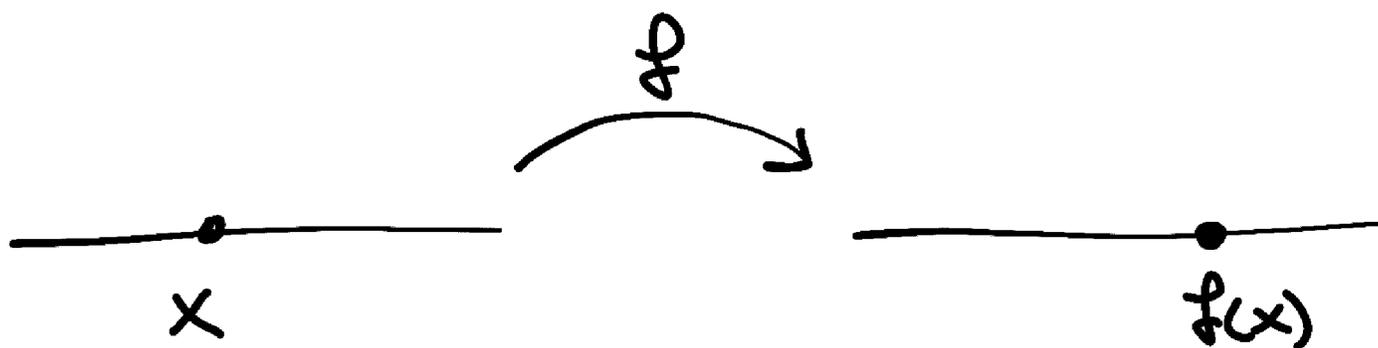
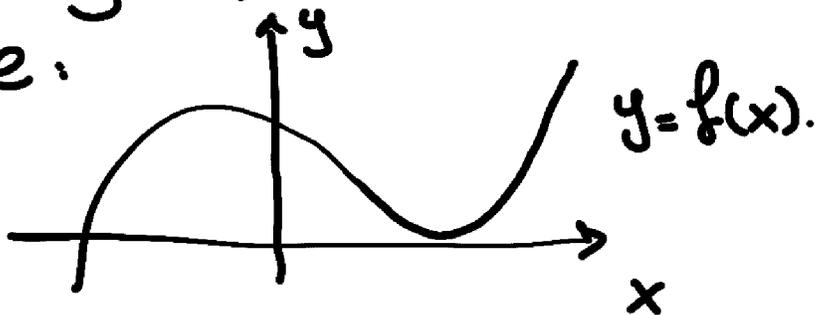


§4.1 Functions of Several Variables

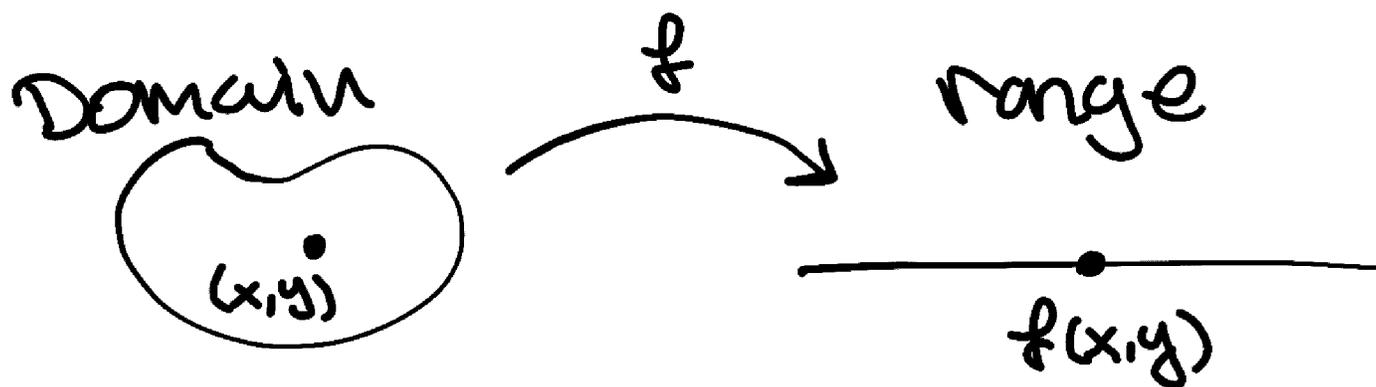
Recall from Calc I/II that a function $f(x)$ (of 1 variable) has 1 input and 1 output; typically both input and outputs are real numbers (scalars):



