)$ -matrix A is symmetric if $A = A^T$.

The transpose of A satisfies:

- (a) $(A+B)^T = A^T + B^T$ for all $(m \times n)$ -matrices A, B
- (b) $(kA)^T = kA^T$ for all $(m \times n)$ -matrices A, B and all scalars k
- (c) $(AB)^T = B^T A^T$ for all $(m \times p)$ -matrices A and all

$(p \times n)$ -matrices B

(d) $\text{rank}(A^T) = \text{rank}(A)$ for all matrices A

(e) $(A^T)^{-1} = (A^{-1})^T$ for all $(n \times n)$ -matrices A .
