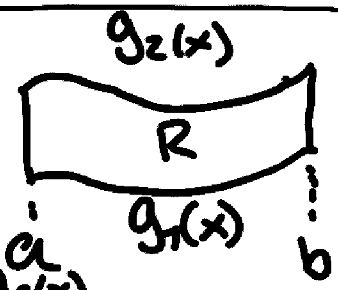
MAT 203 L16 Mon Apr 6

Recall:

• (1)

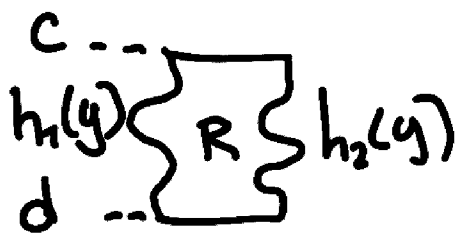


$a \le x \le b$

$g_1(x) \le y \le g_2(x)$

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

(2)



$h_1(y) \le x \le h_2(y)$

$c \le y \le d$

$$\iint_R f(x,y)\,dx\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

• Polar Coord change:

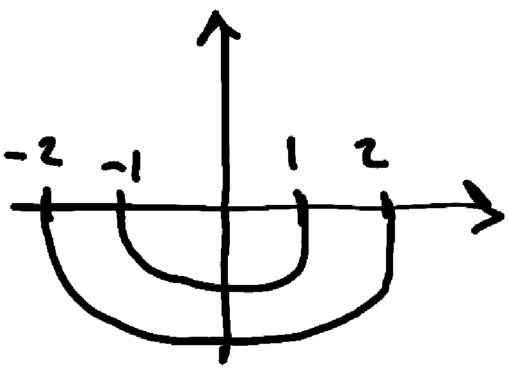
$$dA = dx\,dy = r\,dr\,d\theta.$$

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

<u>Ex</u>: Compute

$$\iint_R x^2+xy \, dxdy \quad \text{where}$$

$$R=\{(r,\theta) \mid 1\le r\le 2,\ \pi\le\theta\le 2\pi\}$$



$$x^2+xy$$

$$=(r\cos\theta)^2+(r\cos\theta)(r\sin\theta)$$

$$= r^2\cos^2\theta + r^2\cos\theta\sin\theta$$

$$\int_\pi^{2\pi}\int_1^2 r^2(\cos^2\theta+\cos\theta\sin\theta)\, r\,dr\,d\theta$$

$$=\left(\int_\pi^{2\pi}\cos^2\theta+\cos\theta\sin\theta\, d\theta\right)\left(\int_1^2 r^3\,dr\right)$$

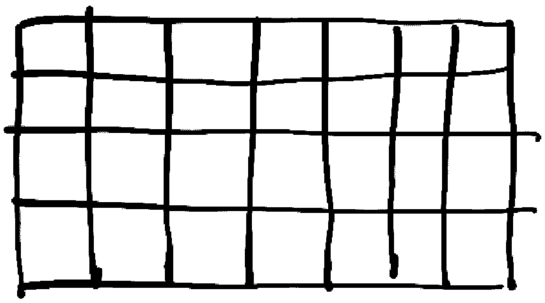
$$=\left[\frac{r^4}{4}\right]_1^2\left(\int_\pi^{2\pi}\frac{1+\cos(2\theta)}{2}+\frac{1}{2}\sin(2\theta)\,d\theta\right)$$

$$=\left(4-\frac{1}{4}\right)\cdot\left[\theta+\frac{\sin(2\theta)}{4}-\frac{\cos(2\theta)}{4}\right]_\pi^{2\pi}$$

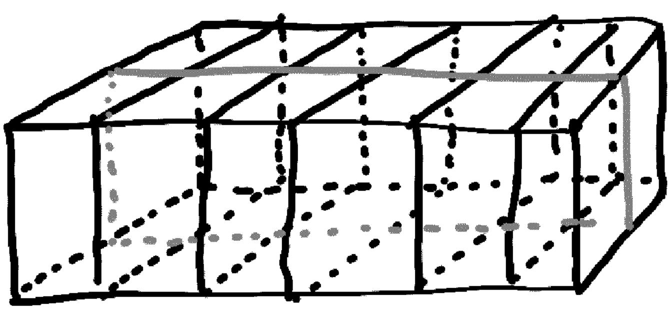
$$=\frac{15}{16}\left(\left(2\pi+0-\frac{1}{4}\right)-\left(\pi+0-\frac{1}{4}\right)\right)=\frac{15\pi}{16}$$

