MAT 203 L11 Wed Mar 11

Recall • f differentiable if

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)-f(a,b)-f_x(a,b)(x-a)-f_y(a,b)(y-b)}{\sqrt{(x-a)^2+(y-b)^2}}$$

$$=0.$$

• Chain rule: $f = f(x,y)$

$$= f(x(u,v), y(u,v)),$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

---

§4.6 Directional derivatives and gradient.

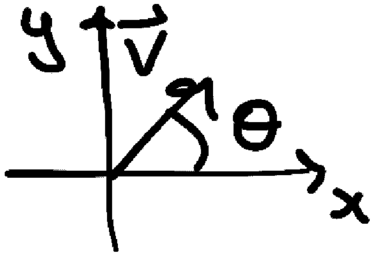
Partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are the rates of change of f in the "x-direction," and the "y-direction," respectively.

Def: Let $\vec{v} = \langle v_x, v_y \rangle$. The directional derivative of $f(x,y)$, in the direction $\vec{v}$, at $(x,y) = (a,b)$, is:

$$D_{\vec{v}} f(a,b) = \lim_{h \to 0} \frac{f(a+hv_x, b+hv_y) - f(a,b)}{h\sqrt{v_x^2 + v_y^2}}$$

Note $\frac{\partial f}{\partial x} = D_{\langle 1,0 \rangle} f$, and $\frac{\partial f}{\partial y} = D_{\langle 0,1 \rangle} f$.

Recall: $\vec{v} = \langle \cos\theta, \sin\theta \rangle$ is the unit vector in the plane making angle θ w/ positive x-axis is



Ex: Let $\vec{v} = \langle 3, 4 \rangle$, and $f(x,y) = x^2 - xy + 3y^2$. Then

$$D_{\vec{v}} f = \lim_{h \to 0} \frac{f(x+3h, y+4h) - f(x,y)}{h\sqrt{3^2+4^2}}$$

Numerator

$$= (x+3h)^2 - (x+3h)(y+4h) + 3(y+4h)^2 - (x^2 - xy + 3y^2)$$

$$= 6xh - 9h^2 - (4xh + 3yh + 12h^2) + 24yh + 48h^2$$

$$= 27h^2 + 2xh + 21yh$$

$$D_{\vec{v}} f = \lim_{h \to 0} \frac{27h^2 + 2xh + 21yh}{5h}$$

$$= \lim_{h \to 0} \frac{27}{5}h + \frac{2x+21y}{5} = \frac{2x+21y}{5}$$

Def: Let $f(x,y)$ be a function such that f_x and f_y exist. The gradient of f is defined as

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle.$$

∇ = "nabla"

<u>Ex</u>: (1) $f(x,y) = x^2 - xy + 3y^2$

$\nabla f(x,y) = \langle 2x - y, -x + 6y \rangle$

(2) $f(x,y) = \sin(3x)\cos(3y)$

$\nabla f(x,y) = \langle 3\cos(3x)\cos(3y), -3\sin(3x)\sin(3y) \rangle$

<u>Ex</u>: $f(x,y) = xy^2 - yx^2$. Then

$\nabla f(x,y) = \langle y^2 - 2xy, 2xy - x^2 \rangle$

<u>Thm</u>: Let $\vec{v}$ be a vector, and $f(x,y)$ a function such that f_x and f_y exist. Then

$$\boxed{D_{\vec{v}} f = \nabla f \cdot \frac{\vec{v}}{\|\vec{v}\|}}$$

Note that if $\vec{v}$ is a unit vector, then

$$D_{\vec{v}} f = \nabla f \cdot \vec{v}.$$

Ex: Earlier we looked at $f(x,y) = x^2 - xy + 3y^2$, and $\vec{v} = \langle 3,4 \rangle$.

We found $D_{\vec{v}} f = \frac{2x+21y}{5}$.

