

Recall:

To solve the ODE

$$ay'' + by' + cy = 0 \quad \text{for}$$

a, b, c constants

we first solve the characteristic equation

$$ar^2 + br + c = 0$$

Three cases:

① If r_1, r_2 are two different real roots, the general sol

is

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

② If $r = \mu \pm \lambda x$ are two non-real roots, the general sol is

$$y(x) = e^{\mu x} (C_1 \cos(\lambda x) + C_2 \sin(\lambda x))$$

③ If r is a repeated root, the general sol is

$$y(x) = (C_1 + C_2 x) e^{rx}$$

Ex: Solve the initial-value problem

$$2y'' + y' - y = 0, \quad y(0) = 0, \quad y'(0) = 1$$

Char eqn

$$2r^2 + r - 1 = 0 \Leftrightarrow r^2 + \frac{1}{2}r - \frac{1}{2} = 0$$

$$r = -\frac{1}{4} \pm \sqrt{\frac{1}{16} + \frac{1}{2}} = -\frac{1}{4} \pm \sqrt{\frac{1+8}{16}}$$

$$= -\frac{1}{4} \pm \frac{3}{4} = \frac{-1 \pm 3}{4}$$

$$r_1 = -1, \quad r_2 = \frac{1}{2}$$

General solution is

$$y(x) = C_1 e^{-x} + C_2 e^{\frac{x}{2}}.$$

Initial conditions:

$$y(0) = C_1 e^0 + C_2 e^0 = C_1 + C_2 = 0$$

$$y'(x) = -C_1 e^{-x} + \frac{C_2}{2} e^{\frac{x}{2}}$$

$$y'(0) = -C_1 e^0 + \frac{C_2}{2} e^0 = -C_1 + \frac{C_2}{2} = 1$$

Two equations:

$$\begin{cases} C_1 + C_2 = 0 \\ -C_1 + \frac{C_2}{2} = 1 \end{cases}$$

First eq $\leadsto C_1 = -C_2$

Insert in second eqn:

$$-(-C_2) + \frac{C_2}{2} = 1$$

$$\Leftrightarrow \frac{3C_2}{2} = 1$$

$$C_2 = \frac{3}{2}$$

$\Rightarrow C_1 = -\frac{3}{2}$ The specific

solution is

$$y(x) = -\frac{3}{2}e^{-x} + \frac{3}{2}e^{\frac{x}{2}}$$

Ex: Solve the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1$$
$$y'(0) = 0$$

Characteristic eq:

$$r^2 - 4r + 4 = 0$$

$$\Leftrightarrow (r-2)^2 = 0$$

repeated solution $r=2$.

General solution:

$$y(x) = (C_1 + C_2 x) e^{2x}.$$

Initial conditions:

$$y(0) = (C_1 + C_2 \cdot 0) e^0 = \boxed{C_1 = 1}$$

$$y'(x) = C_2 e^{2x} + 2(C_1 + C_2 x) e^{2x}$$

$$y'(0) = C_2 e^0 + 2(C_1 + C_2 \cdot 0) e^0$$

$$= C_2 + 2C_1 = C_2 + 2 = 0$$

$$C_1 = 1$$

$$\Rightarrow \boxed{C_2 = -2}$$

Specific solution is

$$\boxed{y(x) = (1 - 2x) e^{2x}}$$

Ex Solve the IVP

$$y'' - 6y' + 10y = 0, y(0) = 1, y'(0) = 0$$

Characteristic eqn:

$$r^2 - 6r + 10 = 0$$

$$r = 3 \pm \sqrt{9 - 10} = 3 \pm \sqrt{-1} = 3 \pm i$$

General solution is

$$y(x) = e^{3x} (C_1 \cos(x) + C_2 \sin(x))$$

Initial conditions:

$$y(0) = e^0 (C_1 \cos(0) + C_2 \sin(0))$$

$$= \boxed{C_1 = 1}$$

$$y'(x) = 3e^{3x} (C_1 \cos x + C_2 \sin x)$$

$$+ e^{3x} (-C_1 \sin x + C_2 \cos x)$$

$$y'(0) = 3e^0(C_1 \cos 0 + C_2 \sin 0) + e^0(-C_1 \sin 0 + C_2 \cos 0)$$

$$= 3C_1 + C_2 = 3 + C_2 = 0$$

$$C_1 \uparrow = 1$$

$$\Rightarrow \boxed{C_2 = -3}$$

Specific solution is

$$\boxed{y(x) = e^{3x}(\cos x - 3 \sin x)}$$
