MAT 203 L9 Wed Mar 4

Recall:

- $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ exists if

$|f(x,y) - L|$ can be made arbitrarily small by making the distence between (x,y) and (a,b) very small

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§4.2 Continuity

$f(x,y)$ is continuous at (a,b) if

(1) $f(a,b)$ exists (2) $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists

(3) $f(a,b) = \lim\limits_{(x,y)\to(a,b)} f(x,y)$

Thm: Let $f(x,y)$ and $g(x,y)$ be continuous at $(x,y) = (a,b)$. Then

(1) $f(x,y) \pm g(x,y)$ is continuous

at $(x,y)=(a,b)$

(2) $f(x,y)g(x,y)$ is continuous at $(x,y)=(a,b)$

(3) If $h(z)$ is continuous at $z=f(a,b)$, then $h(f(x,y))$ is continuous at $(x,y)=(a,b)$

Ex: (1) $f(x,y)=4x^2y^3$, $h(z)=\cos(z)$ are both continuous everywhere, so $h(f(x,y))=\cos(4x^2y^3)$ is also continuous.

(2) $f(x,y)=\begin{cases} \frac{xy}{x^2+y^2} & , (x,y)\neq(0,0) \\ 0 & , (x,y)=(0,0) \end{cases}$

$f(0,0)$ exists, but we saw last time that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ doesn't exist, so $f(x,y)$ is not continuous at $(x,y)=(0,0)$.

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§4.3 Partial derivatives

Def: Let $f(x,y)$ be a function of two variables.

The partial derivative of f, with respect to x, written $\frac{\partial f}{\partial x}$, or f_x is defined as:

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

The partial derivative of f with respect to y is defined as

$$\frac{\partial f}{\partial y}(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

$\frac{\partial f}{\partial x}$ is the derivative of $f(x,y)$ with respect to x, and treating y as if it was a constant!

(and vice versa for $\frac{\partial f}{\partial y}$)

Ex: $\frac{\partial}{\partial x}(3xy + x + 2y) = 3y + 1$

$\frac{\partial}{\partial y}(3xy + x + 2y) = 3x + 2$

All rules for derivatives we learned in Calc I applies.

Ex: (1) $f(x,y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12$

$\frac{\partial f}{\partial x} = f_x = 2x - 3y - 4$

$\frac{\partial f}{\partial y} = f_y = -3x + 4y + 5$

(2) $g(x,y) = \sin(x^2y - 2x + 4)$

$\frac{\partial g}{\partial x} = \cos(x^2y - 2x + 4) \cdot (2xy - 2)$

$\frac{\partial g}{\partial y} = \cos(x^2y - 2x + 4) \cdot (x^2)$.

Everything also works the same w/ 3 or more variables.

<u>EX</u>: $f(x,y,z) = \ln(x^2+y^2+z^2)$

$$f_x = \frac{1}{x^2+y^2+z^2} \cdot 2x$$

$$f_y = \frac{1}{x^2+y^2+z^2} \cdot 2y$$

$$f_z = \frac{1}{x^2+y^2+z^2} \cdot 2z$$

Higher-order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

We also have mixed partial derivatives:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

$\boxed{\text{y first, then x}}$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}.$$

<u>Ex</u>: $f(x,y) = xe^{-3y} + \sin(2x-5y)$

$$f_x = e^{-3y} + \cos(2x-5y)\cdot 2$$

$$f_{xx} = -2\sin(2x-5y)\cdot 2 = -4\sin(2x-5y)$$

$$f_{xy} = -3e^{-3y} - 2\sin(2x-5y)\cdot(-5)$$

$$= -3e^{-3y} + 10\sin(2x-5y)$$

$$f_y = -3xe^{-3y} + \cos(2x-5y)\cdot(-5)$$

$$f_{yy} = 9xe^{-3y} + 5\sin(2x-5y)\cdot(-5)$$

$$= 9xe^{-3y} - 10\sin(2x-5y)$$

$$f_{yx} = -3e^{-3y} + 5\sin(2x-5y)\cdot 2$$

$$= -3e^{-3y} + 10\sin(2x-5y)$$

Note $\boxed{f_{xy} = f_{yx}}$

In fact, this is always true!

<u>Thm</u>: If $f(x,y)$ is a function of 2 variables such that f_{xy} and f_{yx} exist. Then $f_{xy} = f_{yx}$.

<u>Ex</u>: Let $f(x,y) = \frac{1}{2}(e^y - e^{-y})\sin x$.
Show $f_{xx} + f_{yy} = 0$