Graph: Plotting all points (x, y) in the plane such that $y = f(x)$ often gives a curve in the plane.

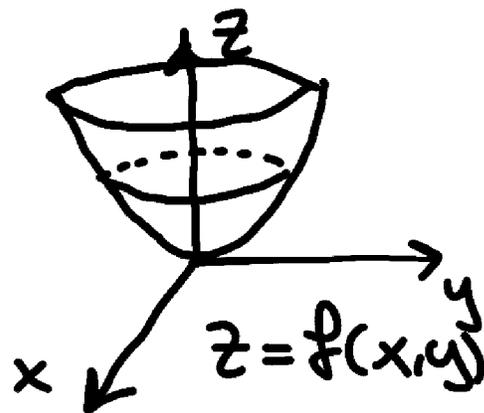


Now: We will allow 2 inputs
 $f(x,y)$. Inputs and output are still
real numbers (scalars)



Graph: The graph will be a
surface in 3D

EX: $f(x,y) = x^2 + y^2$
graph is an elliptic
paraboloid.



Def: A function of two variables
maps each ordered pair (x,y)
in a subset D of the plane \mathbb{R}^2
to a unique real number z .

$D =$ "domain"

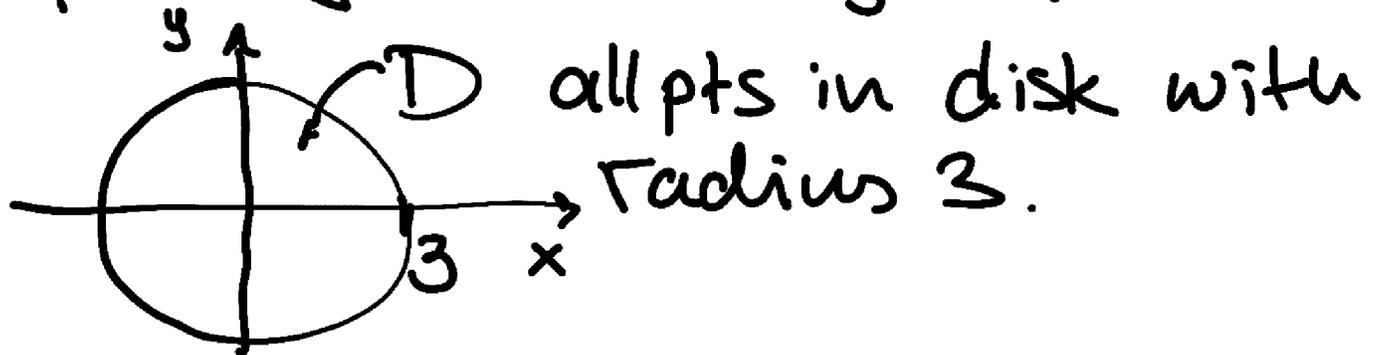
"range" = set of all numbers z such that $z = f(x,y)$, for some (x,y) in D

Typically, the domain for us will be the "largest possible."

Ex The domain of

$f(x,y) = \sqrt{9-x^2-y^2}$ is all (x,y) such that $9-x^2-y^2 \geq 0$ since we do not allow complex numbers as outputs. Domain is

$$9 - x^2 - y^2 \geq 0 \Leftrightarrow x^2 + y^2 \leq 9$$



Range is $[0,3]$ because

$$f(x,y) = \sqrt{9-x^2-y^2} \geq 0, \text{ and}$$

$$9-x^2-y^2 \leq 9, \text{ so } \sqrt{9-x^2-y^2} \leq \sqrt{9} = 3$$

EX: $f(x,y) = 3x + 5y + 2$.

