MAT 127 Lec 5 Tu Sep 10

**Recall:**

- Geometric series

$$\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ if } |r|<1 \text{, diverges otherwise}}$$

- **Divergence test:**

- If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

- If $\lim_{n\to\infty} a_n = 0$ we cannot draw any conclusion.

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**Ratio test** §8.4 Stewart

A geometric series $\sum_{n=1}^{\infty} r^n$ conv if and only if $|r|<1$.

In such a series $a_n = ar^n$

$a_{n+1} = ar^{n+1}$

$$\rightsquigarrow \left|\frac{a_{n+1}}{a_n}\right| = |r|$$

Geometric series converges if this quotient is $< 1$.

What if we have $\sum_{n=0}^{\infty} b_n$. Does $\left|\frac{b_{n+1}}{b_n}\right|$ tell us anything useful about convergence? Yes!

**Thm** (The ratio test)

For the series $\sum_{n=1}^{\infty} b_n$, let

$$\rho = \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right|.$$

- If $\rho < 1$ then $\sum_{n=1}^{\infty} b_n$ converges
- If $\rho > 1$ then $\sum_{n=1}^{\infty} b_n$ diverges
- If $\rho = 1$ we do not know.

Ex Does $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converge?

First check divergence test:

$\lim_{n\to\infty} \frac{n^3}{3^n} = 0$ (exponentials grow faster than polynomials)

⇝ Can not draw any conclusion.

Try the ratio test. $a_n = \frac{n^3}{3^n}$

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^3/3^{n+1}}{n^3/3^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(n+1)^3 \cdot 3^n}{n^3 \cdot 3^{n+1}}\right| = \lim_{n\to\infty} \left|\left(\frac{n+1}{n}\right)^3 \cdot \frac{1}{3}\right|$$

$$= \frac{1}{3} \cdot \left(\lim_{n\to\infty} \frac{n+1}{n}\right)^3 = \frac{1}{3} \cdot 1^3 = \frac{1}{3} < 1$$

so ratio tells us that the series converges.

$\underline{\text{Ex}}$: Does $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converge?

① Div test: $\lim_{n\to\infty} \frac{2^n}{n!} = 0$. No conclusion can be drawn.

② Ratio test: $a_n = \frac{2^n}{n!}$

$$\underline{\underline{\rho}} = \lim_{n\to\infty} \left| \frac{2^{n+1}/(n+1)!}{2^n/n!} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1} \cdot n!}{2^n \cdot (n+1)!} \right|$$

$$= 2 \cdot \lim_{n\to\infty} \frac{n!}{(n+1)!} = 2 \cdot \lim_{n\to\infty} \frac{\boxed{n \cdot (n-1) \cdots 2 \cdot 1}}{(n+1) \cdot \underbrace{n(n-1) \cdots 2 \cdot 1}_{\text{CANCEL}}}$$

$$= 2 \cdot \lim_{n\to\infty} \frac{1}{n+1} = 0 \underline{\underline{< 1}}$$

Ratio test gives that $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges

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$\underline{\text{Ex}}$ Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge?

① Div test: $\lim_{n\to\infty} \frac{1}{n^2} = 0$. No conclusion can be drawn.

② Ratio test: $a_n = \frac{1}{n^2}$.

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1/(n+1)^2}{1/n^2} = \lim_{n\to\infty}\frac{n^2}{(n+1)^2}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = 1$$

$\rightarrow$ Can not draw any conclusion.

(In fact, by other means $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$)

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__Ex__ Does $\sum_{n=1}^{\infty}\frac{(-1)^n (n!)^2}{(2n)!}$ converge?

Ratio test:

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}((n+1)!)^2/(2(n+1))!}{(-1)^n (n!)^2/(2n)!}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)!\cdot(n+1)!\cdot(2n)!}{n!\cdot n!\cdot(2n+2)!}$$

$$= \lim_{n\to\infty}\frac{[(n+1)\cancel{n!}][(n+1)\cancel{n!}]\cdot\cancel{(2n)!}}{\cancel{n!}\cdot\cancel{n!}\cdot[(2n+2)\cdot(2n+1)\cdot\cancel{(2n)!}]}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2}$$

$$= \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4+\frac{6}{n}+\frac{2}{n^2}} = \frac{1}{4} < 1$$

$\longrightarrow$ Series converges.