$$f_x = \frac{1}{2}(e^y - e^{-y})\cos x$$

$$f_{xx} = -\frac{1}{2}(e^y - e^{-y})\sin x$$

$$f_y = \frac{1}{2}(e^y - e^{-y}(-1))\sin x = \frac{1}{2}(e^y + e^{-y})\sin x$$

$$f_{yy} = \frac{1}{2}(e^y + e^{-y}(-1))\sin x$$

$$= \frac{1}{2}(e^y - e^{-y})\sin x = -f_{xx}.$$

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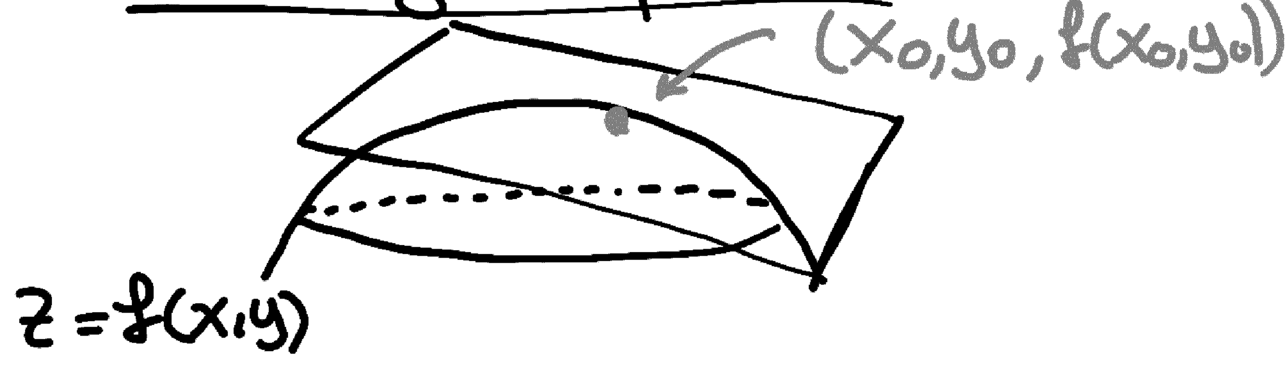
§4.4 Tangent planes



Def: If S is the surface $z = f(x,y)$ in space, the tangent plane at (x_0, y_0) to S is given by the eqn

$$\boxed{f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z - f(x_0,y_0)) = 0}$$

Remark: A normal vector to S is $\langle f_x(x_0,y_0), f_y(x_0,y_0), -1 \rangle$.

Ex: $f(x,y) = 2x^2 - 3xy + 8y^2 + 2x - 4y + 4$

let's find the tangent plane at $(x,y) = (2,-1)$.

$f_x(x,y) = 4x - 3y + 2$

$f_y(x,y) = -3x + 16y - 4$

$f_x(2,-1) = 8 + 3 + 2 = 13$

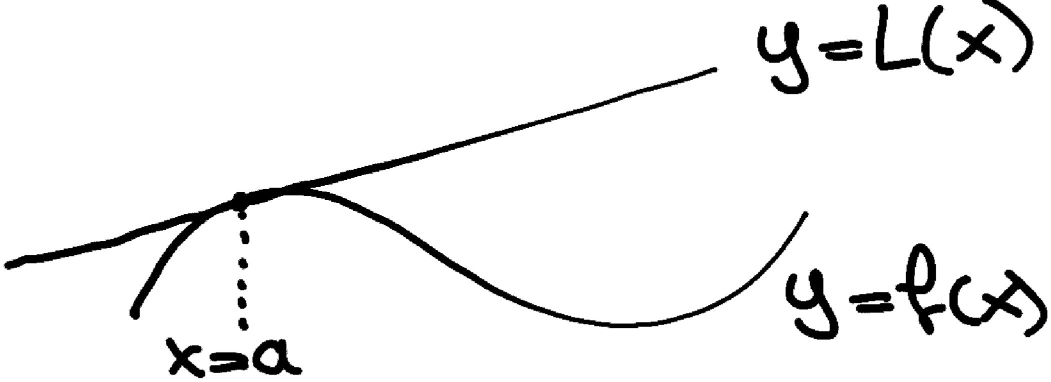
$f_y(2,-1) = -6 - 16 - 4 = -26$

$$f(2,-1) = 2\cdot 4 - 3\cdot 2\cdot(-1) + 8(-1)^2 + 2\cdot 2 - 4(-1) + 4 = 26$$

Tangent plane at $(2,-1)$ is:

$$13(x-2) - 26(y+1) - (z-26) = 0$$

From Calc I again:



$$L(x) = f(a) + f'(a)(x-a)$$

is the linear approximation of $f(x)$ at $x=a$.

Similarly,

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

is the linear approximation of $f(x,y)$ at $(x,y)=(a,b)$.

The idea is that if f is differentiable, it has a linear approximation at $x=a$.

<u>Def</u>: $f(x,y)$ is <u>differentiable</u> at $(x,y)=(a,b)$ if for all points (x,y) near (a,b), we can write

$$f(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)+E(x,y)$$

error term satisfies

$$\lim_{(x,y)\to(a,b)} \frac{E(x,y)}{\sqrt{(x-a)^2+(y-b)^2}}=0$$

<u>Important</u>: Just because f_x and f_y exist, it does <u>not</u> mean f is differentiable.

Ex: $f(x,y) = 2x^2 - 4y$ is differentiable at the origin.

$f_x(x,y) = 4x, \quad f_x(0,0) = 0$

$f_y(x,y) = -4, \quad f_x(0,0) = -4$

$f(0,0) = 0.$

$$E(x,y) = \Big(f(x,y) - \underbrace{f(0,0)}_{=0}\Big) - \underbrace{f_x(0,0)}_{=0}(x-0) - \underbrace{f_y(0,0)}_{=-4}(y-0)$$

$$= 2x^2 - 4y + 4y = 2x^2.$$

Now we have

$$\lim_{(x,y)\to(0,0)} \frac{2x^2}{\sqrt{x^2+y^2}} = 0 \text{ because}$$

$$0 \le \lim_{(x,y)\to(0,0)} \frac{2x^2}{\sqrt{x^2+y^2}} \le \lim_{(x,y)\to(0,0)} \frac{2x^2}{\sqrt{x^2+0}}$$

$$= \lim_{x\to 0} \frac{2x^2}{|x|} = 0$$

$\boxed{\begin{array}{l} E(x,y) \ge 0 \\ \text{and the} \\ \text{square root} \ge 0 \end{array}}$