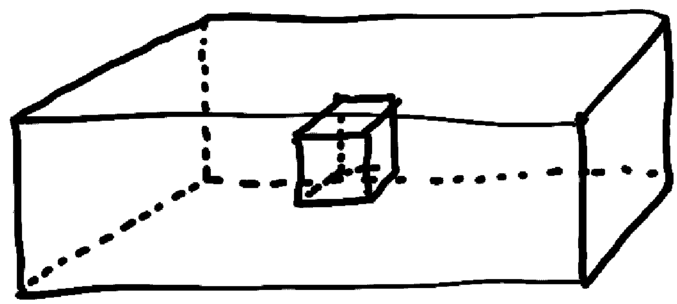
§5.4 Triple integrals:

Similar to how we defined double integrals by subdividing rectangles



we may define triple integrals by subdividing rectangular boxes:





$$\iiint_R f(x,y,z)\,dV, \quad dV = dx\,dy\,dz.$$

The triple integral satisfies the same properties as the double integral. Ex: Compute $\iiint_B x^2yz\,dx\,dy\,dz$

where

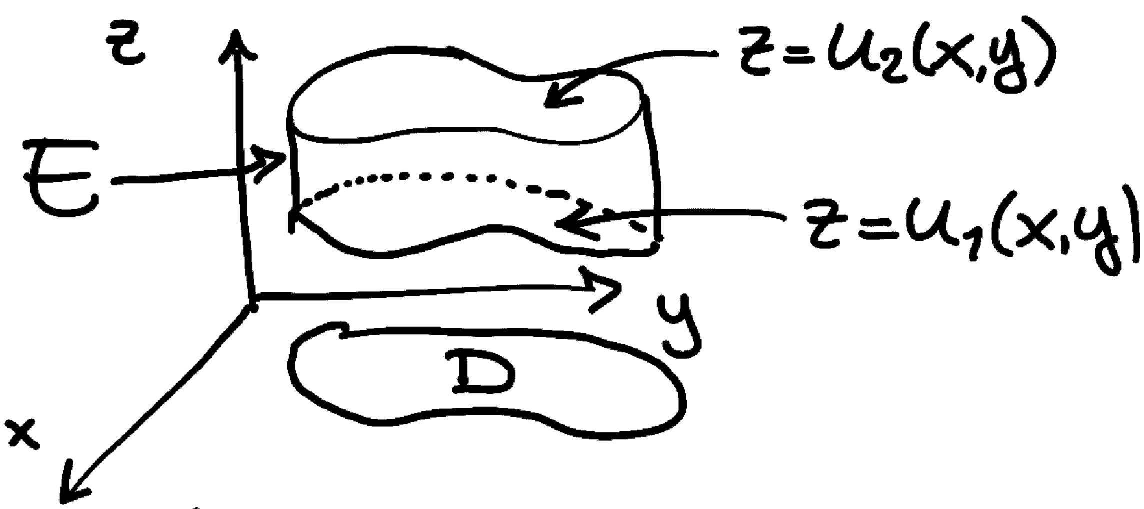
$$B = \{(x,y,z) \mid -2 \le x \le 1,\ 0 \le y \le 3,\ 1 \le z \le 5\}$$

$$\int_{x=-2}^{1} \int_{y=0}^{3} \int_{z=1}^{5} x^2yz\,dz\,dy\,dx$$

$$= \int_{x=-2}^{1} \int_{y=0}^{3} \left[\frac{x^2yz^2}{2}\right]_1^5 dy\,dx = \int_{-2}^{1}\int_{0}^{3} 12x^2y\,dy\,dx$$

$$= \int_{-2}^{1} \left[6x^2y^2\right]_0^3 dx = \int_{-2}^{1} 54x^2\,dx = \left[18x^3\right]_{-2}^{1}$$

$$= 18(1+8) = 162$$



If $E = \{(x,y,z) \mid (x,y) \in D$

$u_1(x,y) \le z \le u_2(x,y)\}$

then

$$\iiint_E f(x,y,z)\,dx\,dy\,dz = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dx\,dy.$$

It works in a similar way when we integrate over a region in space of points (x,y,z) such that