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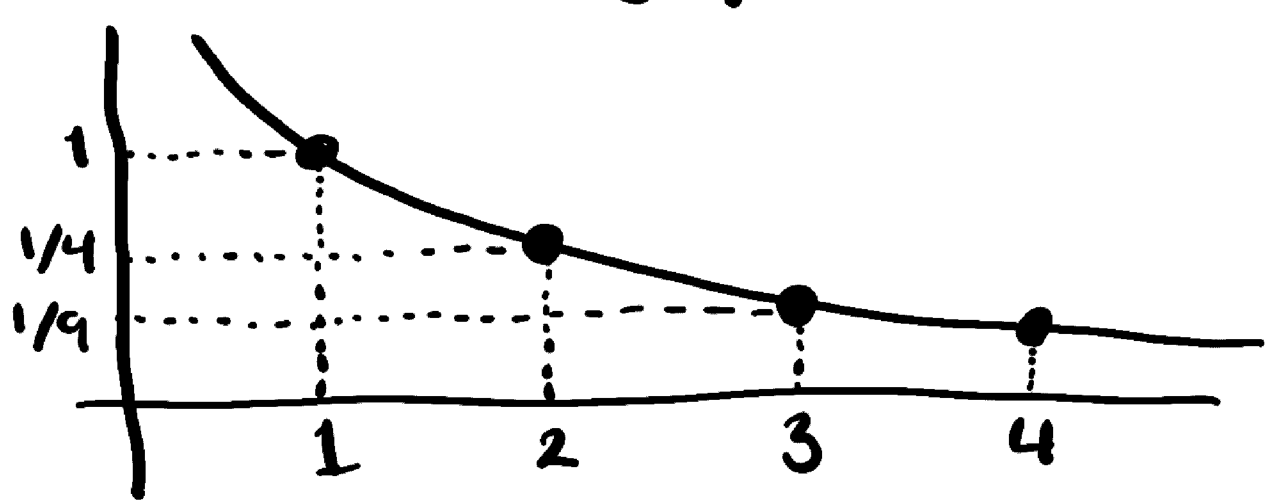
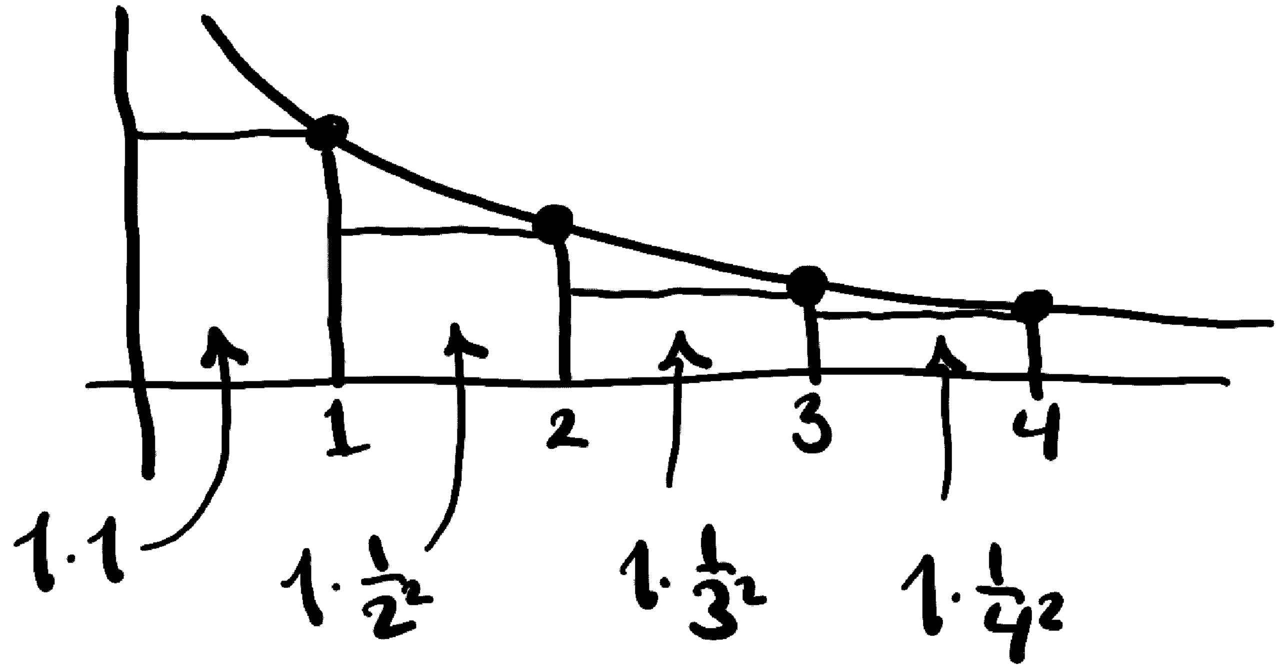
$\boxed{\text{Integral test}}$ $\boxed{\text{§8.3 Stewart}}$

Consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$ Will try to understand it geometrically.

