MAT 127 Lec 25 TU NOV 19

<u>Recall</u> • 2^{nd} order ODE w/ Constant coefficients

$$\boxed{ay'' + by' + cy = 0}$$

• Solutions typically have <u>two</u> Constants

<u>Ex</u>: Let's try to solve

$$y'' - 9y = 0$$

We only changed the "1" in front of y to a "9", so we would guess that there are solutions of the form

e^{rx} for some r.

Let's try to find r!

$y=e^{rx}$, $y'=re^{rx}$, $y''=r^2e^{rx}$

$$y''-9y = r^2e^{rx}-9e^{rx}$$
$$= e^{rx}(r^2-9)=0$$

Since we want this to be $=0$ as a <u>function</u> (so for all x) we must have

$$r^2-9=0 \Leftrightarrow r=\pm 3.$$

Therefore

$y_1=e^{3x}$, $y_2=e^{-3x}$

so $y(x)=C_1e^{3x}+C_2e^{-3x}$

is the general solution to $y''-9y=0$.

<u>Ex</u> $y''-5y'+6y=0.$

Let's again look for solutions of the form $y=e^{rx}$.

$y'=re^{rx}, \quad y''=r^2e^{rx}.$

$$y''-5y'+6y=r^2e^{rx}-5re^{rx}+6e^{rx}$$

$$=e^{rx}(r^2-5r+6)=0$$

For the same reason as before we want

$$r^2-5r+6=0$$

$$r=\frac{5}{2}\pm\sqrt{\left(\frac{5}{2}\right)^2-6}$$

$$=\frac{5}{2}\pm\sqrt{\frac{25-24}{4}}$$

$$=\frac{5}{2}\pm\frac{1}{2}$$

$r_1 = 2$, $r_2 = 3$

so e^{2x} and e^{3x} are solutions.

General solution is

$$y(x) = C_1 e^{2x} + C_2 e^{3x}.$$

Ex: Solve the IVP

$$\boxed{\begin{array}{l} y'' - 3y' + 2y = 0 \\ y(0) = -1, \quad y'(0) = 1 \end{array}}$$

First look for solutions of the eqn of the form e^{rx}.

$$y = e^{rx}, \quad y' = re^{rx}, \quad y'' = r^2 e^{rx}.$$

Then

$$y'' - 3y' + 2y = r^2 e^{rx} - 3re^{rx} + 2e^{rx}$$

$= e^{rx}(r^2-3r+2) = 0$

$\Rightarrow r = \frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - 2} = \frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{8}{4}}$

$= \frac{3}{2} \pm \frac{1}{2}$

$r_1 = 1, \ r_2 = 2$

e^x and e^{2x} are solutions

$\rightsquigarrow$ General solution is

$y = C_1 e^x + C_2 e^{2x}$.

Initial values:

$y(0) = C_1 e^0 + C_2 e^0 = C_1 + C_2 = 1$

$y'(x) = C_1 e^x + 2C_2 e^{2x}$

$y'(0) = C_1 + 2C_2 = 2$

$$\begin{cases} C_1 + C_2 = -1 & (1) \\ C_1 + 2C_2 = 1 & (2) \end{cases}$$

(1) $C_1 = -1 - C_2$ Plug into (2)

$\rightsquigarrow -1 - C_2 + 2C_2 = -1 + C_2 = 1$

$\Leftrightarrow \boxed{C_2 = 2}$

$\Rightarrow \boxed{C_1 = -1 - 2 = -3}$

Specific solution to the IVP is therefore

$$\boxed{y(x) = -3e^x + 2e^{2x}}$$

<u>Recall</u>:

We are interested in solving 2^{nd} order equations of the form

$$\boxed{\begin{array}{c} ay''+by'+cy=0 \\ a,b,c \text{ constants} \end{array}}$$

Have seen that solutions are often of the form

$$y=C_1e^{r_1x}+C_2e^{r_2x}$$

Where r_1, r_2 are two constants determined by the equation.

<u>Ex</u> $y''-4y=0$. Let's find solutions of the form e^{rx}.

$y=e^{rx}$, $y'=re^{rx}$, $y''=r^2e^{rx}$

$y'' - 4y = r^2 e^{rx} - 4e^{rx} = e^{rx}(r^2-4)$

$= 0$ for all x

means $r^2 - 4 = 0$

$\Rightarrow r = \pm 2$, so both e^{2x} and e^{-2x} are solutions to $y'' - 4y = 0$. More generally

$\boxed{y = c_1 e^{2x} + c_2 e^{-2x}}$ is the general solution

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In fact we can solve all equations in a similar way.

$$ay'' + by' + cy = 0$$

then if we look for solutions

of the form $y=e^{rx}$ we get

$y'=re^{rx}$; $y''=r^2e^{rx}$

$$ay''+by'+cy=ar^2e^{rx}+bre^{rx}+ce^{rx}$$

$$=e^{rx}(ar^2+br+c)=0$$

$$\Leftrightarrow \boxed{ar^2+br+c=0}$$

So if r is a root of the polynomial ar^2+br+c then $y=e^{rx}$ is a solution to

$$ay''+by'+cy=0.$$

The equation

$$ar^2+br+c=0$$ is

called the "characteristic equation".

Ex: $2y'' - 3y' + y = 0$.

Characteristic eqn:

$$2r^2 - 3r + 1 = 0$$

$$r^2 - \frac{3}{2}r + \frac{1}{2} = 0$$

$$r = \frac{3}{4} \pm \sqrt{\left(\frac{3}{4}\right)^2 - \frac{1}{2}}$$

$$= \frac{3}{4} \pm \sqrt{\frac{9}{16} - \frac{8}{16}} = \frac{3}{4} \pm \frac{1}{4}$$

$r_1 = 1$, $r_2 = \frac{1}{2}$.

e^x and $e^{\frac{x}{2}}$ are solutions to $2y'' - 3y' + y = 0$ &

$\boxed{y = C_1 e^x + C_2 e^{\frac{1}{2}x}}$ is the general solution