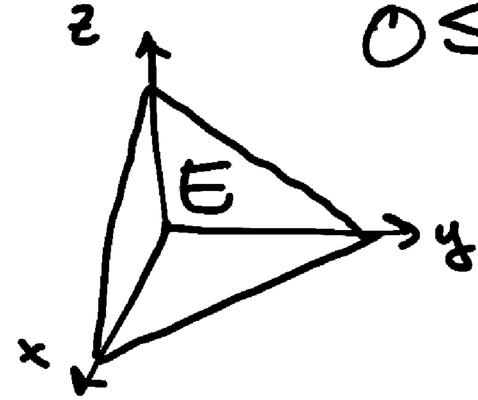
$$(x,z) \in D, \quad v_1(x,z) \le y \le v_2(x,z)$$

or

$$(y,z) \in D, \quad w_1(y,z) \le x \le w_2(y,z).$$

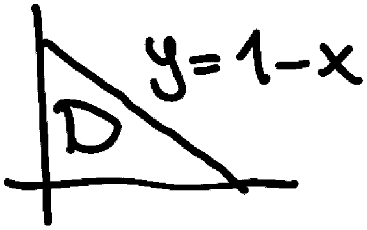
Ex. Compute $\iiint_E x\,dx\,dy\,dz$, where

$$E = \{(x,y,z) \mid y \ge 0,\ x \ge 0,\ x+y \le 1$$
$$0 \le z \le 1 - x - y\}.$$



$y \le 1 - x$

Base:



$$\iiint_E dx\,dy\,dz$$

$$= \iint_D \left(\int_0^{1-x-y} dz \right) dx\,dy = \int_0^1 \int_0^{1-x} \Big[z \Big]_0^{1-x-y} dy\,dx$$

$$= \int_0^1 \int_0^{1-x} 1-x-y \;\; dy\,dx = \int_0^1 \left[(1-x)y - \frac{y^2}{2} \right]_0^{1-x} dx$$

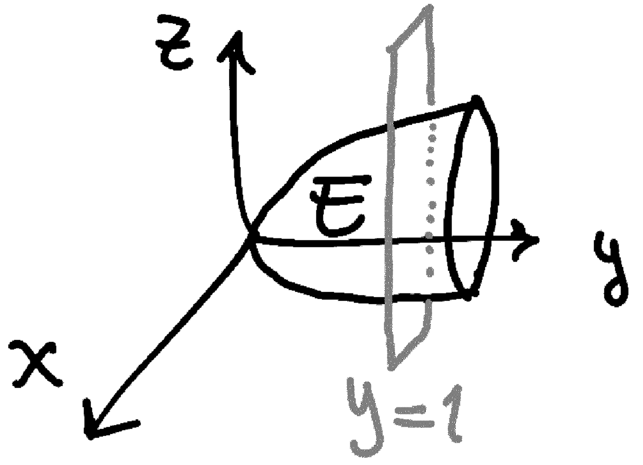
$$= \int_0^1 (1-x)^2 - \frac{(1-x)^2}{2} \, dx = \int_0^1 \frac{(1-x)^2}{2} \, dx$$

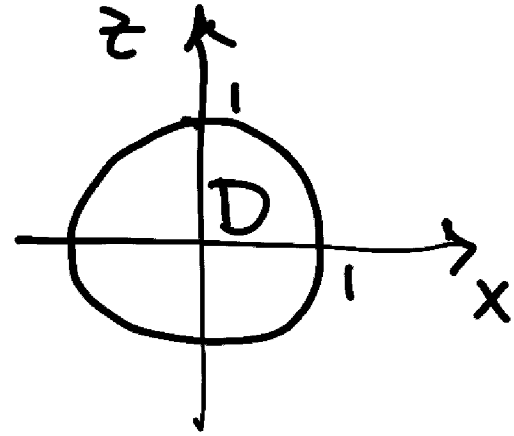
$$= \left[-\frac{(1-x)^3}{6} \right]_0^1 = 0 - \left(-\frac{1}{6} \right) = \frac{1}{6}.$$

Ex: Compute $\iiint_E \sqrt{x^2+z^2}\, dx\,dy\,dz$

Where E is the region bounded by the paraboloid $y=x^2+z^2$ and the plane $y=1$.

