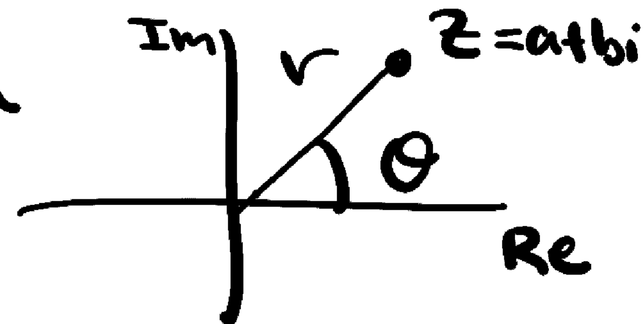
MAT 127 Lec 24 Th Apr 17

<u>Recall</u>: Polar form



$r = |z| = \sqrt{a^2+b^2}$

$\theta = \arg \theta$ = angle between line $0 \longrightarrow z$ an $+$Re-axis.

$$\arg \theta = \begin{cases} \arctan\left(\frac{b}{a}\right), & a>0 \\ \arctan\left(\frac{b}{a}\right)+\pi, & a<0,\ b\geq 0 \\ \arctan\left(\frac{b}{a}\right)-\pi, & a<0,\ b<0 \\ \pi/2, & a=0,\ b>0 \\ -\pi/2, & a=0,\ b<0 \\ \text{undef}, & a=b=0 \end{cases}$$

Euler's formula

Remember

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Let's compute e^{ix}.

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n x^n}{n!}$$

$$= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \cdots$$

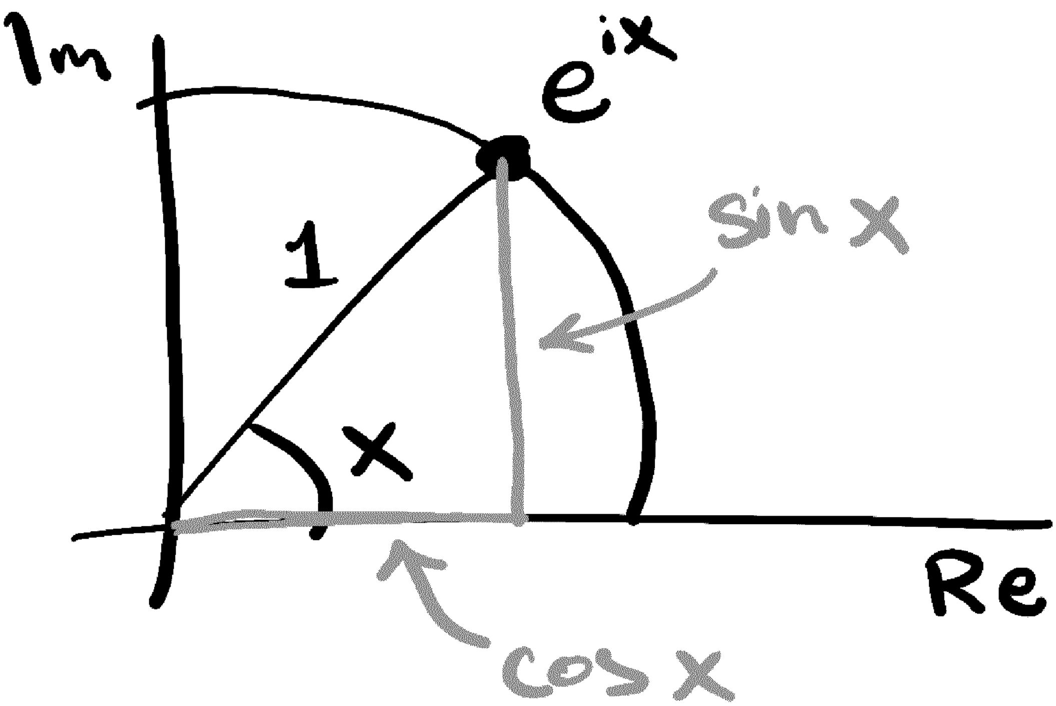
$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= \boxed{\cos x + i \sin x = e^{ix}}$$

