MAT 127 Lec 4, Th Sep 5

Recall: • $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \boxed{\sum_{n=1}^{N} a_n}$ ← partial sums

• Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ <u>diverges</u>.

Geometric Series:

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots$$

$$= a \boxed{\sum_{n=0}^{\infty} r^n}$$ (a, r some constants)

<u>r = 1</u>: $\sum_{k=0}^{\infty} 1^k = 1+1+1+\cdots \longrightarrow \infty$

DIVERGES

So assume $r \neq 1$.

n-th partial sum:

$$S_n = \sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n$$

$$rS_n = r + r^2 + r^3 + \cdots + r^{n+1}$$

$$rS_n + 1 = \underbrace{\boxed{1 + r + r^2 + r^3 + \cdots + r^n}}_{= S_n} + r^{n+1}$$

$$rS_n + 1 = S_n + r^{n+1}$$

$$1 - r^{n+1} = S_n - rS_n = S_n(1-r$$

$$\boxed{S_n = \frac{1-r^{n+1}}{1-r}}$$

we know $\lim\limits_{n\to\infty} \frac{1-r^{n+1}}{1-r}$

$$= \frac{1 - \lim\limits_{n\to\infty} r^{n+1}}{1-r} \boxed{=} \frac{1}{1-r}$$

if $-1 < r < 1$
(diverges otherwise)

$$\boxed{\sum_{k=0}^{\infty} ar^n = \frac{a}{1-r} \text{ if } |r| < 1}$$

Ex: $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{1-2/3} = \frac{1}{1/3} = 3$

Ex: $\displaystyle\sum_{n=0}^{\infty}\frac{2^{n+1}}{5^n} = \sum_{n=0}^{\infty}\frac{2\cdot 2^n}{5^n} = 2\cdot\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n$

$\displaystyle = 2\cdot\frac{1}{1-2/5} = 2\cdot\frac{1}{3/5} = 2\cdot\frac{5}{3} = \frac{10}{3}.$

Ex: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{e^{2n}} = \sum_{n=1}^{\infty}\frac{1}{(e^2)^n} = \sum_{n=1}^{\infty}\left(\frac{1}{e^2}\right)^n$

Almost geometric with $r=\frac{1}{e^2}$ but Sum starts at $n=1$.

$\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{e^2}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{e^2}\right)^{n+1} = \frac{1}{e^2}\sum_{n=0}^{\infty}\left(\frac{1}{e^2}\right)^n$

$\displaystyle = \frac{1}{e^2}\cdot\frac{1}{1-1/e^2} = \frac{1}{e^2}\cdot\frac{1}{\frac{e^2}{e^2}-\frac{1}{e^2}} = \frac{1}{\cancel{e^2}}\cdot\frac{\cancel{e^2}}{e^2-1}$

$\displaystyle = \frac{1}{e^2-1}$

Telescoping series

Ex: Compute $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Note (by partial fraction decomposition)

$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, so

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= 1 \underbrace{- \frac{1}{2} + \frac{1}{2}}_{\text{cancel}} \underbrace{- \frac{1}{3} + \frac{1}{3}}_{\text{cancel}} \underbrace{- \frac{1}{4} + \frac{1}{4}}_{\text{cancel}} - \cdots$$

N-th partial sum

$$S_N = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 \underbrace{- \frac{1}{2}\right) + \cdots + \left(\frac{1}{N}}_{\text{All these cancel}} - \frac{1}{N+1}\right)$$

$$= 1 - \frac{1}{N+1}.$$

$$\lim_{N \to \infty} S_N = \lim_{N \to \infty} 1 - \frac{1}{N+1} = 1$$

so $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Thm: If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Warning! If we know $\lim_{n\to\infty} a_n = 0$ then we can not draw any conclusion!

Ex: Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges but $\lim_{n\to\infty} \frac{1}{n} = 0$.

Ex: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ as seen previously and $\lim_{n\to\infty} \frac{1}{n(n+1)} = 0$.

A logically equivalent version of the theorem:

Test for divergence

If $\lim_{n\to\infty} a_n$ does not exist, or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

When faced with a problem of finding out whether a series converges or diverges, we should always try this test first!

<u>Ex</u>: Does $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ converge or diverge?

<u>Div test</u>:

$$\lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5+\frac{4}{n^2}} = \frac{1}{5} \neq 0$$

So the series <u>diverges</u>.

Let us prove it!

<u>Thm</u>: If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Proof: $\sum_{n=1}^{\infty} a_n$ convergent means that if $S_N = \sum_{n=1}^{N} a_n$ is the N-th partial sum, the limit $\boxed{\lim_{N\to\infty} S_N \text{ exists}}$.

Note

$$S_N = a_1 + a_2 + \cdots + a_{N-1} + a_N$$

$$S_{N-1} = a_1 + a_2 + \cdots + a_{N-1}$$

$\rightarrow S_N - S_{N-1} = a_N$ so

$$\lim_{N\to\infty} a_N = \lim_{N\to\infty}(S_N - S_{N-1})$$

$$= \underbrace{\lim_{N\to\infty} S_N}_{=L} - \underbrace{\lim_{N\to\infty} S_{N-1}}_{=L} = L - L = 0$$

$\square$

Ex: Does $\sum_{n=1}^{\infty} \frac{3^n}{2^n+1}$ Converge?

NO! Because $\lim_{n\to\infty} \frac{3^n}{2^n+1}$

$= \lim_{n\to\infty} \frac{1}{\left(\frac{2}{3}\right)^n + 1} = 1 \neq 0.$

Ex: Does $\sum_{n=1}^{\infty} \frac{3^n - 2^n}{e^n}$ converge or diverge?

$\lim_{n\to\infty} \frac{3^n-2^n}{e^n} = \lim_{n\to\infty} \frac{3^n}{e^n} - \frac{2^n}{e^n} = \lim_{n\to\infty} \left(\frac{3}{e}\right)^n$

$- \lim_{n\to\infty} \left(\frac{2}{e}\right)^n = \infty - 0 = \infty$

so divergence test gives

$\sum_{n=1}^{\infty} \frac{3^n-2^n}{e^n}$ diverges.