MAT 127 Lec 2, Th Aug 29

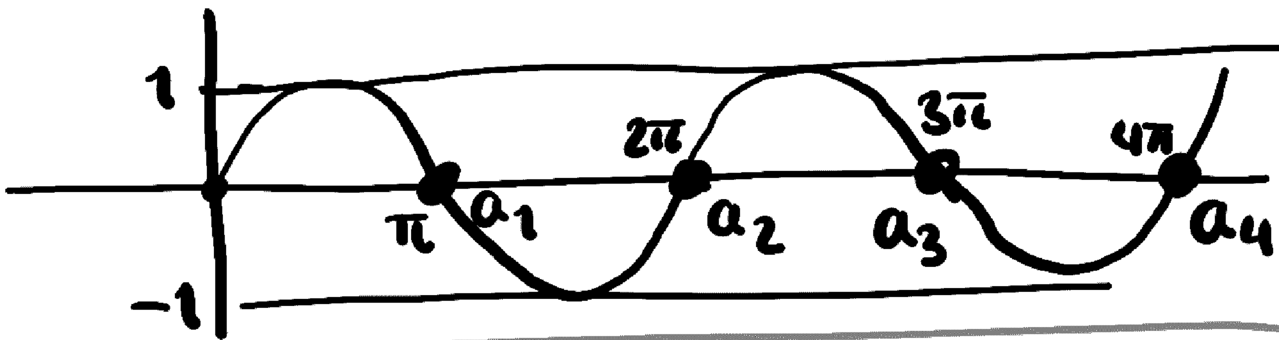
<u>Recall</u>: • $\{a_n\}$ sequence $\lim_{n\to\infty} a_n = L$ if a_n approaches L as n becomes large. It's convergent if L exists, else divergent.

• If $a_n = f(n)$ for some function and $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$

> <u>Warning</u>: If $a_n = f(n)$ for a function, and $f(x)$ diverges as $x \to \infty$ we DO NOT know whether $\lim_{n\to\infty} a_n$ exists or not!

Ex: $a_n = \sin(\pi n)$, $n = 1, 2, \dots$

Note $a_n = f(n)$, $f(x) = \sin(\pi x)$ for real x.



$\boxed{\lim_{x\to\infty} \sin(\pi x) \text{ does not exist as it oscillates between } -1 \text{ and } 1}$

However $a_n = \sin(\pi n) = 0$ for all positive integers n.

So $\boxed{\lim_{n\to\infty} \sin(\pi n) = 0}$.

Ex: $a_n = (-1)^n$.

$a_1 = -1,\ a_2 = 1,\ a_3 = -1, \dots$

$\{a_n\}$ does not approach a number as $n \to \infty$ since it oscillates between ± 1.

$\lim_{n\to\infty} (-1)^n$ does not exist,

so $\{(-1)^n\}$ is divergent.

Ex: Find $\lim_{n\to\infty} \frac{n^2}{\sqrt{n^3+4n}}$. Divide numerator & denominator w/ highest power of n. It's n^2 because as n is large, denominator is approximated by $\sqrt{n^3} = n^{\frac{3}{2}}$.

$$\lim_{n\to\infty} \frac{n^2/n^2}{\frac{\sqrt{n^3+4n}}{n^2}} = \lim_{n\to\infty} \frac{1}{\sqrt{\frac{n^3+4n}{n^4}}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{1}{\frac{n^3+4n}{n^4}}} = \lim_{n\to\infty} \sqrt{\frac{1}{\frac{1}{n}+\frac{4}{n^3}}}$$

$$= \sqrt{\frac{1}{\lim\limits_{n\to\infty}\frac{1}{n} + \lim\limits_{n\to\infty}\frac{4}{n^3}}} = \infty.$$

approaches 0

<u>Thm</u> If $\lim\limits_{n\to\infty} a_n = L$ and the function f is continuous at L,

$$\lim_{n\to\infty} f(a_n) = f(L).$$

<u>Ex</u>: Find $\lim\limits_{n\to\infty} \sin\left(\frac{\pi}{n}\right)$.

$a_n = \frac{\pi}{n}$ and we want to find $\lim\limits_{n\to\infty} f(a_n)$ for $f(x) = \sin x$.

Note $\lim_{n\to\infty} \frac{\pi}{n} = 0$.

$f(x) = \sin x$ is continuous at 0 (its continuous at all points)

so $\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right)$

$= \sin(0) = 0$.

<u>Ex</u>: <u>Geometric sequence</u>

For what values of r is $a_n = r^n$ a convergent sequence?

$a_n = f(n)$, $f(x) = r^x$.

Recall

$$\lim_{x\to\infty} r^x = \begin{cases} \infty & \text{for } r > 1 \\ 0 & \text{for } 0 \leq r < 1 \end{cases}$$

Boundary case: $\lim_{x\to\infty} 1^x = 1$

We will also allow negative values of r, so:

$$\boxed{\lim_{n\to\infty} r^n = \begin{cases} 1, & r=1 \\ 0, & -1<r<1 \\ \infty, & \text{else} \end{cases}}$$

<u>Def</u> A sequence $\{a_n\}$ is

- <u>increasing</u> if $a_n \le a_{n+1}$ for all $n \ge 1$
- <u>decreasing</u> if $a_n \ge a_{n+1}$ for all $n \ge 1$
- <u>monotonic</u> if it's either increasing or decreasing

Ex: $a_n = \frac{3}{n+5}$ is decreasing:

$$a_{n+1} = \frac{3}{(n+1)+5} = \frac{3}{n+6}$$

$$a_n = \frac{3}{n+5} \geq \frac{3}{n+6} = a_{n+1} \text{ for all } n \geq 1.$$

bigger denominator
→ smaller quotient.

