MAT 127 Lec 8 Th Sep 19

Recall:

$$\sum_{n=1}^{\infty} c_n (x-a)^n \text{ power series}$$

centered at $x=a$.

- interval of convergence is always $-R < x-a < R$ for some R. ($R=0$ and $R=\infty$ allowed)
R = radius of conv.

Ex: Find radius and interval of conv of $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n^2}$.

Ratio test:

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(3x-2)^{n+1}/(n+1)^2}{(3x-2)^n/n^2} \right|$$

$$= \lim_{n\to\infty} |3x-2| \cdot \frac{n^2}{(n+1)^2} = |3x-2|$$

$\underline{\text{Conv}}$ if $\rho = |3x-2| < 1$

$\Leftrightarrow -1 < 3x-2 < 1$

$\Leftrightarrow 1 < 3x < 3$

$\Leftrightarrow \frac{1}{3} < x < 1$

$\underline{\text{div}}$ if $\rho = |3x-2| > 1 \Leftrightarrow$

$x < \frac{1}{3}$ or $x > 1$.

<u>Boundary points:</u>

$\boxed{x=1}$ $\sum_{n=1}^{\infty} \frac{(3\cdot 1-2)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

p-series w/ p=2, so it's <u>conv</u>.

$\boxed{x=\frac{1}{3}}$ $\sum_{n=1}^{\infty} \frac{(3\cdot\frac{1}{3}-2)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

$a_n = \frac{1}{n^2}$ decreasing and

$\lim_{n\to\infty} \frac{1}{n^2} = 0$ so alternating series test gives $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ <u>conv</u>.

Interval of conv: $[\frac{1}{3}, 1]$

Radius of conv? $\boxed{R = \frac{1}{3}}$

$|3x-2| < 1 \iff |x - \frac{2}{3}| < \frac{1}{3}$

Functions as power series

(§8.6 Stewart)

$$\boxed{\sum_{n=1}^{\infty} x^n = \frac{1}{1-x} \quad \text{if } |x| < 1.}$$

could view this as representing the function $f(x) = \frac{1}{1-x}$ in terms of a power series.

We can also use this formula to find power series of other functions!

<u>Ex</u>: $f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)}$

$$= \sum_{n=0}^{\infty} (-x)^n \qquad \underset{|x|}{\underset{\shortparallel}{|-x|}} < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$

__Ex__: $f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$

$$= \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Converges when $|-x^2| < 1$

$\iff x^2 < 1$

$\iff |x| < 1$

Interval of conv is $(-1,1)$ (can check that it diverges at $x = \pm 1$).

__Ex__ $f(x) = \frac{1}{2-x}$. Trick: Factor out the 2.

$$\frac{1}{2-x} = \frac{1}{2(1-\frac{x}{2})} = \frac{1}{2} \cdot \frac{1}{1-\frac{x}{2}}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}$$

Conv when $\left|\frac{x}{2}\right| < 1 \Leftrightarrow |x| < 2$.

Interval of convergence: $(-2,2)$

Look at $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

We have $\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}(1-x)^{-1}$

$$= -(1-x)^{-2}\cdot(-1) = \frac{1}{(1-x)^2}.$$

We can find a power series rep by differentiating $\sum_{n=0}^{\infty} x^n$:

$$\frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}\frac{d}{dx}x^n$$

$$= \frac{d}{dx}\left(1 + x + x^2 + \cdots\right) = 1 + 2x + 3x^2 + \cdots$$

$$= \boxed{\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}}$$

Starting at 1

Interval of conv is the same as for the Series we started with. Namely $|x|<1$.

<u>Thm</u>: If $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of conv $R>0$, then

$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ is differentiable on $|x-a|<R$, and

(i) $f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$ Start at 1

(ii) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

Both Series have interval of conv $|x-a|<R$.

Ex: We calculated

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Then

$$\underbrace{\int \frac{1}{1+x}\,dx}_{=\ln(1+x)} = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$\ln(1+x)$. w/ interval of Conv $|x|<1$.

Find Const by plugging in $x=0$.

$$\underbrace{\ln(1)}_{=0} = C + \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{0^{n+1}}{n+1}}_{=0} = C$$

So $\boxed{C=0}$ hence

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

for $|x|<1$.

Boundary points:

$\underline{x=1}$: $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{1^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Converges by alternating series test (saw this previously)

$\underline{x=-1}$: $\ln(1+x)$ undef at $x=-1$, so series has no chance at converging.

so

$$\boxed{\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \text{ for } x \in [-1,1]}$$

$$\boxed{\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}}$$