MAT 211 L1 Mon Aug 24

§Admin stuff: (see syllabus)

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Office hours: MW 2:30-3:30pm

Tue 5-6 pm (MLC)

Course web page:

math.stonybrook.edu/~jasplund/mat211_fall26

Textbook: Linear Algebra with Applications (5th ed). Otto Bretscher

Quizzes:

Almost weekly (see course webpage for dates), we will have an in-class quiz.

- 15 minutes (end of lecture)
- 1-2 problems
- Worst score dropped at the end of the course.

Important dates:

- Midterm 1: Wed Oct 7
- Midterm 2: Wed Nov 11

- Final exam: Mon Dec 14
- Quizzes: First one Mon Aug 31, see syllabus/course schedule for all dates

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Final grade:

Quiz: 15%
Midterms: 25% each
Final: 35%.

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§1.1 Systems of linear equations

Solving linear systems of equations such as

$$\begin{cases} x-2y=3 \\ 2x-3y=6 \end{cases} \quad \text{or} \quad \begin{cases} x+y+3z=3 \\ x+3y+2z=3 \\ 3x+y+z=0 \end{cases}$$

is ubiquitous and important not only in mathematics, but also in physics, chemistry, computer science, economics, etc.

<u>Ex</u>: $\begin{cases} x-2y=3 \\ 2x-3y=6 \end{cases}$ can be solved

by solving for one variable in one of the equations, and substituting it into the other one:

Eq 1: $x-2y=3 \iff x=3+2y$.

Sub into eq 2:

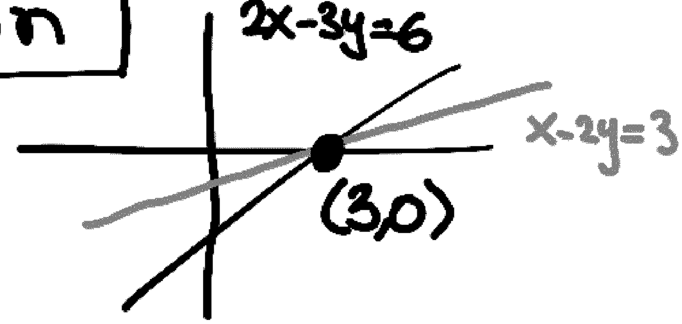
$$2x-3y=6 \iff 2(3+2y)-3y=6$$

$\iff \cancel{6}+4y-3y=\cancel{6} \iff \boxed{y=0}$.

Then $x=3+2y=3$. So the system has a single solution

$\begin{cases} x=3 \\ y=0 \end{cases}$

Geometrically:



<u>Ex</u>: Let's consider instead

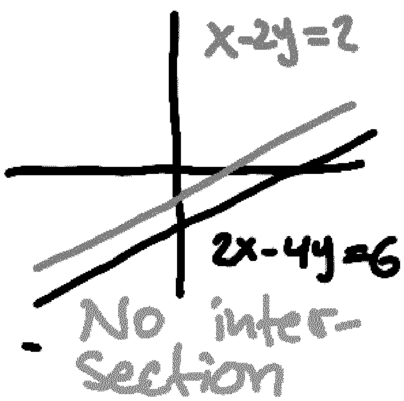
$\begin{cases} x-2y=2 \\ 2x-4y=6 \end{cases}$. Eq 1: $x=2+2y$

Sub into eq 2: $2(2+2y)-4y=6$

$\iff 4+4y-4y=6 \iff 4=6$.

But this equation is not true

$\rightsquigarrow$ <u>The system has no solution</u>. System is <u>inconsistent</u>.



<u>Ex</u>: Now consider

$\begin{cases} x-2y=3 \\ 2x-4y=6 \end{cases}$. Eq 1: $x=3+2y$

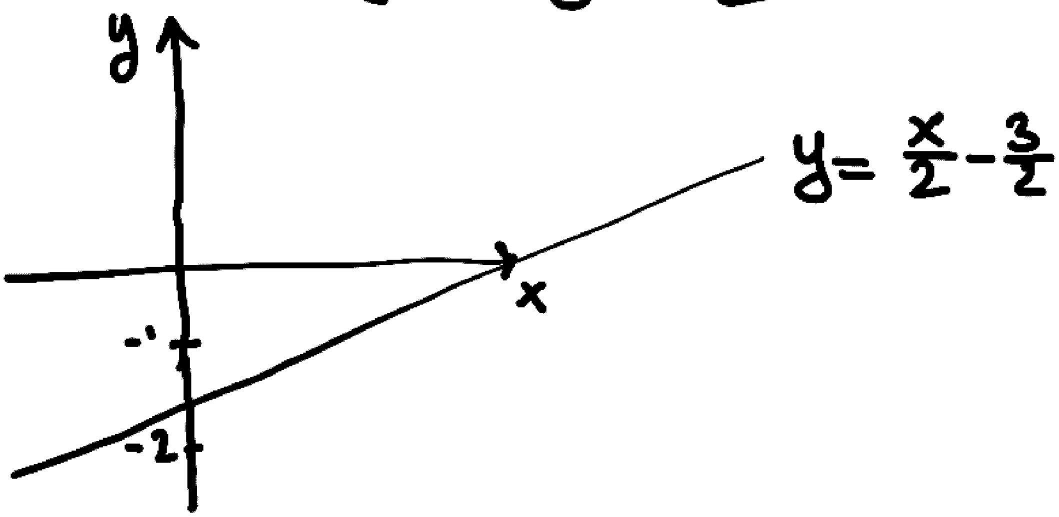
Sub into eq 2:

$2(3+2y)-4y=6 \iff 6+4y-4y=6$

$\Leftrightarrow 6=6$. This equation is true for all values of y.

Conclusion: All values of (x,y) so that $x=3+2y$ (e.g. $(x,y)=(3,0),(5,1)$ $(1,-1)$, ...) are solutions. There's an infinite number of solutions, and they all lie on the line

$x=3+2y \Leftrightarrow y=\frac{x}{2}-\frac{3}{2}$ in the plane



The system $\begin{cases} x+y+3z=3 \\ x+3y+2z=3 \\ 3x+y+z=0 \end{cases}$ can be studied similarly, but it's more laborsome. A better way to solve the system is by adding entire equations together to eliminate variables.

Ex: $\begin{cases} x-2y=3 \\ 2x-3y=6 \end{cases}$ We will use the first equation to eliminate $2x$ from the second one.

1. Multiply eq 1 by -2.
2. Add eq 1 to eq 2.

$$\begin{cases} x-2y=3 \\ 2x-3y=6 \end{cases} \xrightarrow{1.} \begin{cases} -2x+4y=-6 \\ 2x-3y=6 \end{cases}$$

$$\xrightarrow{2.} \begin{cases} -2x+4y=-6 \\ y=0 \end{cases}$$

All in one go would be to add $(-2)\cdot$ Eq 1 to Eq 2:

$$(-2)\ + \begin{cases} x-2y=3 \\ 2x-3y=6 \end{cases} \longrightarrow \begin{cases} -2x+4y=-6 \\ y=0 \end{cases}$$

Finally read off $y=0$ & sub into first eq to arrive at $x=3$.

Ex: Now let's solve

$$\begin{cases} x+y+3z=3 \\ x+3y+2z=3 \\ 3x+y+z=2 \end{cases}$$ using this method.

We first use eq 1 to eliminate x & $3x$ from eqs 2 and 3:

$$-3 \quad -1 \quad \begin{cases} x+y+3z=3 \\ x+3y+2z=3 \\ 3x+y+z=0 \end{cases}$$

$$\longrightarrow \begin{cases} x+y+3z=3 \\ 2y-z=0 \\ -2y-8z=-9 \end{cases}$$

Then add eq 2 to eq 3

$$\begin{cases} x+y+3z=3 \\ 2y-z=0 \\ -2y-8z=-9 \end{cases} \longrightarrow \begin{cases} x+y+3z=3 \\ 2y-z=0 \\ -9z=-9 \end{cases}$$

Read off $\boxed{z=1}$ sub into eq 2

gives $y=1/2$, and subbing into eq 1

gives $x = 3 - y - 3z = 3 - \frac{1}{2} - 3 = -1/2$

solution is $\begin{cases} x = -1/2 \\ y = 1/2 \\ z = 1 \end{cases}$

Geometrically, each of the eqns

$$\begin{aligned} x + y + 3z &= 3 \\ x + 3y + 2z &= 3 \\ 3x + y + z &= 2 \end{aligned}$$

describe a plane in 3d space, and they have a common intersection point at $(x, y, z) = (-\frac{1}{2}, \frac{1}{2}, 1)$.

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§1.2 Matrices & Gauss-Jordan elimination

When solving the system $\begin{cases} x + y + 3z = 3 \\ x + 3y + 2z = 3 \\ 3x + y + z = 2 \end{cases}$ it's cumbersome to keep writing x, y and z all the time. It's more efficient to just keep track of the coefficients.

$$\begin{cases} x+y+3z=3 \\ x+3y+2z=3 \\ 3x+y+z=2 \end{cases} \longleftrightarrow \left[\begin{array}{ccc:c} 1 & 1 & 3 & 3 \\ 1 & 3 & 2 & 3 \\ 3 & 1 & 1 & 2 \end{array}\right]$$

dotted line just reminds us that the last column corresponds to the RHS of the equations

Def: A <u>matrix</u> is an array of real numbers.

