

Alternating Series test

Ex $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \dots$
 [Actually converges .

All series test

If $\sum_{n=1}^{\infty} (-1)^n a_n$ is so that

(i) $a_{n+1} \leq a_n$ for all n
 (a_n is decreasing)

(ii) $\lim_{n \rightarrow \infty} a_n = 0$

then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges .

Ex $a_n = \frac{1}{n}$ decreasing &

$\lim_{n \rightarrow \infty} a_n = 0$

so $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges .

Ex $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$.

$a_n = \frac{3n}{4n-1}$. Is it decreasing?

$f(x) = \frac{3x}{4x-1}$. Want to see if $f'(x) < 0$

for $x \geq 1$.

$$f'(x) = \frac{3(4x-1) - 4 \cdot 3x}{(4x-1)^2} = -\frac{3}{(4x-1)^2} < 0$$

so a_n is decreasing. However

$$\lim_{n \rightarrow \infty} \frac{3n}{4n-1} = \lim_{n \rightarrow \infty} \frac{3}{4 - \frac{1}{n}} = \frac{3}{4}$$

so the alternating series test doesn't apply!

However the div test is conclusive in this case.

$$\lim_{n \rightarrow \infty} (-1)^n \cdot \frac{3n}{4n-1} = \left(\lim_{n \rightarrow \infty} (-1)^n \right) \cdot \left(\lim_{n \rightarrow \infty} \frac{3n}{4n-1} \right)$$

$$= \frac{3}{4} \cdot \lim_{n \rightarrow \infty} (-1)^n \text{ doesn't exist}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{4n-1} \text{ diverges.}$$

Power Series (§8.5 Stewart)

Until now: only numerical series:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \text{ is } \underline{\text{numerical}}$$

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots = \text{some number.}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ is a } \underline{\text{power series}}$$

(x is a variable.)

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} = \dots$$

For each x it's a numerical series.

Def. A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

where $\{c_n\}$ is a sequence of real numbers, x is a variable, and a is a constant.

"power series centered at a "

Ex: We have already seen & studied $\sum_{n=0}^{\infty} x^n$ for varying values of x .

Geometric Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

$$= \frac{1}{1-x} \text{ for } |x| < 1$$

and it diverges if $|x| \geq 1$.

Ex: For what values of x does $\sum_{n=0}^{\infty} n! x^n$ converge?

Ratio test: $a_n = n! x^n$.

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)\cancel{n!}}{\cancel{n!}} \cdot x \right| = \lim_{n \rightarrow \infty} (n+1) |x|$$

If $|x| \neq 0$, then $\rho = \infty$
& ratio test tells us that it
div.

If $|x| = 0$ then $\rho = 0$

& ratio test tells us that it
conv.

$\sum_{n=0}^{\infty} n! x^n$	conv if $ x = 0$
	div if $ x \neq 0$

Ex: For which values does
 $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converge?

Ratio test: $a_n = \frac{x^n}{n}$.

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}/(n+1)}{x^n/n} \right| \\ &= \lim_{n \rightarrow \infty} |x| \cdot \frac{n}{n+1} = |x| \cdot \underbrace{\lim_{n \rightarrow \infty} \frac{n}{n+1}}_{=1} = |x| \end{aligned}$$

• $\rho = |x| < 1$ then
 $\sum_{n=1}^{\infty} \frac{x^n}{n}$ conv

• $\rho = |x| > 1$ then
 $\sum_{n=1}^{\infty} \frac{x^n}{n}$ div.

• $\rho = |x| = 1$ (meaning $x = \pm 1$)
INCONCLUSIVE.

CHECK THESE VALUES SEPARATELY!

$$\boxed{x=1} \quad \sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \quad \underline{\text{div}}$$

$$\boxed{x=-1} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad \underline{\text{conv}} \quad \text{by the alternating series test.}$$

(see previous example)

To summarize:

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{conv. for } -1 \leq x < 1$$

and div. otherwise.

Thm: For a power series $\sum_{n=1}^{\infty} c_n(x-a)^n$ there is 1 of 3 possibilities:

① it converges only at $x=a$

② it converges for all x

③ There's a positive number R such that it conv for

$|x-a| < R$ and diverges
for $|x-a| > R$

(And anything can happen
for $|x-a| = R \Leftrightarrow x = a \pm R$)

$R =$ radius of convergence

in case ① $R = 0$

in case ② $R = \infty$

The interval of convergence is the interval that consists of all values of x for which the series conv.