Domain is all (x,y) in \mathbb{R}^2 , since the expression is defined for all x,y .

Range is all real numbers:

Given any z , we can always find x,y such that

$$\boxed{z = 3x + 5y + 2}$$

Of course we can also consider functions that depend on 3 variables (or more...), too:

$f(x,y,z)$, $f(x,y,z,w)$, etc.

EX: $f(x,y,z) = y \sin(x) + z^3 xy$

§4.2 Limits and Continuity

Recall limits from Calc I again:

If L is a real number sth

$L = \lim_{x \rightarrow a} f(x)$, it means that
"when x is very close to a ,
 $f(x)$ will be very close to L "

A little more precisely, the distance
 $|f(x) - L|$ can be made as small
as we wish, by making the
distance $|x - a|$ very very small.

Def: Let L be a real number.
We say $L = \lim_{(x,y) \rightarrow (a,b)} f(x,y)$ if

$|f(x,y) - L|$ can be made arbitrarily
small by making the distance
between (x,y) and (a,b) small.

"As (x,y) gets close to (a,b) , $f(x,y)$
will get close to L ."

Ex: $\lim_{(x,y) \rightarrow (1,1)} x + 2y = 3$

because when (x,y) is very close to $(1,1)$, the value of $f(x,y) = x+2y$ is very close to 3.

We now need some limit laws to help us to compute limits.

Theorem:

Let $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$, $C =$
constant
 $\lim_{(x,y) \rightarrow (a,b)} g(x,y) = M$

$$(1) \lim_{(x,y) \rightarrow (a,b)} C = C$$

$$(2) \lim_{(x,y) \rightarrow (a,b)} x = a, \quad \lim_{(x,y) \rightarrow (a,b)} y = b$$

$$(3) \lim_{(x,y) \rightarrow (a,b)} (f(x,y) \pm g(x,y)) = L \pm M$$

$$(4) \lim_{(x,y) \rightarrow (a,b)} C f(x,y) = CL$$

$$(5) \lim_{(x,y) \rightarrow (a,b)} f(x,y)g(x,y) = LM$$

$$(6) \lim_{(x,y) \rightarrow (a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M} \quad \text{if } M \neq 0$$

$$(7) \lim_{(x,y) \rightarrow (a,b)} (f(x,y))^n = L^n, \quad n \geq 1 \text{ integer}$$

Ex: (1) $\lim_{(x,y) \rightarrow (1,1)} (x+y) \cdot 3x$

$$= \left(\lim_{(x,y) \rightarrow (1,1)} x+y \right) \left(\lim_{(x,y) \rightarrow (1,1)} 3x \right)$$

$$= \left[\left(\lim_{(x,y) \rightarrow (1,1)} x \right) + \left(\lim_{(x,y) \rightarrow (1,1)} y \right) \right] \left(\lim_{(x,y) \rightarrow (1,1)} 3x \right)$$

$$= [1 + 1] (3) = 6.$$

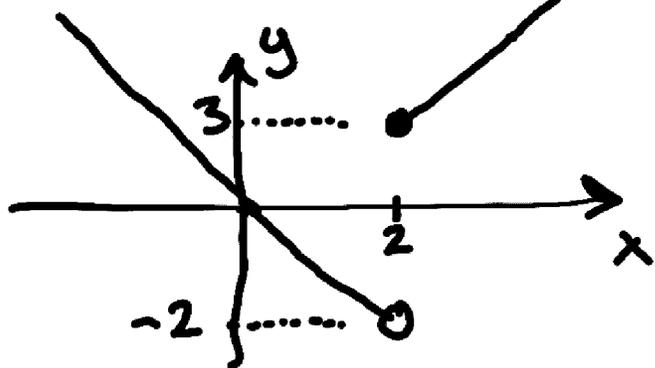
$$(2) \lim_{(x,y) \rightarrow (2,-1)} \frac{2x+3y}{4x-3y} = \frac{\lim_{(x,y) \rightarrow (2,-1)} 2x+3y}{\lim_{(x,y) \rightarrow (2,-1)} 4x-3y}$$

