MAT 127 Lec 23 Tu Nov 12

Ex: $y' = x^2 y$. It's separable, so we can solve it that way, but let's use power series.

$$y = \sum_{n=0}^{\infty} c_n x^n$$

shift n

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

multiply

$$x^2 y = x^2 \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+2}$$

$$= \sum_{n=2}^{\infty} c_{n-2} x^n$$

$$y' - x^2 y = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=2}^{\infty} c_{n-2} x^n \quad (*)$$

To gather the terms we need

the series to start at the same index. To "fix" it, we write out the first two terms in the former series.

$$(*) = C_1 X + 2C_2 X^2 + \sum_{n=2}^{\infty} (n+1) C_{n+1} X^n$$
$$- \sum_{n=2}^{\infty} C_{n-2} X^n$$

$$= \underset{=0}{C_1} + \underset{=0}{2C_2 X} + \sum_{n=2}^{\infty} \left[(n+1) C_{n+1} - C_{n-2} \right] X^n$$

$$= 0$$

Immediately get $\boxed{c_1 = 0, \quad c_2 = 0.}$

Then $\boxed{(n+1) C_{n+1} - C_{n-2} = 0 \quad n \geq 2}$

$$C_{n+1} = \frac{C_{n-2}}{n+1} \quad n \geq 2$$

$\underline{n=2}$ $\quad C_3 = \dfrac{C_0}{3}$

$\underline{n=3}$ $c_4 = \frac{c_1}{4} = 0$

$\underline{n=4}$ $c_5 = \frac{c_2}{5} = 0$

$\underline{n=6}$ $c_6 = \frac{c_3}{6} = \frac{c_0}{6\cdot 3} = \frac{c_0}{3^2\cdot 2\cdot 1} = \frac{c_0}{3^2\cdot 2!}$

$\underline{n=7}$ $c_7 = \frac{c_4}{7} = 0$

$\underline{n=8}$ $c_8 = \frac{c_5}{8} = 0$

$\underline{n=9}$ $c_9 = \frac{c_6}{9} = \frac{c_0}{9\cdot 3^2\cdot 2!} = \frac{c_0}{3^3\cdot 3!}$

Pattern seems to be

$$c_{3n} = \frac{c_0}{3^n\cdot n!}$$ and $c_n = 0$ for n not a multiple of 3.

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \cdots$$

$$= c_0 + c_3 x^3 + c_6 x^6 + c_9 x^9 + \cdots$$

$$= c_0\left(1 + \frac{x^3}{3\cdot 1!} + \frac{x^6}{3^2\cdot 2!} + \frac{x^9}{3^3\cdot 3!} + \cdots\right)$$

$$= c_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{3^n n!} = c_0 \sum_{n=0}^{\infty} \frac{\left(\frac{x^3}{3}\right)^n}{n!}$$

$$= c_0 e^{\frac{x^3}{3}}$$

This is also the solution obtained via separation of variables!

<u>Next topic</u>: second order ODEs "with constant coefficients".

In general we can in fact solve

$a y'' + b y' + c y = 0$ where

a, b, c are <u>constants</u>

Before delving into this topic we will review complex numbers.

Complex numbers

Recall that the solutions to the eqn $x^2-4=0$ are $x=\pm 2$.

Only working with real numbers makes it impossible to solve $x^2+4=0$.

Imaginary unit: $i=\sqrt{-1}$.

Then

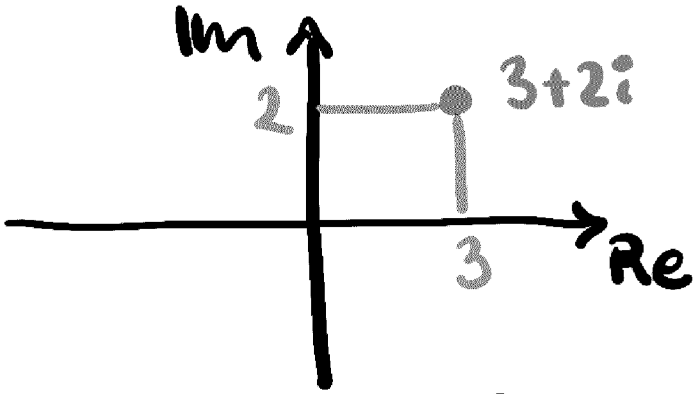
$$x^2=-4 \Rightarrow x=\pm\sqrt{-2}=\pm 2i$$

So it has solutions in the complex numbers.