Sol:





$(x,z) \in D =$ unit disk in xz-plane

$= \{ x^2 + z^2 \leq 1 \}$

$$\iiint_E \sqrt{x^2+z^2}\, dx\,dy\,dz = \iint_D \left(\int_{x^2+z^2}^{1} \sqrt{x^2+z^2}\, dy \right) dx\,dz$$

$$= \iint_D \sqrt{x^2+z^2}\,(1-(x^2+y^2))\,dx\,dz$$

Now change to polar coords:

$D = \{ (r,\theta) \mid 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi \}$

$$\iint_D \sqrt{x^2+z^2}\,(1-(x^2+y^2))\,dx\,dz = \int_0^{2\pi}\int_0^1 r(1-r^2)\,r\,dr\,d\theta$$

$$= 2\pi \int_0^1 r^2 - r^4\, dr\, d\theta = 2\pi \left[\frac{r^3}{3} - \frac{r^5}{5} \right]_0^1$$

$= 2\pi\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{4\pi}{15}.$

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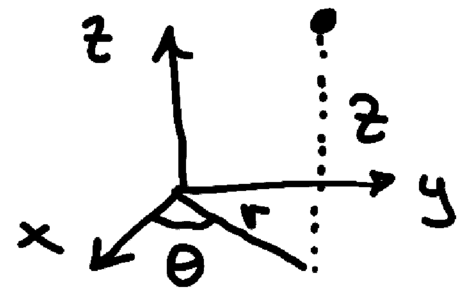
§5.5 Cylindrical and spherical coords.

Cylindrical coords: (r, θ, z)

$$\begin{cases} x = r\cos\theta & 0 \le r < \infty \\ y = r\sin\theta & 0 \le \theta \le 2\pi \\ z = z & -\infty < z < \infty \end{cases}$$



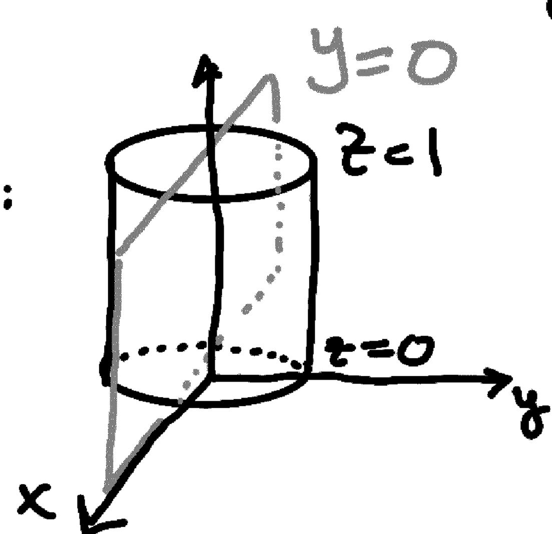
Similar to polar coords, we can show:

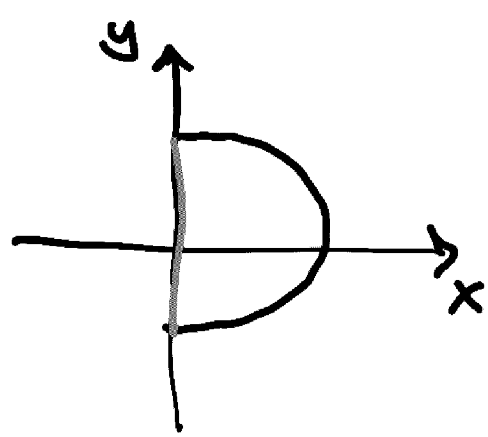
$$\boxed{dV = dx\,dy\,dz = r\,dr\,d\theta\,dz}$$

Ex: Compute $\iiint_C yz\,dx\,dy\,dz$,

where C is the region bounded by $0 \le z \le 1$, $x^2+y^2 \le 1$, $y = 0$

Sol:





In cylindrical coords,

$$C = \{(r,\theta,z) \mid 0 \le r \le 1,\ -\pi \le \theta \le \pi,\ 0 \le z \le 1\}$$

$$\iiint_C xz\,dx\,dy\,dz = \int_0^1 \int_{-\pi}^{\pi} \int_0^1 z\,r\cos\theta\, r\,dr\,d\theta\,dz$$

$$= \left(\int_0^1 z\,dz\right)\left(\int_0^1 r^2\,dr\right)\left(\int_{-\pi}^{\pi} \cos\theta\,d\theta\right)$$

$$= \left[\frac{z^2}{2}\right]_0^1 \left[\frac{r^3}{3}\right]_0^1 \left[\sin\theta\right]_{-\pi}^{\pi}$$

$$= \left(\tfrac{1}{2}\right)\cdot\left(\tfrac{1}{3}\right)\left(1-(-1)\right) = \frac{1}{3}$$