Geometric explanation

$z = e^{ix}$, $|z| = 1$, $\arg z = x$



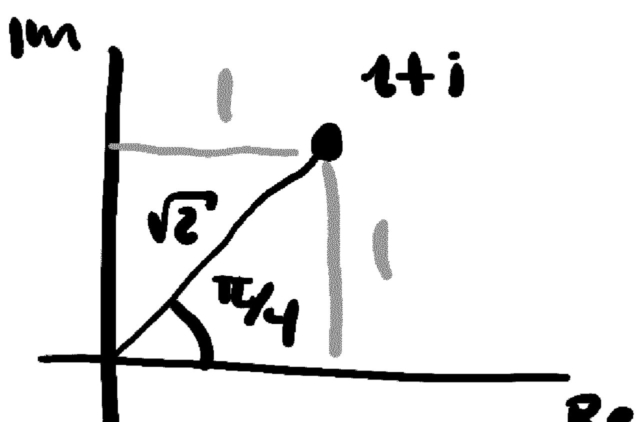
$e^{ix} = a+bi$, then $\begin{cases} a = \cos x \\ b = \sin x \end{cases}$

<u>Ex</u>: To compute $(1+i)^{10}$ we first write $1+i$ on the form $re^{i\theta}$.

$r = |1+i| = \sqrt{1+1} = \sqrt{2}$

$\arg(1+i) = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$, so

$1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$

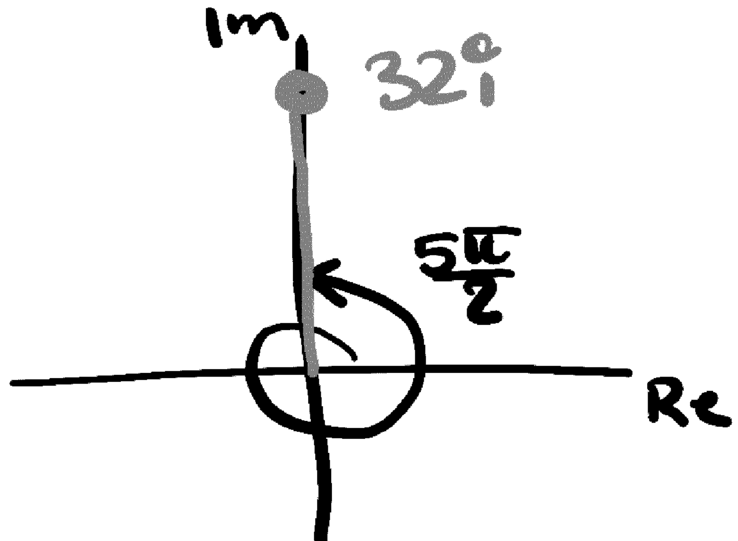


Then $(1+i)^{10} = \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^{10}$

$= \sqrt{2}^{10} e^{i\frac{\pi}{4}\cdot 10}$

$= \left(\sqrt{2}^{2}\right)^{5} e^{i\frac{5\pi}{2}} = 2^{5} e^{i\left(2\pi + \frac{\pi}{2}\right)}$

$= 32 e^{i\frac{\pi}{2}} = 32i$



If $z = re^{i\theta}$ then $z^2 = \left(re^{i\theta}\right)^2$

$= r^2 e^{2i\theta} = r^2\left(\cos(2\theta) + i\sin(2\theta)\right)$

can also multiply out:

$\left(re^{i\theta}\right)^2 = \left(r(\cos\theta + i\sin\theta)\right)^2$

$= r^2 (\cos\theta + i\sin\theta)^2$

$= r^2 \left((\cos^2\theta - \sin^2\theta) + 2\cos\theta\sin\theta\, i \right)$

so this derives the identities

$$\begin{cases} \cos(2\theta) = \cos^2\theta - \sin^2\theta \\ \sin(2\theta) = 2\cos\theta\sin\theta \end{cases}.$$

In general we have de Moivre's theorem:

If $z = re^{i\theta}$, then

$$\boxed{z^n = r^n \left(\cos(n\theta) + i\sin(n\theta)\right)}.$$

Note $e^{ix} = \cos x + i \sin x$

$e^{-ix} = \cos(-x) + i\sin(-x)$

$= \cos x - i \sin x$

$\boxed{\begin{array}{l} \cos(-x) = \cos x \\ \sin(-x) = -\sin x \end{array}}$

So

$$e^{ix}+e^{-ix}=(\cos x+i\sin x)$$
$$+(\cos x-i\sin x)$$
$$=2\cos x$$

$$\boxed{\cos x=\frac{e^{ix}+e^{-ix}}{2}}$$

Similarly: $e^{ix}-e^{-ix}=2i\sin x$

$$\boxed{\sin x=\frac{e^{ix}-e^{-ix}}{2i}}$$

Last topic of the semester:

Linear homogeneous 2nd order ODES with constant coefficients.

These are fancy words used to describe 2^{nd} order ODES of the form:

$$\boxed{\begin{gathered} Ay'' + By' + Cy = 0 \\ A, B, C \text{ constants} \end{gathered}}$$

When solving a first order ODE (e.g. via separation of variables) we get a general solution depending on one constant. This is to say there is a single degree of freedom (we can choose whatever value of the constant as we wish).

- For initial-value problems such as

$$e^y y' = 2x, \quad y(0) = 0$$

there is a single initial condition because there is 1 constant to determine.

2^{nd} order ODEs:

- We will always expect two constants, and solutions are of the form

$$y = C_1 y_1(x) + C_2 y_2(x).$$

- Initial-value problems need to come with two conditions:

For instance

$$\boxed{y'' + y = 0, \quad y(0) = 1, \quad y'(0) = -1}$$

we saw two lectures ago using power series that

$$y = c_1 \cos x + c_2 \sin x$$ is

a general solution to $y'' + y = 0$.

so via the initial conditions we can determine both constants:

$$y(0) = c_1 = 1$$

$$y'(x) = -c_1 \sin x + c_2 \cos x$$

$$y'(0) = c_2 = -1$$

Specific solution:

$$y = \cos x - \sin x.$$

Ex $y''=0$ is a 2^{nd} order ODE of the kind we will consider.

Integrate twice to solve it!

$y' = C_1$

$$\boxed{y = C_1 x + C_2}$$

Note that since we integrated twice we got two constants.

Ex $y''-y=0$.

Let's guess some solutions!

- $y_1(x) = e^x \Rightarrow y_1'(x) = e^x$

$\Rightarrow y_1''(x) = e^x$

So $y_1'' = y_1$.

- $y_2(x) = e^{-x} \Rightarrow y_2'(x) = -e^{-x}$

$$\Rightarrow y_2''(x) = e^{-x}$$

so $y_2'' = y_2$

In fact we see that

$c_1 y_1(x)$ and $c_2 y_2(x)$

constants

are both solutions too!
(They don't affect the derivatives...)
The general solution is their linear combination

$$y(x) = c_1 e^{x} + c_2 e^{-x}.$$

This is generally true:

If $y_1(x)$ and $y_2(x)$ are both solutions to

$$Ay'' + By' + Cy = 0$$

then so is

$$c_1 y_1(x) + c_2 y_2(x)$$

<u>Proof</u>:

$$y(x) := c_1 y_1(x) + c_2 y_2(x)$$

$$y'(x) = c_1 y_1'(x) + c_2 y_2'(x)$$

$$y''(x) = c_1 y_1''(x) + c_2 y_2''(x)$$

so

$$Ay'' + By' + Cy$$

$$= A\left(c_1 y_1''(x) + c_2 y_2''(x)\right)$$
$$+ B\left(c_1 y_1'(x) + c_2 y_2'(x)\right)$$
$$+ C\left(c_1 y_1(x) + c_2 y_2(x)\right)$$
$$= c_1\left(A y_1''(x) + B y_1'(x) + C y_1(x)\right)$$
$$+ c_2\left(A y_2''(x) + B y_2'(x) + C y_2(x)\right)$$
$$= c_1 \cdot 0 + c_2 \cdot 0 = 0. \qquad \square$$