

Recall:  $\{a_n\}$  sequence

- increasing:  $a_n \leq a_{n+1}$   $n \geq 1$
- decreasing:  $a_n \geq a_{n+1}$   $n \geq 1$
- bounded:  $m \leq a_n \leq M$   $n \geq 1$

• squeeze thm:

$\{a_n\}, \{b_n\}, \{c_n\}$  sequences

such that

$$a_n \leq b_n \leq c_n \quad n \geq 1$$

If  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$  exists,

then  $\lim_{n \rightarrow \infty} b_n = L$ .

Here is a useful fact that we can prove as a consequence:

Thm If  $\{a_n\}$  is a sequence such that  $\lim_{n \rightarrow \infty} |a_n| = 0$ , then  $\lim_{n \rightarrow \infty} a_n = 0$ .

Proof: We always have

$-|a_n| \leq a_n \leq |a_n|$  for all  $n \geq 1$   
for any sequence  $\{a_n\}$ , so  
if  $\lim_{n \rightarrow \infty} |a_n| = 0$ , also  $\lim_{n \rightarrow \infty} -|a_n| = 0$

The squeeze thm then implies  
 $\lim_{n \rightarrow \infty} a_n = 0$ .

□

---

Ex: Find  $\lim_{n \rightarrow \infty} \frac{\sin(n)}{n}$ .

$b_n = \frac{\sin(n)}{n}$ . We know that

$-1 \leq \sin(n) \leq 1$ , so  $\underbrace{-\frac{1}{n}}_{a_n} \leq \underbrace{\frac{\sin(n)}{n}}_{b_n} \leq \underbrace{\frac{1}{n}}_{c_n}$

and  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = 0$ .

So the squeeze thm gives

$$\lim_{n \rightarrow \infty} \frac{\sin(n)}{n} = 0.$$

---

# §8.2 Stewart

## § Infinite Series

Comes naturally eg from the desire to describe numbers with infinite decimal expansions such as  $\pi$  or  $e$ .

$$\pi = 3.14159265\dots$$

$$= 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \dots$$

Series  $\neq$  Sequence

infinite  
sum

infinite list

Sequence:  $\{a_n\} = \left\{ \frac{1}{2^n} \right\}_{n=1}^{\infty}$

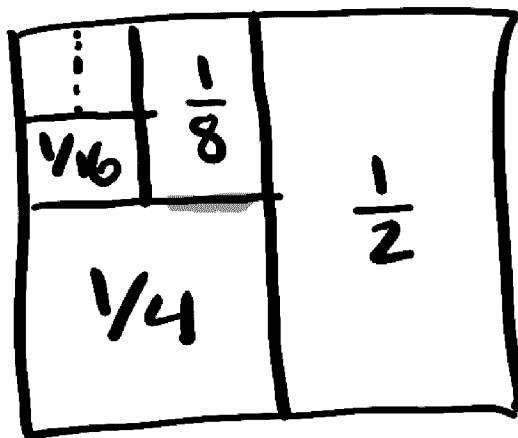
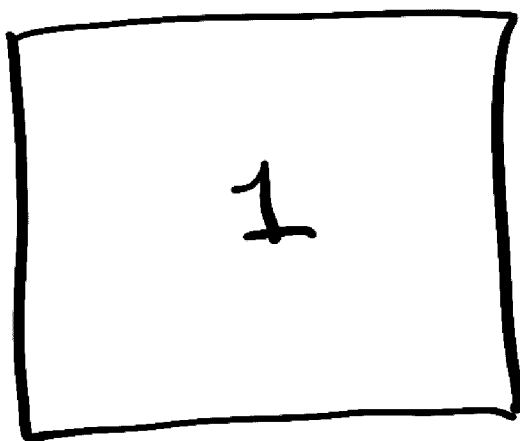
Series:

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

$$= \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{2^k}$$

$$= \lim_{n \rightarrow \infty} \underbrace{\left(1 + \frac{1}{2} + \dots + \frac{1}{2^n}\right)}_{\text{geometric sum}} = 2$$

Geometrically:



Algebraically:

We have the sequence

$$b_n = \sum_{k=1}^n a_k = \sum_{k=1}^n \frac{1}{2^k}$$

$$\{b_n\} = 1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{4}, 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}, \dots$$

$$\{b_n\} = 1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16} \dots$$

$$b_n = \frac{2^{n+1} - 1}{2^n} \text{ \& we find}$$
$$\lim_{n \rightarrow \infty} \frac{2^{n+1} - 1}{2^n} = \lim_{n \rightarrow \infty} \frac{1 - \frac{1}{2^{n+1}}}{\frac{1}{2}} = 2.$$

---

Def: An (infinite) series

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \dots + a_n + \dots$$

There is a sequence of partial sums  $\{S_n\}$  where

$$S_n = \sum_{k=1}^n a_k. \text{ Then we define}$$

$$\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} S_n \quad \text{if it converges}$$

If  $\lim_{n \rightarrow \infty} S_n$  doesn't exist, then  $\sum_{k=1}^{\infty} a_k$  diverges.

---

Ex: Consider  $\sum_{k=1}^n k$ .

Then its partial sums are

$$S_1 = \sum_{k=1}^1 k = 1$$

$$S_2 = \sum_{k=1}^2 k = 1+2 = 3$$

$$S_3 = \sum_{k=1}^3 k = 1+2+3 = 6$$

⋮

$$S_n = \sum_{k=1}^n k = 1 + \dots + n = \frac{n(n-1)}{2}$$

ARITHMETIC  
SUM

And of course

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{n(n-1)}{2} = \infty$$

So  $\sum_{k=1}^{\infty} k$  diverges (to  $\infty$ ).

Ex: Consider  $\sum_{k=1}^{\infty} (-1)^{k+1}$

Partial sums:

$$S_1 = 1, \quad S_2 = 1 - 1 = 0, \quad S_3 = 1 - 1 + 1 = 1$$

$$\text{So } \{S_n\} = 1, 0, 1, 0, 1, \dots$$

$\lim_{n \rightarrow \infty} S_n$  does not exist, so

$$\sum_{k=1}^{\infty} (-1)^{k+1} \text{ is divergent.}$$

---

## Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Grows very slowly.

- Takes over 12,000 terms for  $S_n \geq 10$ .
- To get  $S_n \geq 54$  it requires as many terms as stars in the universe  $\sim 1.6 \cdot 10^{23}$

Thm The harmonic series diverges to  $\infty$ .

Proof: Look at partial sums  $S_2, S_4, S_8, S_{16}, S_{32}, \dots, S_{2^n}$

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \boxed{\frac{1}{3} + \frac{1}{4}} \geq 1 + \frac{1}{2} + \boxed{\frac{1}{4} + \frac{1}{4}}$$

$$= 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2}$$

$$S_8 = 1 + \frac{1}{2} + \boxed{\frac{1}{3} + \frac{1}{4}} + \boxed{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}$$

$$\geq 1 + \frac{1}{2} + \frac{1}{2} + \boxed{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \quad \frac{1}{2}$$

$$= 1 + \frac{3}{2} \quad (8 = 2^3) \quad \text{so}$$

$$\boxed{S_{2^n} \geq 1 + \frac{n}{2}}$$



As  $n$  increases, partial sums  
go to  $\infty$ , so  $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$

□

---