

MAT 127

L22-23 TH Nov 6

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Power series solutions to ODEs

It's not always possible to find a solution to an ODE in terms of elementary functions.

For example

$$y'' - 2xy' + y = 0$$

But it's important to be able to solve them.

Can obtain a power series representation of the solution y .

Ex. Let's consider

$y'' + y = 0$ and let's try to use power series to solve it.

Assume there's a solution near $x=0$ and that

$$y(x) = \sum_{n=0}^{\infty} b_n x^n.$$

Then

$$\begin{cases} y'(x) = \sum_{n=1}^{\infty} n b_n x^{n-1} \\ y''(x) = \sum_{n=2}^{\infty} n(n-1) b_n x^{n-2}. \end{cases}$$

To compare the series for y and y'' we shift the index:

$$\begin{aligned} y''(x) &= \sum_{n=2}^{\infty} n(n-1) b_n x^{n-2} \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) b_{n+2} x^n. \end{aligned}$$

Therefore

$$y'' + y = \sum_{n=0}^{\infty} (n+2)(n+1) b_{n+2} x^n$$

$$+ \sum_{n=0}^{\infty} b_n x^n$$

$$= \sum_{n=0}^{\infty} [(n+2)(n+1)b_{n+2} + b_n] x^n = 0$$

This power series being zero must mean we need to find the coefficients b_0, b_1, b_2, \dots so that

$$\boxed{(n+2)(n+1)b_{n+2} + b_n = 0} \text{ for all } n$$

recursion relation

$$\boxed{b_{n+2} = -\frac{b_n}{(n+2)(n+1)}}$$

$$\underline{n=0} \quad b_2 = -\frac{b_0}{2 \cdot 1}$$

$$\underline{n=1} \quad b_3 = -\frac{b_1}{3 \cdot 2}$$

$$\underline{n=2} \quad b_4 = -\frac{b_2}{4 \cdot 3} = -\frac{1}{4 \cdot 3} \cdot \left(-\frac{b_0}{2 \cdot 1}\right)$$

$$= \frac{b_0}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{b_0}{4!}$$

$$\underline{n=3} \quad b_5 = -\frac{b_3}{5 \cdot 4} = \left(-\frac{1}{5 \cdot 4}\right) \left(-\frac{b_1}{3 \cdot 2}\right)$$
$$= \frac{b_1}{5!}$$

$$\underline{n=4} \quad b_6 = -\frac{b_4}{6 \cdot 5} = -\frac{b_0}{6 \cdot 5 \cdot 4!} = -\frac{b_0}{6!}$$

Pattern:

$$\begin{cases} b_{2n} = (-1)^n \frac{b_0}{(2n)!} \\ b_{2n+1} = (-1)^n \frac{b_1}{(2n+1)!} \end{cases} \quad (*)$$

Let's check:

$$b_2 = (-1)^1 \frac{b_0}{2!} = -\frac{b_0}{2!}$$

$$b_4 = (-1)^2 \frac{b_0}{4!} = \frac{b_0}{4!}$$

$$b_6 = (-1)^3 \frac{b_0}{6!} = -\frac{b_0}{6!}$$

Ok

$$b_3 = (-1)^1 \frac{b_1}{3!} = -\frac{b_1}{3!} \quad \text{ok.}$$

$$b_5 = (-1)^2 \frac{b_1}{5!} = \frac{b_1}{5!}$$

Now let's plug in (*) into

$$y(x) = \sum_{n=0}^{\infty} b_n x^n$$

$$= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \dots$$

$$= b_0 + b_1 x - \frac{b_0}{2!} x^2 - \frac{b_1}{3!} x^3$$

$$+ \frac{b_0}{4!} x^4 + \frac{b_1}{5!} x^5 + \dots$$

$$= \left(b_0 - \frac{b_0}{2!} x^2 + \frac{b_0}{4!} x^4 - \frac{b_0}{6!} x^6 + \dots \right)$$

$$+ \left(b_1 x - \frac{b_1}{3!} x^3 + \frac{b_1}{5!} x^5 - \dots \right)$$

$$\begin{aligned}
&= b_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right) \\
&\quad + b_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right) \\
&= b_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + b_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\
&= b_0 \cos x + b_1 \sin x.
\end{aligned}$$

Therefore $y = b_0 \cos x + b_1 \sin x$
for any constants b_0, b_1 is
the general solution to
 $y'' + y = 0$

Note: Usually we can not
solve the recurrence relation
involving the coefficients in the
power series.

