MAT 127 Lec6 Th Sep 12

<u>Recall</u>:

- Div test

  - $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges

  - $\lim_{n\to\infty} a_n = 0 \Rightarrow$ INCONCLUSIVE

- Ratio test: $\sum_{n=1}^{\infty} a_n$

  $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

  - $\rho > 1 \Rightarrow \sum a_n$ divergent

  - $\rho < 1 \Rightarrow \sum a_n$ convergent

  - $\rho = 1 \Rightarrow$ INCONCLUSIVE

- <u>Integral test</u>: f continuous, positive, decreasing function.

  $a_n = f(n)$

- $\int_1^{\infty} f(x)\,dx$ convergent

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ convergent}$$

- $\int_1^{\infty} f(x)\,dx$ divergent

$$\Rightarrow \sum_{n=1}^{\infty} a_n \quad \text{divergent}$$

- p-series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$

and diverges otherwise.

<u>Ex</u>: Does $\sum_{n=1}^{\infty} \frac{n}{n^2-1}$ converge or diverge?

① Div test: $\lim_{n \to \infty} \frac{n}{n^2-1} = 0$

<u>INCONCLUSIVE</u>

② Ratio test: $a_n = \frac{n}{n^2-1}$

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)/((n+1)^2-1)}{n/(n^2-1)}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)(n^2-1)}{n((n+1)^2-1)} = \lim_{n\to\infty}\frac{n^3+n^2-n-1}{n(n^2+2n)}$$

$$= \lim_{n\to\infty}\frac{1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}}{1+\frac{2}{n}} = 1$$

INCONCLUSIVE

③ Integral test?

$f(x) = \frac{x}{x^2-1}$ is continuous and positive for $x \geq 1$.

Decreasing? $f'(x) = \frac{(x^2-1)-x(2x)}{(x^2-1)^2}$

$$= \frac{-1-x^2}{(x^2+1)^2} < 0 \text{ for } x \geq 1.$$

So it's decreasing. We can use the integral test.

$$\int_{x=1}^{x=\infty} \frac{x}{x^2-1}\,dx = \begin{bmatrix} u = x^2-1 \\ du = 2x\,dx \\ x=1 \Rightarrow u=0 \\ x\to\infty \Rightarrow u\to\infty \end{bmatrix} = \int_{u=0}^{u=\infty} \frac{1}{2}\cdot\frac{1}{u}\,du$$

$$= \frac{1}{2}\Big[\ln|u|\Big]_{u=0}^{\infty} = \infty$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{n}{n^2-1}$ diverges.

Comparison test

The idea: the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ looks very similar to a series that we know, namely $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$.

We notic that the terms in $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ are <u>smaller</u> (larger denominator) than those in $\sum_{n=1}^{\infty} \frac{1}{2^n}$

so $\sum_{n=1}^{\infty} \frac{1}{2^n+1} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = \underbrace{\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n}_{\text{geometric series with } r=\frac{1}{2}} = \frac{1}{1-\frac{1}{2}} = 2$

therefore $\sum_{n=1}^{\infty} \frac{1}{2^n+1} \leq 2$ so it must converge.

<u>Thm</u> (Comparison test)

If $\sum a_n$ and $\sum b_n$ are series w/ positive terms and $a_n \leq b_n$ for all $n \geq 1$, then

① If $\sum b_n$ is convergent then $\sum a_n$ is convergent

② If $\sum a_n$ is divergent then $\sum b_n$ is divergent.

<u>Ex</u> Consider $\sum_{n=1}^{\infty} \frac{n}{n^2-1}$ from before.

It "almost" looks like $\sum \frac{n}{n^2} = \sum \frac{1}{n}$ so we would like to compare the series.

$$\sum_{n=1}^{\infty} \frac{n}{n^2-1} \geq \sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

larger denominator $\rightarrow$ smaller quotient

$$\Downarrow$$

$$\sum_{n=1}^{\infty} \frac{n}{n^2-1} \text{ divergent.}$$

<u>Ex</u>: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

Changing -1 to $+1$ should not change its convergence properties, but the comparison

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1} \leq \sum_{n=1}^{\infty} \frac{n}{n^2} = \boxed{\sum_{n=1}^{\infty} \frac{1}{n}}$$

DIVERGENT

but because $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ is <u>smaller</u> we can not use the comparison test. However:

$$\frac{n}{n^2+1} \geq \frac{n}{n^2+n^2} = \frac{n}{2n^2} = \frac{1}{2n}$$

$\boxed{\text{Since } n \geq 1}$

$\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n} = \infty$ so

$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges by the comparison test.