Spherical coords:

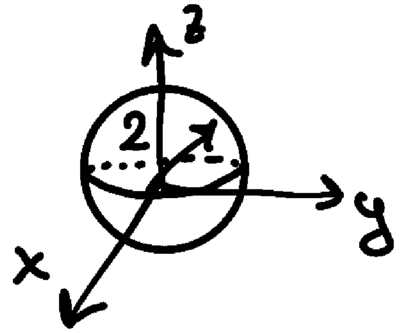
$$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases} \qquad \begin{aligned} 0 &\le r < \infty \\ 0 &\le \theta \le 2\pi \\ 0 &\le \varphi \le \pi \end{aligned}$$

$$\boxed{dV = r^2 \sin\varphi\, dr\, d\theta\, d\varphi}$$

<u>Ex</u>: Compute $\iiint_S z \, dx\,dy\,dz$

Where S is the inside of the sphere, centered at (0,0,0), with radius 2.

<u>Sol</u>:



In spherical coords:

$$S = \{(r,\theta,\varphi) \mid 0 \le r \le 2, \ 0 \le \theta \le 2\pi, \ 0 \le \varphi \le \pi\}.$$

$$\iiint_S z \, dx\,dy\,dz = \int_0^{\pi}\int_0^{2\pi}\int_0^{2} r\cos\varphi \, (r^2 \sin\varphi) \, dr\,d\theta\,d\varphi$$

$$= \left(\int_0^2 r^3 \, dr\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\varphi\cos\varphi \, d\varphi\right)$$

$$= \left[\frac{r^4}{4}\right]_0^2 (2\pi) \int_0^{\pi} \tfrac{1}{2}\sin(2\varphi)\, d\varphi$$

$$= 8\pi \left[-\frac{\cos(2\varphi)}{4}\right]_0^{\pi} = 8\pi\left(-\tfrac{1}{4} - \left(-\tfrac{1}{4}\right)\right) = 0$$

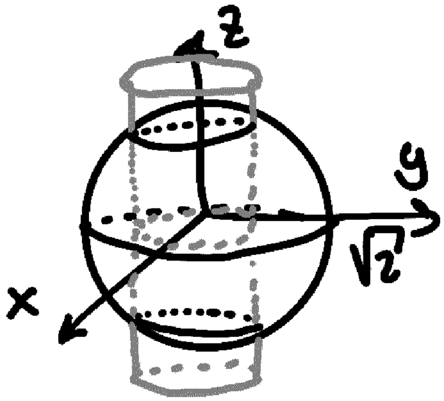
<u>Ex</u>: Let E be the region between the cylinder

$x^2+y^2=1$ and outside the

sphere $x^2+y^2+z^2=2$. Compute

$$\iiint_S \frac{1}{x^2+y^2+z^2}\,dx\,dy\,dz.$$

<u>Sol</u>: We first want to describe E using spherical coords.

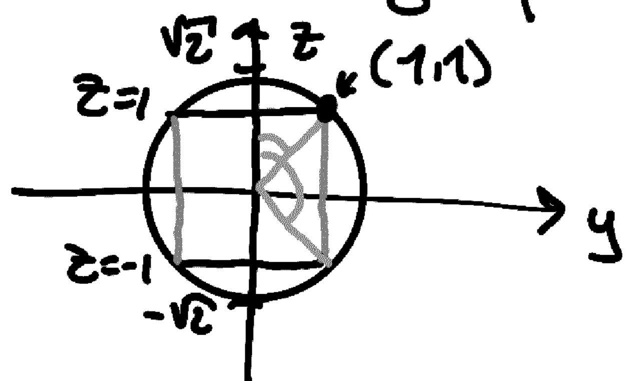


The bounds on θ is $0 \leq \theta \leq 2\pi$.
To find bounds on φ, let's find the intersection circles of the cylinder and the sphere.