The imaginary unit satisfies $\boxed{i^2=-1}$ so if $x=2i$ we get

$$x^2=(2i)^2=2^2\cdot i^2=-4$$ and this is indeed a solution to $x^2+4=0$.

A general complex number is one of the form $\boxed{a+bi}$



a= real part

b= imaginary part.

– Both a and b are real.

<u>Operations</u>: "Treat i as a variable"

- $(a+bi)+(c+di) = (a+c)+(b+d)i$
- $(a+bi)-(c+di) = (a-c)+(b-d)i$
- $(a+bi)(c+di) = ac+adi+bci+bdi^2$

$=(ac-bd)+(ad+bc)i$

- $\dfrac{a+bi}{c+di} = \dfrac{(a+bi)(c-di)}{(c+di)(c-di)}$

$$= \frac{ac - adi + bci - bdi^2}{c^2 - d^2 i^2}$$

$$= \frac{(ac+bd) + (bc-ad)i}{c^2 + d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i$$

$z = a+bi$ then its <u>conjugate</u> is $\bar{z} = a - bi$

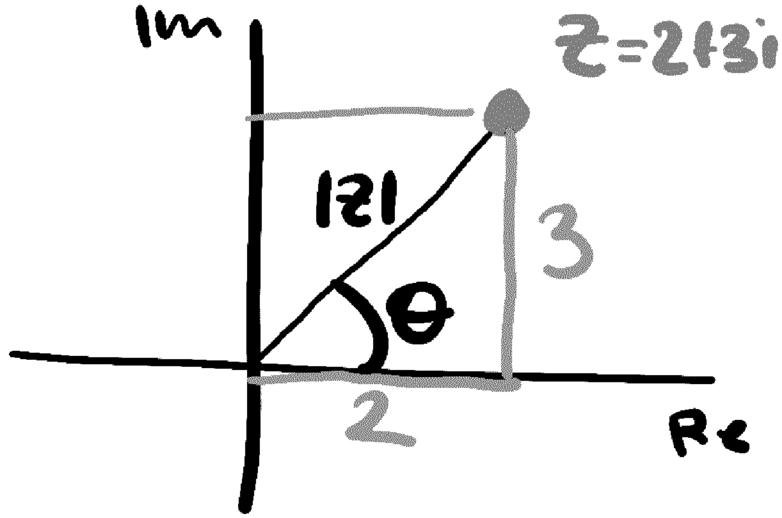
note that $z\bar{z}$ is always a real number!

$$z\bar{z} = (a+bi)(a-bi) = a^2 - (bi)^2$$

$$= a^2 - b^2 i^2 = a^2 + b^2$$

. $z = a+bi$ then its modulus ("size") is

$|z| := \sqrt{a^2+b^2}$



Can also measure the angle θ: it's called the <u>argument</u>

$\arg z = \theta = \arctan\left(\frac{b}{a}\right)$

[<u>Note</u>: This formula is true when z belongs to the 1st quadrant.]