Ex $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$. Find radius & interval of conv.

Ratio test: (this is similar to the previous calculation)

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-3)^{n+1} / (n+1)}{(x-3)^n / n} \right|$$

$$= \lim_{n \rightarrow \infty} |x-3| \cdot \frac{n}{n+1} = |x-3|.$$

Conu if $\rho = |x-3| < 1$

$$\Leftrightarrow -1 < x-3 < 1$$

$$\Leftrightarrow 2 < x < 4$$

div if $\rho = |x-3| > 1$

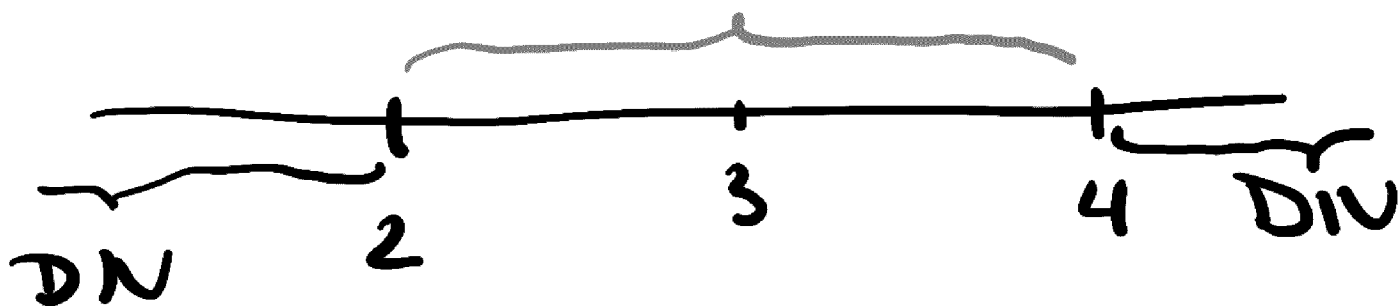
$$\Leftrightarrow x-3 < -1 \text{ or}$$

$$x-3 > 1$$

$$\Leftrightarrow x < 2 \text{ or}$$

$$x > 4$$

Conu



Need to check

$x=2$ and $x=4$ Separately.

$$\underline{x=2}: \sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad \underline{\text{Conu}}$$

$$\underline{x=4}: \sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \quad \underline{\text{div.}}$$

- Radius of convergence is 1.
 - Interval of convergence is $[2, 4)$
-

EX: $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ find radius & interval of conv.

Ratio test:

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1} / (n+1)!}{x^n / n!} \right| = \lim_{n \rightarrow \infty} |x| \cdot \frac{1}{n} = 0$$

for all values of x .

So $R = \infty$ and interval of convergence is $(-\infty, \infty)$.

Ex: Find interval & radius of conv
of $\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^n}$.

Ratio test:

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x+2)^{n+1} / 3^{n+1}}{n(x+2)^n / 3^n} \right|$$
$$= \lim_{n \rightarrow \infty} \frac{n+1}{n} \cdot |x+2| \cdot \frac{1}{3} = \frac{1}{3} \cdot |x+2|.$$

Conv if $\rho = \frac{1}{3}|x+2| < 1$

$$|x+2| < 3 = R$$

$$\Leftrightarrow -3 < x+2 < 3$$

$$\Leftrightarrow -5 < x < 1$$

div if $\rho = \frac{1}{3}|x+2| > 1$

$$\Leftrightarrow |x+2| > 3$$

Endpoints:

$$\underline{x = -5}$$

$$\sum_{n=1}^{\infty} \frac{n \cdot (-5+2)^n}{3^n} = \sum_{n=1}^{\infty} (-1)^n n$$

div by div test

$\lim_{n \rightarrow \infty} (-1)^n n$ doesn't exist!

$$\underline{x=1} \quad \sum_{n=1}^{\infty} \frac{n \cdot (1+2)^n}{3^n} = \sum_{n=1}^{\infty} n$$

also div by div test.

R (radius of conv) = 3

interval of conv = $(-5, 1)$.