Can compute this using $D_{\vec{v}} f = \nabla f \cdot \frac{\vec{v}}{\|\vec{v}\|}$ instead of the definition.

$$\nabla f = \langle 2x-y, -x+6y \rangle$$

$$\|\vec{v}\| = \sqrt{3^2+4^2} = \sqrt{25} = 5.$$

$$D_{\vec{v}} f = \langle 2x-y, -x+6y \rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

$$= \frac{3}{5}(2x-y) + \frac{4}{5}(-x+6y)$$

$$= \left(\frac{6}{5} - \frac{4}{5}\right)x + \left(-\frac{3}{5} + \frac{24}{5}\right)y$$

$$= \frac{2}{5}x + \frac{21}{5}y.$$

Properties of the gradient:

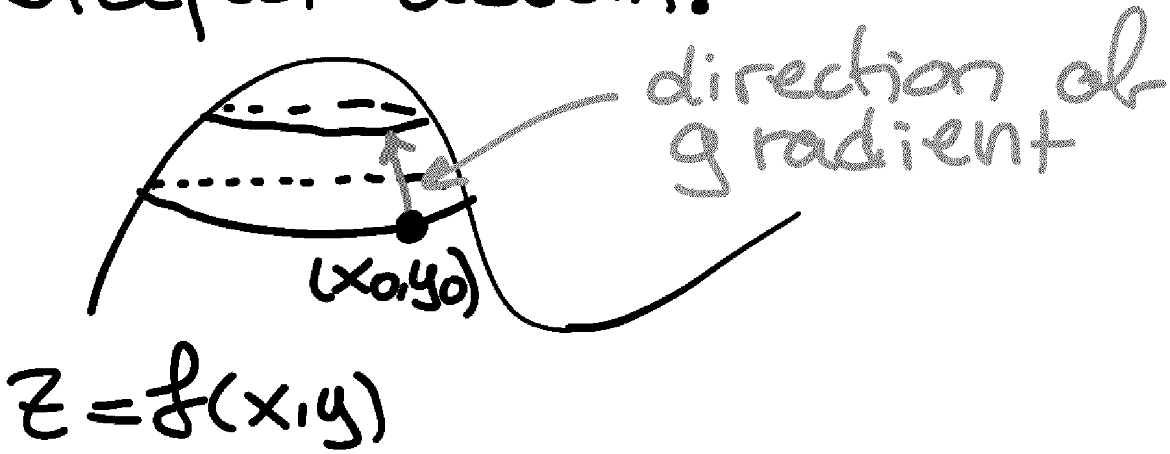
(1) If $\nabla f(x,y) = 0$, then $D_{\vec{v}} f(x,y) = 0$ for all $\vec{v}$.

(2) If $\nabla f(x,y) \neq 0$, then

(a) $D_{\vec{v}} f$ is maximized when $\vec{v}$ and $\nabla f(x,y)$ point in the same direction.
This maximum is $\|\nabla f(x,y)\|$.

(b) $D_{\vec{v}} f$ is minimized when $\vec{v}$ and $\nabla f(x,y)$ point in the opposite directions. This minimum is $-\|\nabla f(x,y)\|$.

"steepest ascent."



Ex: Find the direction for which the directional derivative of $f(x,y) = 3x^2 - 4xy + 2y^2$ at $(-2,3)$ is a maximum. What's the max value?

Sol: By the above properties, the direction is $\nabla f(-2,3)$. We compute:

$\nabla f(x,y) = \langle 6x - 4y, -4x + 4y \rangle$

$\nabla f(-2,3) = \langle -12-12, 8+12 \rangle = \langle -24, 20 \rangle$

The max value is $\|\nabla f(-2,3)\|$

$= \sqrt{24^2+20^2}$, so

$D_{\vec{v}} f$ is maximal in the direction $\langle -24, 20 \rangle$, with value

