MAT 127 Lec 18 Tu Oct 29

<u>Recall</u> ODE is <u>separable</u> if it can be written in the form

$$\boxed{\frac{dy}{dx} = f(y)g(x)}$$

We solve it by collecting everything "involving" y in the LHS & everything "involving" x in the RHS.

$$\frac{1}{f(y)}\,dy = g(x)\,dx$$

Then integrate:

$$\int \frac{1}{f(y)}\,dy = \int g(x)\,dx$$

<u>Ex</u>: $y' = x e^{-y} \Leftrightarrow e^{y}\,dy = x\,dx$

$\Leftrightarrow \int e^{y}\,dy = \int x\,dx \Leftrightarrow e^{y} = \frac{x^2}{2} + C$

$\Leftrightarrow y = \log\left(\frac{x^2}{2} + C\right)$

<u>Ex</u> $y' + xy = x$

$$\frac{dy}{dx} = x - xy = x(1-y)$$

$$\Leftrightarrow \frac{1}{1-y}\,dy = x\,dx$$

$$\Leftrightarrow -\ln|1-y| = \frac{x^2}{2} + C$$

$$\Leftrightarrow \ln|1-y| = -\frac{x^2}{2} + D$$

$$\Leftrightarrow 1-y = e^{-\frac{x^2}{2}+D} = E e^{-\frac{x^2}{2}}$$

$$\Leftrightarrow \boxed{y = 1 - E \cdot e^{-\frac{x^2}{2}}}$$

<u>Mixing</u> <u>problems</u>

<u>Ex</u>. A tank contains 20 kg of salt dissolved in 5000 L of water.

Brine that contains 0.03 kg salt/L enters the tank at the rate of 25 L/min. The solution is kept thoroughly mixed & it drains with the same rate.

How much salt is in the tank after 30 min?.

Let $y(t)$ = amt of salt in kg at time t (min)

we know $y(0) = 20$ & need to find $y(30)$.

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

salt that enters tank per min

salt that exits tank per min

$$\text{rate in} = \left(0.03 \cdot \frac{kg}{L}\right) \cdot \left(25 \cdot \frac{L}{min}\right)$$

$$= 0.75 \, \frac{kg}{min}$$

To calculate rate out, we need to know the current amt of salt.

$$\text{rate out} = \left(\begin{array}{l}\text{curr salt} \\ \text{concentration}\end{array} \frac{kg}{L}\right) \cdot \left(25 \frac{L}{min}\right)$$

$$\left[\begin{array}{l}\text{curr salt} \\ \text{concentration}\end{array} = \frac{y(t)}{5000} \, \frac{kg}{L}.\right]$$

$$= \frac{y(t)}{5000} \cdot 25 = \frac{y(t)}{200}$$

Hence

$$\boxed{\frac{dy}{dt} = \frac{3}{4} - \frac{y(t)}{200}} \quad \boxed{y(0) = 20}$$

Let's now solve it!

$$200 \, \frac{dy}{dt} = 150 - y(t)$$

$$\Leftrightarrow \int \frac{200}{150 - y} \, dy = \int dt$$

$$\Leftrightarrow -200\ln|150-y| = t + C$$

$$\Leftrightarrow \ln|150-y| = -\frac{t}{200} + D$$

$$\Leftrightarrow 150 - y = e^{-\frac{t}{200}+D} = E \cdot e^{-\frac{t}{200}}$$

$$\Leftrightarrow \boxed{y = 150 - E \cdot e^{-\frac{t}{200}}}$$

$$y(0) = 150 - E \cdot e^{0} = 20$$

$$\Leftrightarrow E = 130$$

$$y(t) = 150 - 130\, e^{-\frac{t}{200}}$$

$$y(30) = 150 - 130\, e^{-\frac{30}{200}}$$

$$\approx 38.108 \text{ kg}$$

<u>Ex</u> A vat with 500 gallons of beer contains 4% alcohol

(by volume). Beer with 6% alcohol is pumped into the vat at a rate of 5 gal/min & the mixture is pumped out at the same rate.

What's the % of alcohol after 1 hour?

Let $P(t)$ = alcohol (gal) at time t in the vat.

$$P(0) = 500 \cdot 4\% = 500 \cdot 0.04 = 20 \text{ gal}$$

$$\frac{dP}{dt} = \text{rate in} - \text{rate out}$$

$$\text{rate in} = 5 \cdot 0{,}06 \ \frac{\text{gal alc}}{\text{min}}$$

$$= 0.3 \text{ gal/min}$$

rate out $= 5 \cdot \frac{P(t)}{500}$ gal alc/min

$= \frac{P(t)}{100}$ gal/min

$$\frac{dP}{dt} = 0.3 - \frac{P(t)}{100}$$

Therefore want to solve the IVP

$$\boxed{\frac{dP}{dt} = 0.3 - \frac{P(t)}{100}, \quad P(0) = 20}$$

$$\int \frac{1}{0.3 - \frac{P(t)}{100}} dP = \int dt$$

$$\Leftrightarrow \frac{1}{-1/100} \ln\left|0.3 - \frac{P(t)}{100}\right| = t + C$$

$$\Leftrightarrow \ln\left|0.3 - \frac{P(t)}{100}\right| = -\frac{t}{100} + D$$

$$\Leftrightarrow 0.3 - \frac{P(t)}{100} = e^{-\frac{t}{100} + D} = E e^{-\frac{t}{100}}$$

$$\Leftrightarrow 30 - P(t) = F e^{-\frac{t}{100}}$$

$$\Longleftrightarrow P(t) = 30 - Fe^{-\frac{t}{100}}$$

$$P(0) = 20 \Longleftrightarrow 30 - F = 20$$

$$\Longleftrightarrow F = 10$$

$$P(t) = 30 - 10e^{-\frac{t}{100}}$$

want to find $\frac{P(60)}{500}$.

$$\frac{P(60)}{500} = \frac{30 - 10e^{-\frac{6}{10}}}{500} \approx 0.0529$$

$$= 5.29\%.$$