Ex: $y'' - 2xy' + y = 0$

Let's use power series to find a power series representation of the general solution.

$$y = \sum_{n=0}^{\infty} C_n x^n$$

shift n
↓
∞

$$y' = \sum_{n=1}^{\infty} n C_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) C_{n+1} x^n$$

$$y'' = \sum_{n=1}^{\infty} (n+1)n C_{n+1} x^{n-1}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n$$

shift n

$$2xy' = 2x \sum_{n=0}^{\infty} (n+1) C_{n+1} x^n$$

multiply

$$= 2 \sum_{n=0}^{\infty} (n+1) C_{n+1} x^{n+1}$$

$$\rightarrow = 2 \sum_{n=1}^{\infty} n c_n x^n$$

shift n

$$= 2 \sum_{n=0}^{\infty} n c_n x^n$$

add back (n=0)-term (which is =0)

Therefore

$$y'' - 2xy' + y = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - 2 \sum_{n=0}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=0}^{\infty} [(n+2)(n+1) c_{n+2} - 2n c_n + c_n] x^n$$

$$= \sum_{n=0}^{\infty} [(n+2)(n+1) c_{n+2} - (2n-1) c_n] x^n$$

$$= 0$$

we get the recurrence

$$(n+2)(n+1)C_{n+2} - (2n-1)C_n = 0$$

$$C_{n+2} = \frac{(2n-1)C_n}{(n+2)(n+1)}$$

Let's find a pattern.

$$\underline{n=0} \quad C_2 = \frac{-C_0}{2 \cdot 1} = -\frac{C_0}{2!}$$

$$\underline{n=1} \quad C_3 = \frac{1 \cdot C_1}{3 \cdot 2} = \frac{C_1}{3!}$$

$$\underline{n=2} \quad C_4 = \frac{3 \cdot C_2}{4 \cdot 3} = -\frac{3 \cdot C_0}{4 \cdot 3 \cdot 2!} = -\frac{3C_0}{4!}$$

$$\underline{n=3} \quad C_5 = \frac{5C_3}{5 \cdot 4} = \frac{5 \cdot C_1}{5 \cdot 4 \cdot 3!} = \frac{5 \cdot C_1}{5!}$$

$$\underline{n=4} \quad C_6 = \frac{7 \cdot C_4}{6 \cdot 5} = -\frac{7 \cdot 3 \cdot C_0}{6 \cdot 5 \cdot 4!} = -\frac{7 \cdot 3 \cdot C_0}{6!}$$

$$\underline{n=5} \quad C_7 = \frac{9 \cdot C_5}{7 \cdot 6} = \frac{9 \cdot 5 \cdot C_1}{7 \cdot 6 \cdot 5!} = \frac{9 \cdot 5 \cdot C_1}{7!}$$

The patterns are

$$\begin{cases} C_{2n} = \frac{-3 \cdot 7 \cdot 11 \cdots (4n-5) C_0}{(2n)!} \\ C_{2n+1} = \frac{5 \cdot 9 \cdot 13 \cdots (4n-3) C_1}{(2n+1)!} \end{cases}$$