Can describe complex numbers using <u>polar coordinates</u> consisting of its modulus and its argument

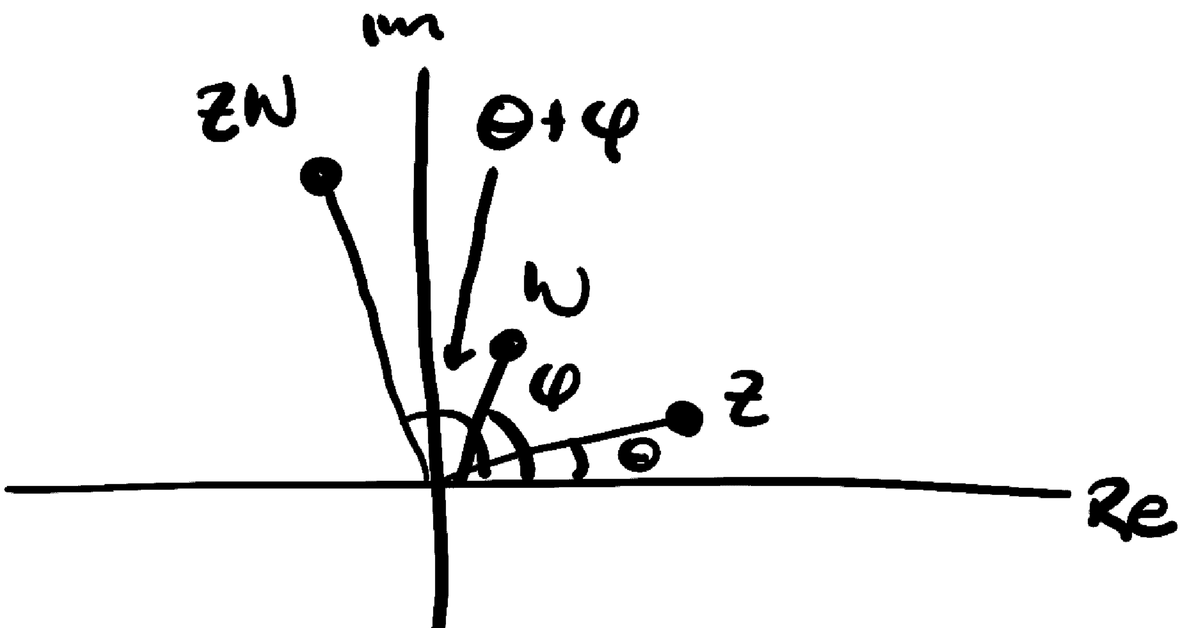
$z = a+bi$, $r = |z|$, $\theta = \arg z$

then $\boxed{z = re^{i\theta}}$

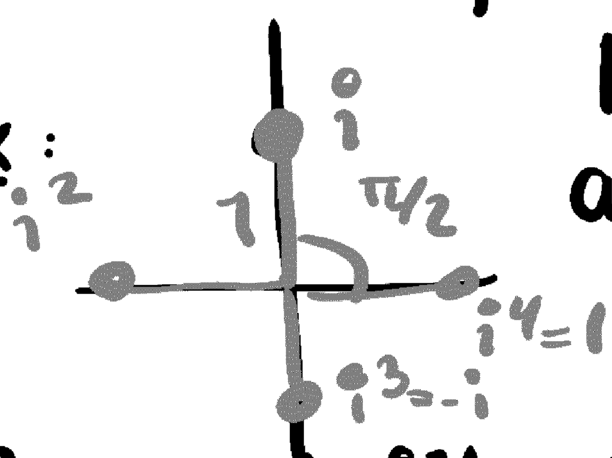
multiplication is easier:

$z = re^{i\theta}$, $w = se^{i\varphi}$ then

$zw = (re^{i\theta})(se^{i\varphi}) = rse^{i(\theta+\varphi)}$



Ex:



$|i| = 1$

$\arg i = \frac{\pi}{2}$

$i^2 = -1$, $|i^2| = 1$, $\arg i^2 = \pi$

$$i^3 = i \cdot i^2 = i \cdot (-1) = -i$$

$$\arg i^3 = \frac{3\pi}{2}$$

$$i^4 = (i^2)^2 = (-1)^2 = 1$$

$$\arg i^4 = 0 \quad (\text{or } 2\pi)$$

$$\frac{1}{i} = \frac{-i}{i(-i)} = \frac{-i}{-i^2} = \frac{-i}{1} = -i$$