MAT 127 Lec 26 Th Nov 21

<u>EX</u> $y''+y=0$. We have already solved this equation before using power series. The general solution is

$$y = c_1 \cos x + c_2 \sin x. \quad (*)$$

Now suppose we consider the characteristic equation:

$r^2+1=0$. It's solutions are $r = \pm i$.

Then e^{ix} and e^{-ix} are solutions! So the general solution is

$$y = c_1 e^{ix} + c_2 e^{-ix}$$

But how is this consistent w/ $(*)$?

Remember Euler's formula

$$e^{ix} = \cos x + i \sin x.$$

$$\rightarrow e^{-ix} = e^{i(-x)} = \cos(-x) + i \sin(-x)$$

$$= \cos x - i \sin x$$

$$\boxed{\begin{array}{l} \cos(-x) = \cos x \\ \sin(-x) = -\sin x \end{array}}$$

So

$$y = c_1 e^{ix} + c_2 e^{-ix}$$

$$= c_1(\cos x + i \sin x)$$

$$+ c_2(\cos x - i \sin x)$$

$$= (c_1 + c_2)\cos x + i(c_1 - c_2)\sin x$$

$$= D_1 \cos x + D_2 \sin x$$

CONSTANTS

More generally:

If r_1, r_2 are two complex roots of ar^2+br+c of the form

$$r_1=\mu+\lambda i, \quad r_2=\mu-\lambda i$$

then the general solution is

$$y=C_1 e^{(\mu+\lambda i)x}+C_2 e^{(\mu-\lambda i)x}$$

We will not present this function like this, but we will use Euler's formula like before:

$$C_1 e^{(\mu+\lambda i)x}+C_2 e^{(\mu-\lambda i)x}$$

$$=C_1 e^{\mu x}\cdot e^{i(\lambda x)}+C_2 e^{\mu x} e^{i(-\lambda x)}$$

$$=e^{\mu x}\Big(C_1(\cos(\lambda x)+i\sin(\lambda x))$$
$$+C_2(\cos(\lambda x)-i\sin(\lambda x))$$

$$= e^{\mu x}\Big((C_1+C_2)\cos(\lambda x) + i(C_1-C_2)\sin(\lambda x)\Big)$$

$$= e^{\mu x}\big(D_1\cos(\lambda x) + D_2\sin(\lambda x)\big)$$

constants

<u>Ex</u>: $y'' - 6y' + 10y = 0$.

Characteristic eqn

$$r^2 - 6r + 10 = 0$$

$$r = 3 \pm \sqrt{3^2 - 10} = 3 \pm i$$

$$\mu = \mathrm{Re}(r) = 3$$

$$\lambda = \mathrm{Im}(r) = 1$$

$\Rightarrow$ General solution is

$$y = e^{3x}(C_1\cos x + C_2 \sin x).$$

Let us summarize:

Want to solve

$$ay'' + by' + cy = 0.$$

First find solutions to the characteristic equation

$$\boxed{ar^2 + br + c = 0}$$

- If r_1, r_2 are two real solutions, the general solution is

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

C_1, C_2 are constants

- If $r = \mu \pm \lambda i$ are non-real solutions, the general solution is

Therefore:

$$ay'' + by' + cy = 0$$

if characteristic eqn

$$ar^2 + br + c = 0$$

has two real solutions r_1, r_2

general sol is

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

if it has two non-real solutions

$r = \mu \pm \lambda i$ then general sol

is

$$y = e^{\mu x}\left(C_1 \cos(\lambda x) + C_2 \sin(\lambda x)\right)$$

Remaining case of

a single solution (double root)

is discussed next time!

$$\boxed{y(x) = e^{\mu x}\left(C_1 \cos(\lambda x) + C_2 \sin(\lambda x)\right)}.$$

<u>Ex</u>: $y'' + 2y' + 5y = 0$

<u>Char equ</u>: $r^2 + 2r + 5 = 0$.

<u>solutions</u>:

$$r = -1 \pm \sqrt{1-5} = -1 \pm \sqrt{-4}$$
$$= -1 \pm 2i$$

$\mu = -1, \ \lambda = 2$

General solution is:

$$\boxed{y(x) = e^{-x}\left(C_1 \cos(2x) + C_2 \sin(2x)\right)}$$

There is one remaining case:
The characteristic eqn
$ar^2 + br + C = 0$ can have
a double/repeated solution

<u>Ex</u> $y'' + 2y' + y = 0$

<u>Char equ</u>:

$r^2 + 2r + 1 = (r+1)^2 = 0$

has the only solution

$r = -1$.

$y_1(x) = e^{-x}$ is a solution

to $y'' + 2y' + y = 0$. Can

verify this by differentiating:

$$\begin{cases} y_1'(x) = -e^{-x} \\ y_1''(x) = e^{-x} \end{cases}$$

$\rightsquigarrow$ $e^{-x} + 2(-e^{-x}) + e^{-x} = 0$ ✓

Can we guess another solution?

It's tricky!

In fact another solution is
$y_2(x) = xe^{-x}$.

$y_2'(x) = e^{-x} - xe^{-x}$

$y_2''(x) = -e^{-x} - (e^{-x} - xe^{-x})$

$= -2e^{-x} + xe^{-x}$

$y_2''(x) + 2y_2'(x) + y_2(x)$

$= (\cancel{-2e^{-x}} + xe^{-x}) + \cancel{2(e^{-x}} - xe^{-x})$

$+ (xe^{-x})$

$= xe^{-x} - 2xe^{-x} + xe^{-x} = 0$ ✓

General solution is therefore

$y(x) = c_1 e^{-x} + c_2 xe^{-x}$

$= (c_1 + c_2 x) e^{-x}$.

This happens in general!

Ex: $y''-4y'+4y=0$

characteristic equation is

$$r^2-4r+4=0$$
$$\shortparallel$$
$$(r-2)^2$$

$r=2$ repeated solution

General solution is therefore

$$\boxed{y(x)=(c_1+c_2x)e^{2x}.}$$

Let's summarize!

To solve the ODE

$ay''+by'+cy=0$ for

a,b,c constants

we first solve the characteristic equation

$$\boxed{ar^2 + br + c = 0}$$

Three cases:

① If r_1, r_2 are two different real roots, the general sol is

$$\boxed{y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}}$$

② If $r = \mu \pm \lambda$ are two non-real roots, the general sol is

$$\boxed{y(x) = e^{\mu x}\left(C_1 \cos(\lambda x) + C_2 \sin(\lambda x)\right)}$$

③ If r is a repeated root, the general sol is

$$\boxed{y(x) = (C_1 + C_2 x)\, e^{rx}}$$