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \dots$$

$$= C_0 + C_1 x - \frac{C_0}{2!} x^2 + \frac{C_1}{3!} x^3$$

$$- \frac{3C_0}{4!} x^4 + \frac{5 \cdot C_1}{5!} x^5 - \frac{3 \cdot 7 C_0}{6!} x^6 + \dots$$

$$= C_0 \left(1 - \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{3 \cdot 7 x^6}{6!} - \dots \right)$$

$$+ C_1 \left(x + \frac{x^3}{3!} + \frac{5x^5}{5!} + \frac{5 \cdot 9 x^7}{7!} + \dots \right)$$

This is the answer & these power series do not have a description in terms of "elementary functions"

(like $\sin x$, $\cos x$, etc)

Ex: $y' = x^2 y$. It's separable, so we can solve it that way, but let's use power series.

$$y = \sum_{n=0}^{\infty} C_n x^n$$

$$y' = \sum_{n=1}^{\infty} n C_n x^{n-1} \stackrel{\text{shift } n}{=} \sum_{n=0}^{\infty} (n+1) C_{n+1} x^n$$

$$x^2 y = x^2 \sum_{n=0}^{\infty} C_n x^n \stackrel{\text{multiply}}{=} \sum_{n=0}^{\infty} C_n x^{n+2}$$

$$= \sum_{n=2}^{\infty} C_{n-2} x^n$$

$$y' - x^2 y = \sum_{n=0}^{\infty} (n+1) C_{n+1} x^n - \sum_{n=2}^{\infty} C_{n-2} x^n \quad (*)$$

To gather the terms we need

the series to start at the same index. To "fix" it, we write out the first two terms in the former series.

$$\begin{aligned} (*) &= C_1 x + 2C_2 x^2 + \sum_{n=2}^{\infty} (n+1)C_{n+1} x^n \\ &\quad - \sum_{n=2}^{\infty} C_{n-2} x^n \\ &= \underbrace{C_1}_{=0} + \underbrace{2C_2}_{=0} x + \sum_{n=2}^{\infty} [(n+1)C_{n+1} - C_{n-2}] x^n \\ &= 0 \end{aligned}$$

Immediately get $C_1 = 0, C_2 = 0.$

Then $(n+1)C_{n+1} - C_{n-2} = 0 \quad n \geq 2$

$$C_{n+1} = \frac{C_{n-2}}{n+1} \quad n \geq 2$$

$$\underline{n=2} \quad C_3 = \frac{C_0}{3}$$

$$\underline{n=3} \quad C_4 = \frac{C_1}{4} = 0$$

$$\underline{n=4} \quad C_5 = \frac{C_2}{5} = 0$$

$$\underline{n=6} \quad C_6 = \frac{C_3}{6} = \frac{C_0}{6 \cdot 3} = \frac{C_0}{3^2 \cdot 2 \cdot 1} = \frac{C_0}{3^2 \cdot 2!}$$

$$\underline{n=7} \quad C_7 = \frac{C_4}{7} = 0$$

$$\underline{n=8} \quad C_8 = \frac{C_5}{8} = 0$$

$$\underline{n=9} \quad C_9 = \frac{C_6}{9} = \frac{C_0}{9 \cdot 3^2 \cdot 2!} = \frac{C_0}{3^3 \cdot 3!}$$

Pattern seems to be

$$C_{3n} = \frac{C_0}{3^n \cdot n!} \quad \text{and } C_n = 0$$

for n not a multiple of 3.

$$y = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 +$$

$$C_4 x^4 + C_5 x^5 + C_6 x^6 + \dots$$

$$= C_0 + C_3 x^3 + C_6 x^6 + C_9 x^9 + \dots$$

$$= C_0 \left(1 + \frac{x^3}{3 \cdot 1!} + \frac{x^6}{3^2 \cdot 2!} + \frac{x^9}{3^3 \cdot 3!} + \dots \right)$$

$$= C_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{3^n n!} = C_0 \sum_{n=0}^{\infty} \frac{\left(\frac{x^3}{3}\right)^n}{n!}$$

$$= C_0 e^{\frac{x^3}{3}}$$

This is also the solution
obtained via separation of
variables!