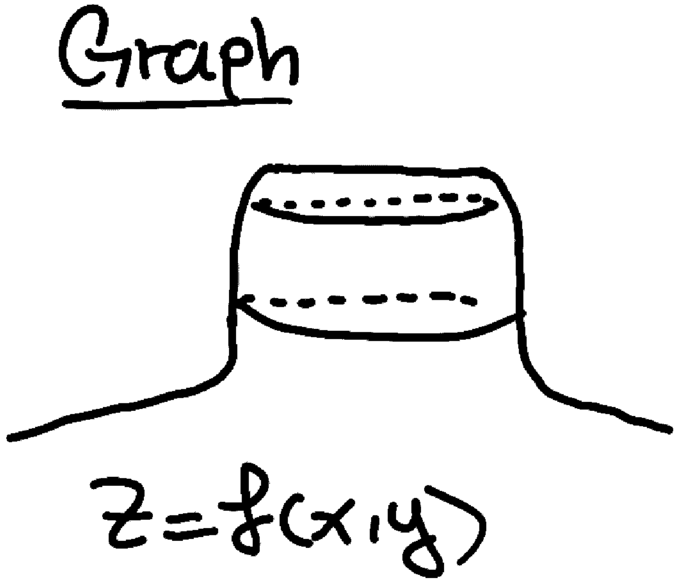
$\boxed{\sqrt{24^2+20^2}}$

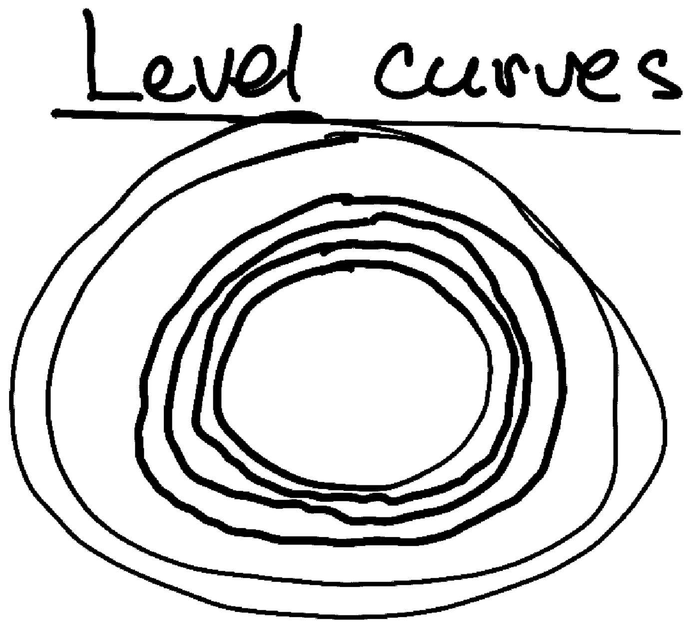
Level curves: For a function $f(x,y)$, the level curve for

$C \in \mathbb{R}$, is the set of points (x,y) in the plane s.th.

$f(x,y) = C$.

<u>Ex</u>

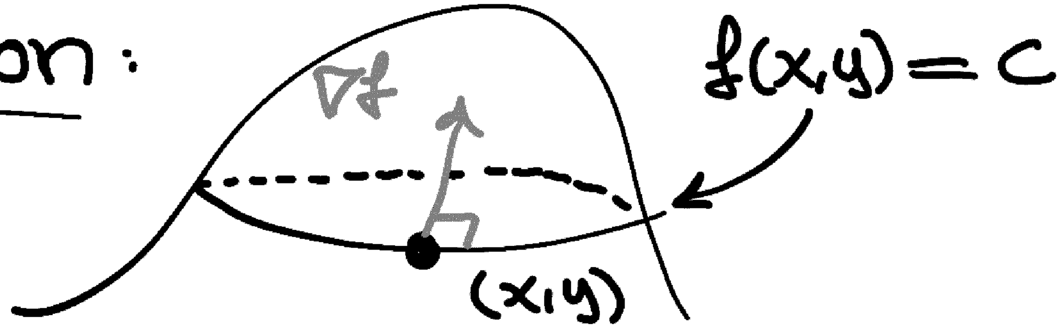




level curves are used to depict height differences in maps of landscapes.

<u>Thm</u>: If $f(x,y)$ is differentiable at (x,y) and $\nabla f(x,y) \neq 0$, then $\nabla f(x,y)$ is normal to the level curve of f at (x,y).

Intuition:



Moving along a level curve is by definition keeping the value $f(x,y)$ constant (so rate of change $=0$)

But ∇f points in the direction along which the rate of change of f is maximal.

Gradients are computed in the same way for functions of more than 2 variables.

Ex: $f(x,y,z) = xy - \ln(z)$.

$$\nabla f(x,y,z) = \left\langle y, x, -\frac{1}{z} \right\rangle$$