Ex: $\begin{bmatrix} 1 & 1 & 3 & 3 \\ 1 & 3 & 2 & 3 \\ 3 & 1 & 1 & 2 \end{bmatrix}$ 3x4 matrix

#rows #columns

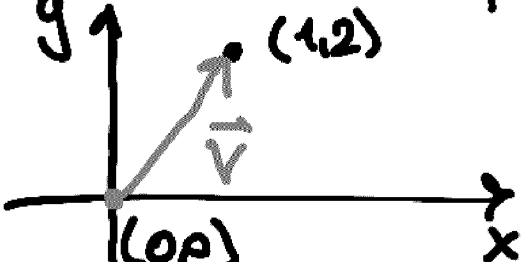
$$\underset{4\times 1}{\begin{bmatrix} \pi \\ 0.1 \\ 0 \\ 0 \end{bmatrix}}, \quad \underset{1\times 2}{\begin{bmatrix} 5 & 4 \end{bmatrix}}, \quad \underset{2\times 2}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}, \quad \underset{1\times 1}{\begin{bmatrix} -2 \end{bmatrix}}$$

Def: A $(n\times 1)$-matrix (just 1 column)

$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ is called a <u>column vector</u> or just <u>vector</u>.

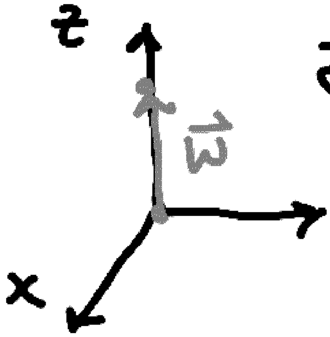
A $(1 \times m)$-matrix (just 1 row) $[b_1 \cdots b_m]$ is called a <u>row vector</u> (but never just a vector! that term is reserved for column vectors)

<u>Ex</u>: $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ represented as an arrow in the plane:



Starts at $(0,0)$ & ends at $(1,2)$.

$\vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



from $(0,0,0)$ to $(0,0,1)$.

We will later discuss how to visualize matrices that are not vectors.

Let's go back to systems of linear equations:

$$\begin{cases} x+y+3z=3 \\ x+3y+2z=3 \\ 3x+y+z=0 \end{cases} \longleftrightarrow \left[\begin{array}{ccc:c} 1 & 1 & 3 & 3 \\ 1 & 3 & 2 & 3 \\ 3 & 1 & 1 & 0 \end{array}\right]$$

We call this the "augmented matrix" associated with the system.

Performing row operations to the augmented matrix is now done similar as before:

$$\begin{array}{c} -1 \\ \\ \\ \end{array}\left[\begin{array}{ccc:c} 1 & 1 & 3 & 3 \\ 1 & 3 & 2 & 3 \\ 3 & 1 & 1 & 0 \end{array}\right] \longrightarrow \begin{array}{c} -3 \\ \\ \\ \end{array}\left[\begin{array}{ccc:c} 1 & 1 & 3 & 3 \\ 0 & 2 & -1 & 0 \\ 3 & 1 & 1 & 0 \end{array}\right]$$

$$\longrightarrow \begin{array}{c} \\ 1 \\ \\ \end{array}\left[\begin{array}{ccc:c} 1 & 1 & 3 & 3 \\ 0 & 2 & -1 & 0 \\ 0 & -2 & -8 & -9 \end{array}\right] \longrightarrow \begin{array}{c} \\ 1 \\ \\ \end{array}\left[\begin{array}{ccc:c} 1 & 1 & 3 & 3 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & -9 & -9 \end{array}\right]$$

We can either stop here & translate back to the system of linear equs

$$\begin{cases} x+y+3z=3 \\ 2y-z=0 \\ -9z=-9 \end{cases}$$

& solve the equs one at a time.

We can also continue:

$$\begin{array}{c} \\ \\ -\frac{1}{9} \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & -9 & -9 \end{array}\right] \longrightarrow \begin{array}{c} \\ \\ 3/9 \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -9 & -9 \end{array}\right]$$

$$\longrightarrow \begin{array}{c} \\ -1/2 \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -9 & -9 \end{array}\right] \longrightarrow \begin{array}{c} \\ \frac{1}{2} \\ -\frac{1}{9} \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -9 & -9 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

this gives the solution to the system directly:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}.$$

Notice the shape:

Leading elements

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Zeros above leading elements

Staircase

- This form is called "row-reduced echelon form" (RREF).
- The procedure we just performed is called <u>Gauss-Jordan elimination</u>.