$$\begin{cases} x^2+y^2+z^2=2 \\ x^2+y^2=1 \end{cases} \Rightarrow z^2=1 \Rightarrow z=\pm 1.$$

so in the yz-plane

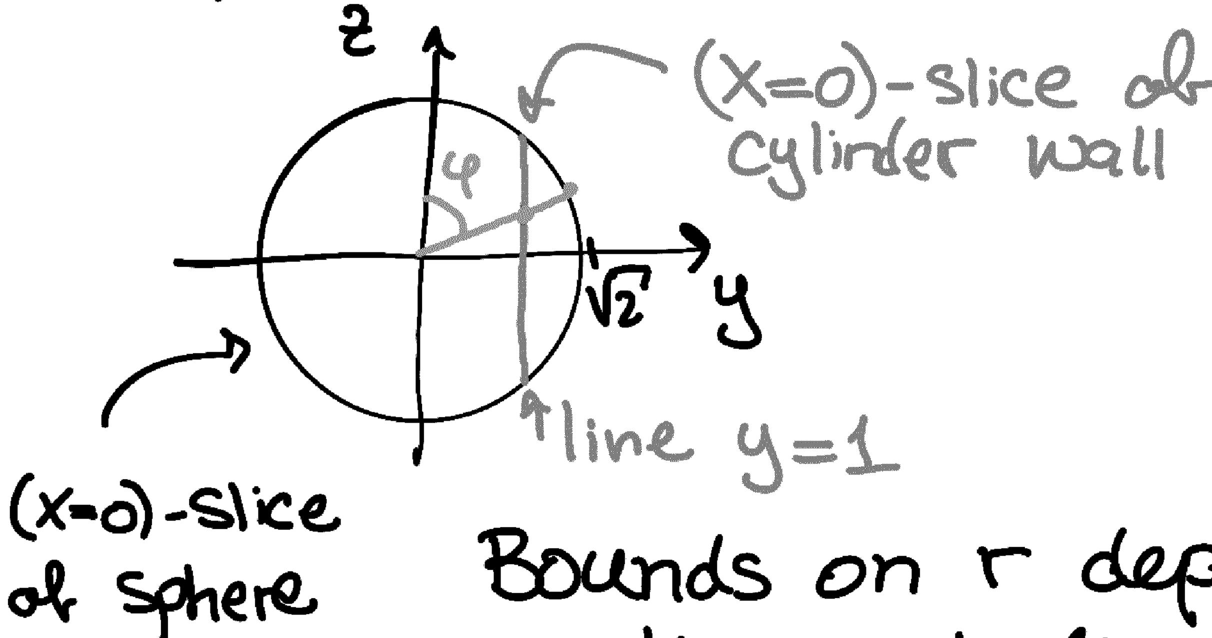
it looks line:



Smaller angle $= \frac{\pi}{4}$

larger angle $= \frac{3\pi}{4}$ so $\frac{\pi}{4} \le \varphi \le \frac{3\pi}{4}$.

Now, we need to find r.



Bounds on r depend on the angle φ.

For $\frac{\pi}{4} \le \varphi \le \frac{3\pi}{4}$, r goes between the line $y=1$ and $r=\sqrt{2}$.

The line $y=1$ in the plane is described by $y = r\sin\varphi = 1$

$\iff r = \frac{1}{\sin\varphi}$.

So $\frac{1}{\sin\varphi} \le r \le \sqrt{2}$.

The integral is therefore:

$$\iiint_S \frac{1}{x^2+y^2+z^2}\,dx\,dy\,dz$$

$$= \iiint_S \frac{r^2 \sin\varphi}{r^2}\,dr\,d\theta\,d\varphi$$

$$= \int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\left(\int_{1/\sin\varphi}^{\sqrt{2}} dr\right)\sin\varphi\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\left(\sqrt{2}-\frac{1}{\sin\varphi}\right)\sin\varphi\,d\varphi\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_{\pi/4}^{3\pi/4} \sqrt{2}\sin\varphi - 1\,d\varphi\right)$$

$$= 2\pi\left[-\sqrt{2}\cos\varphi - \varphi\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$= 2\pi\left(\left(\frac{\sqrt{2}}{\sqrt{2}}-\frac{3\pi}{4}\right)-\left(-\frac{\sqrt{2}}{\sqrt{2}}-\frac{\pi}{4}\right)\right)$$

$$= 2\pi\left(2-\frac{\pi}{2}\right).$$