Look at the graph $y = \frac{1}{x^2}$



Drawing rectangles under the graph:

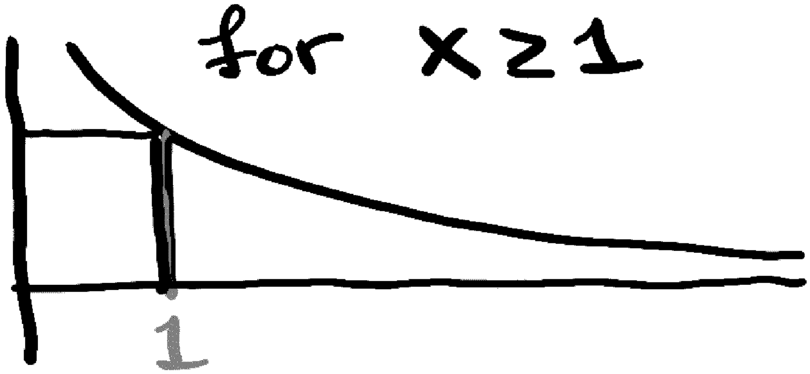



Sum of areas of all such rectangles

$$= \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Remember that

$$\int_1^{\infty} \frac{1}{x^2} dx = \text{area under graph for } x \geq 1$$



Comparing the figures we arrive at

$$0 \le \sum_{n=1}^{\infty} \frac{1}{n^2} \le \underline{\underline{1}} + \underline{\underline{\int_1^{\infty} \frac{1}{x^2}\,dx}}$$

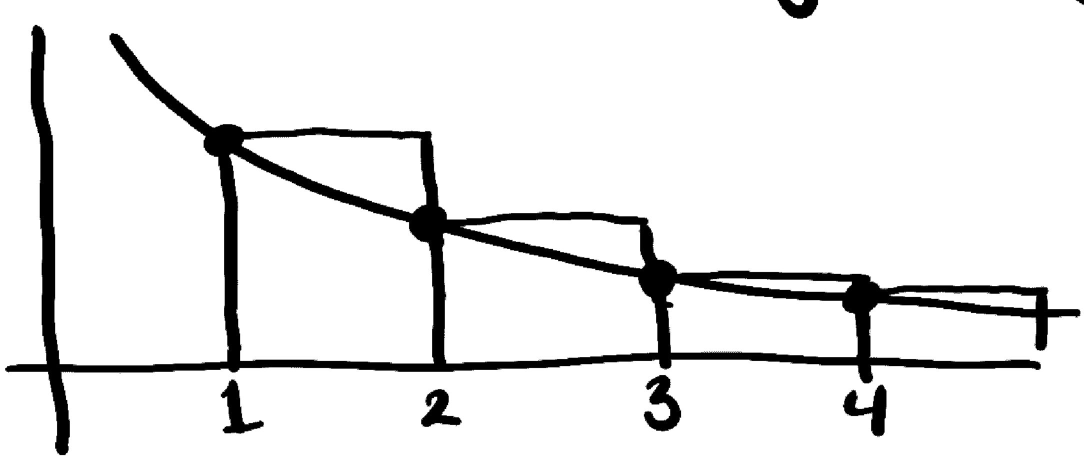
<u>Recall:</u>

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^{\infty} = -\lim_{x\to\infty} \frac{1}{x} + 1$$

$$= 1$$

so $0 \le \sum_{n=1}^{\infty} \frac{1}{n^2} \le 2$ and this series must converge!

Similar way of summing rectangles:



$$\sum_{n=1}^{\infty} \frac{1}{n^2} \geq \int_1^{\infty} \frac{1}{x^2}\,dx = 1.$$

If this integral would diverge, then the series would too.

<u>Thm</u> (The integral test)

If $f$ is continuous, positive, and decreasing on $[1,\infty)$.

(a) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} f(n)$ is convergent.

(b) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} f(n)$ is divergent.

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Consider $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for different values of $p$.

Easy values:

If $p<0$ then $\lim_{n\to\infty} \frac{1}{n^p} = \infty$

so div. test $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges

If $p=0$ then $\frac{1}{n^0} = 1$

and $\sum_{n=1}^{\infty} \frac{1}{n^0} = \sum_{n=1}^{\infty} 1$ diverges.

If $p>0$ we use the integral test:

$$\int_1^{\infty} \frac{1}{x^p}\,dx$$

If $p=1$ $\int_1^{\infty} \frac{1}{x}\,dx = \left[\ln|x|\right]_1^{\infty}$

$= \left(\lim_{x\to\infty} \ln|x|\right) - 1 = \infty$

So $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (of course, we already know this!)

If $p \neq 1$,

$$\int_1^\infty \frac{1}{x^p} = \int_1^\infty x^{-p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^\infty$$

$$= \left(\lim_{x\to\infty} \frac{1}{1-p}\cdot\frac{1}{x^{p-1}}\right) - \frac{1}{1-p}$$

$$= \begin{cases} \frac{1}{p-1}, & p>1 \\ \infty, & 0<p\leq 1 \end{cases}$$

so the integral test gives:

$$\rightarrow \boxed{\begin{array}{l} \sum_{n=1}^\infty \frac{1}{n^p} \text{ convergent for } p>1. \\ \qquad\qquad \text{divergent for } 0<p<1 \end{array}}$$