$$= \frac{2 \cdot 2 + 3(-1)}{4 \cdot 2 - 3(-1)} = \frac{1}{11}$$

To understand how limits can

fail to exist, let's consider the case w/ 1 variable:

$$\text{Ex: } f(x) = \begin{cases} x+1, & x \geq 2 \\ -x, & x < 2 \end{cases}$$



$$\lim_{x \rightarrow 2^+} f(x) = 3$$

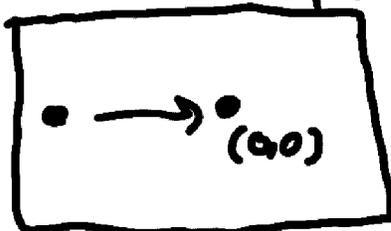
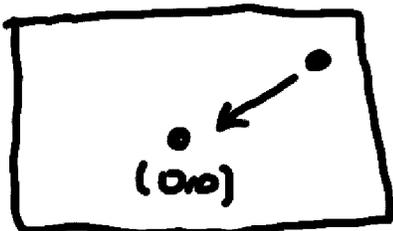
$$\lim_{x \rightarrow 2^-} f(x) = -2$$

When we approach $x=2$ from the left & from the right, we get different results.



For limits where $(x,y) \rightarrow (\infty)$, there are infinitely many directions along which (x,y) can approach

(∞) :

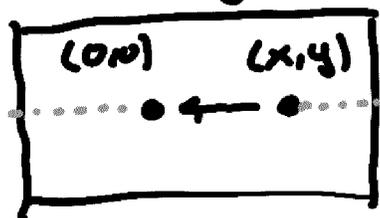


Limits that fail to exist:

Ex (1) $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2}$.

$$f(x,y) = \frac{xy}{x^2+y^2}$$

If $(x,y) \rightarrow (0,0)$ along the x-axis

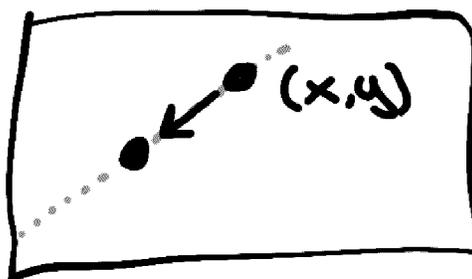


, meaning if $y=0$
and $x \rightarrow 0$,

$$f(x,0) = \frac{x \cdot 0}{x^2 + 0^2} = 0, \text{ then}$$

$$\lim_{x \rightarrow 0} f(x,0) = 0.$$

If $(x,y) \rightarrow (0,0)$ along the
line $x=y$ in the plane,



$$f(x,x) = \frac{x^2}{x^2+x^2} = \frac{1}{2}$$

$$\lim_{x \rightarrow 0} f(x,x) = \frac{1}{2}.$$

therefore, if $(x,y) \rightarrow (0,0)$ along x-axis, and along the line $x=y$, we get two different results, so $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2}$ doesn't exist.

$$(2) \lim_{(x,y) \rightarrow (0,0)} \frac{4xy^2}{x^2+3y^4}$$

$$f(x,y) = \frac{4xy^2}{x^2+3y^4} \quad \text{If } x=y,$$

$$f(x,x) = \frac{4x^3}{x^2+3x^4} = \frac{4x}{1+3x^2}$$

$$\rightarrow \frac{0}{1+0} = 0 \text{ as } x \rightarrow 0$$

$$\text{If } x=y^2, f(y^2,y) = \frac{4y^4}{y^4+3y^4}$$

$$= \frac{4y^4}{4y^4} = 1 \rightarrow 1 \text{ as } y \rightarrow 0$$

So $\lim_{(x,y) \rightarrow (0,0)} \frac{4xy^2}{x^2+3y^4}$ doesn't exist.