MAT 127 Lec 19 Thu Oct 31

Recall

Separable ODEs:

$$\frac{dy}{dx} = g(x) f(y)$$

$$\Leftrightarrow \quad \frac{1}{f(y)} dy = g(x)\, dx$$

Integrate

$$\boxed{\int \frac{1}{f(y)} dy = \int g(x) dx}$$

Solve for y (if possible)

Ex suppose a cup of freshly brewed coffee w/ temperature 200 °F is placed in a room thats 70°F.

Newton's law of Cooling

states

rate of temp change $= k \cdot$ (temp $-$ room temp)

Suppose that $k = -\frac{1}{5}$ in our case. How long do we have to wait until our coffee is 130 °F, the perfect drinking temp?

$$\frac{dT}{dt} = -\frac{1}{5}\left(T(t) - 70\right)$$

$$\Leftrightarrow \quad 5\,\frac{dT}{dt} = 70 - T(t)$$

$$\Leftrightarrow \int \frac{5}{70 - T(t)}\,dT = \int dt$$

$$\Leftrightarrow -5\ln|70 - T(t)| = t + C$$

$$\Leftrightarrow \ln|70 - T(t)| = -\frac{t}{5} + D$$

$$\Leftrightarrow 70 - T(t) = e^{-\frac{t}{5}+D} = E \cdot e^{-\frac{t}{5}}$$

$$\Leftrightarrow T(t) = 70 - E \cdot e^{-\frac{t}{5}}$$

Now, we know $T(0) = 200$:

$$T(0) = 70 - E = 200$$

$$\Leftrightarrow E = -130$$

$$T(t) = 70 + 130 e^{-\frac{t}{5}}$$

want to solve

$$T(t) = 130$$

$$\Leftrightarrow 70 + 130 e^{-\frac{t}{5}} = 130$$

$$\Leftrightarrow 130 e^{-\frac{t}{5}} = 60$$

$$\Leftrightarrow e^{-\frac{t}{5}} = \frac{60}{130} = \frac{6}{13}$$

$$\Leftrightarrow -\frac{t}{5} = \ln\left(\frac{6}{13}\right)$$

$$t = -5 \ln\left(\frac{6}{13}\right) = 5 \ln\left(\frac{13}{6}\right)$$

$= 3.87$ min

Exponential growth and decay

Sometimes

$$\frac{dy}{dt} = ky \qquad k \text{ constant}$$

is called the law of natural growth $(k>0)$

or the law of natural decay $(k<0)$

This is since it's separable, and by solving it we find

$$\int \frac{1}{ky}\, dy = \int dt$$

$$\Leftrightarrow \frac{1}{k} \ln|y| = t + C$$

$$\Leftrightarrow \ln|y| = kt + D$$

$$\Leftrightarrow \boxed{y = e^{kt+D} = y_0 e^{kt}}$$

constant

We have already seen some examples: population growth for instance.

Ex: In 1900 the world's population was 1650 million.

In 1960 it was 3040 million

In 2020 it was 7755 million.

Let's set up an ODE modeling this.

We assume the world's population follows the natural law of growth.

$P(t)$ = million people, t years after 1900.

$\frac{dP}{dt} = kP$. The general sol (as seen previously) is

$$P(t) = P_0 e^{kt}.$$

$P_0 = P(0) = 1650$ = initial population.

To find k, we use the data point

$$P(60) = 3040.$$

$$P(60) = 1650 e^{60k} = 3040$$

$$\Leftrightarrow e^{60k} = \frac{3040}{1650} = \frac{304}{165}$$

$$\Rightarrow k = \frac{1}{60} \ln\left(\frac{304}{165}\right) \approx 0.01 = \frac{1}{10}.$$

Let's now estimate $P(120)$ (population in 2020).

$$P(120) = 1650 \cdot e^{\frac{1}{10} \cdot 120} = 1650 \cdot e^{12}$$

$$\approx 5601$$

which is quite a bit smaller than 7755.

<u>The logistic equation</u>:

<u>Ex</u> suppose that a population is modeled using the logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

with $k = 0.08$, $M = 1000$, and initial population $P(0) = 100$.

$\boxed{\text{Find the population at time } t = 40 \text{ and } t = 80.}$

We first solve the IVP.

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)$$

$$\Leftrightarrow 1000\frac{dP}{dt} = 0.08\,P(1000 - P)$$

$$\Leftrightarrow \int \frac{1000}{P(1000-P)}\,dP = \int 0.08\,dt$$

We find the integral via partial fraction decomposition:

$$\frac{1}{P(1000-P)} = \frac{1}{1000}\left(\frac{1}{P} + \frac{1}{1000-P}\right)$$

$$\Leftrightarrow \quad \int\left(\frac{1}{P} + \frac{1}{1000-P}\right) dP = 0.08t + C$$

$$\Leftrightarrow \quad \log|P| - \log|1000-P| = 0.08t + C$$

$$\Leftrightarrow \log\left|\frac{P}{1000-P}\right| = 0.08t + C$$

$$\Leftrightarrow -\log\left|\frac{1000-P}{P}\right| = 0.08t + C$$

$$\Leftrightarrow \log\left|\frac{1000-P}{P}\right| = -0.08t - C$$

$$\Leftrightarrow \quad \frac{1000-P}{P} = e^{-0.08t-C} = D \cdot e^{-0.08t}$$

$$\Leftrightarrow \quad \frac{1000}{P} - 1 = D \cdot e^{-0.08t}$$

$$\Leftrightarrow \frac{1000}{P} = D \cdot e^{-0.08t} + 1$$

$$\Leftrightarrow \boxed{P = \frac{1000}{1 + D \cdot e^{-0.08t}}}$$

Initial cond: $P(0) = \dfrac{1000}{1 + D \cdot e^{0}} = \dfrac{1000}{1+D}$

$$= 100$$

$$\Leftrightarrow \frac{1000}{100} = 1 + D$$

$$\Leftrightarrow D = 9$$

$$P(t) = \frac{1000}{1 + 9 \cdot e^{-0.08t}}$$

$$P(40) = \frac{1000}{1 + 9e^{-0.08 \cdot 40}} \approx 731.6$$

$$P(80) \approx 985.3$$