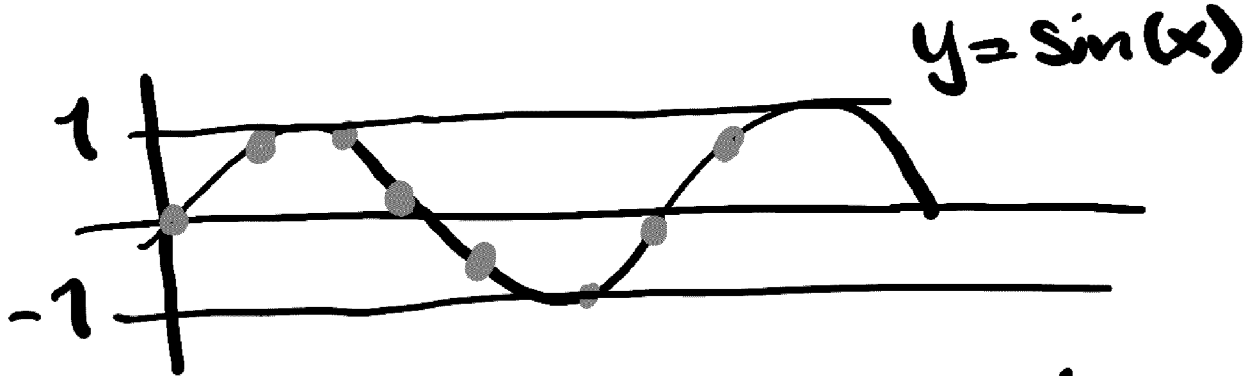
Def A sequence is

- bounded above if there's a number M such that $a_n \leq M$ for all $n \geq 1$

- bounded below if there's a number m such that $a_n \geq m$ for all $n \geq 1$
- bounded if it's both bounded above and below.

Ex $a_n = \sin(n)$ is bounded:

$$-1 \leq \sin(n) \leq 1$$



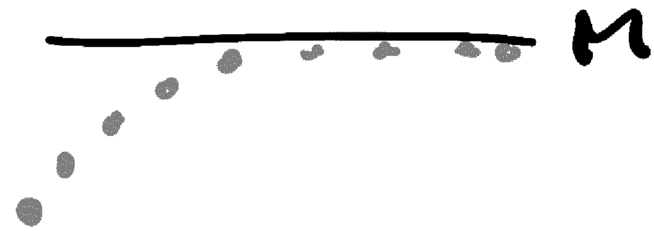
Note $\lim_{n\to\infty} a_n$ does not exist

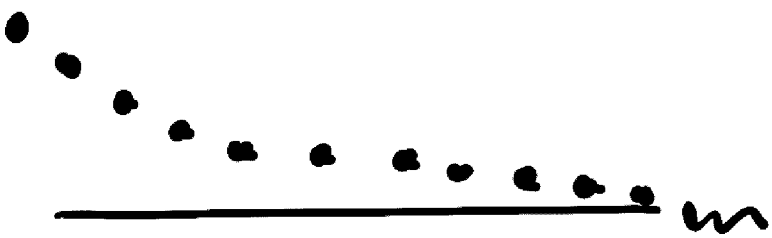
Thm (Monotone convergence theorem)

Let $\{a_n\}$ be a sequence.

- If $\{a_n\}$ is increasing and bounded above, then it is convergent.
- If $\{a_n\}$ is decreasing and bounded below, then it is convergent

Illustration:





Ex: Let a_n be defined by

$$a_1 = 7, \quad a_n = \frac{a_{n-1}}{n} \text{ for } n \geq 2.$$

$$a_2 = \frac{7}{2}, \quad a_3 = \frac{7/2}{3} = \frac{7}{2 \cdot 3}, \quad a_4 = \frac{7/2\cdot 3}{4} = \frac{7}{2 \cdot 3 \cdot 4} = \frac{7}{4!}$$

$$a_5 = \frac{7/2\cdot 3 \cdot 4}{5} = \frac{7}{5!}$$

$$a_{n+1} = \frac{a_n}{n+1} \leq \frac{a_n}{1} = a_n$$

smaller denominator

→ larger quotient

So sequence is decreasing

Also $a_n \geq 0$ is "obviously" true since we only take quotients of positive integers.

So the monotone convergence thm guarantees that $\lim_{n \to \infty} a_n$ exists.

<u>Ex</u> (alternative sol)

a_n as above.

Note $a_n = \frac{7}{n!}$.

factorial $n! = n \cdot (n-1) \cdot \dots \cdot 3 \cdot 2 \cdot 1$

we then see $\lim_{n \to \infty} \frac{7}{n!} = 0$.

Suppose $\{a_n\}$ and $\{b_n\}$ are two sequences such that $a_n \leq b_n$ for all $n \geq 1$.

- If $\lim_{n\to\infty} b_n = B$ exists then we have $\lim_{n\to\infty} a_n \leq B$ if the limit exists.
- If $\lim_{n\to\infty} a_n = A$ exists, then we have $\lim_{n\to\infty} b_n \geq A$ if the limit exists.
- If $a_n \to +\infty$ then $b_n \to +\infty$.

<u>Thm</u> (squeeze theorem)

Let $\{a_n\}, \{b_n\}, \{c_n\}$ be three sequences such that
$a_n \le b_n \le c_n$ for all $n \ge 1$.

If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ exists

then $\lim_{n\to\infty} b_n$ exists and is equal to L.