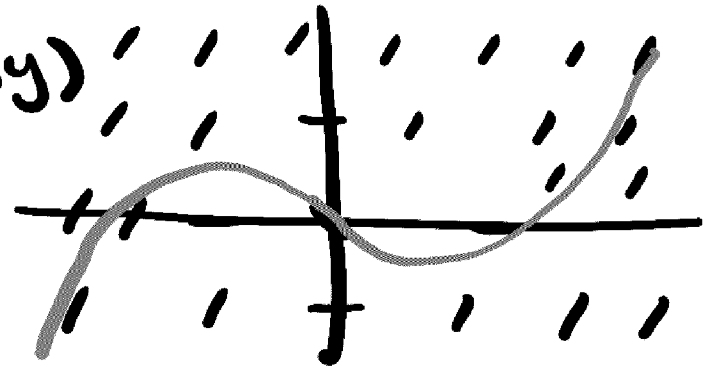
MAT 127 Lec 17 Thu 24 Oct

<u>Recall</u> • Slope fields:

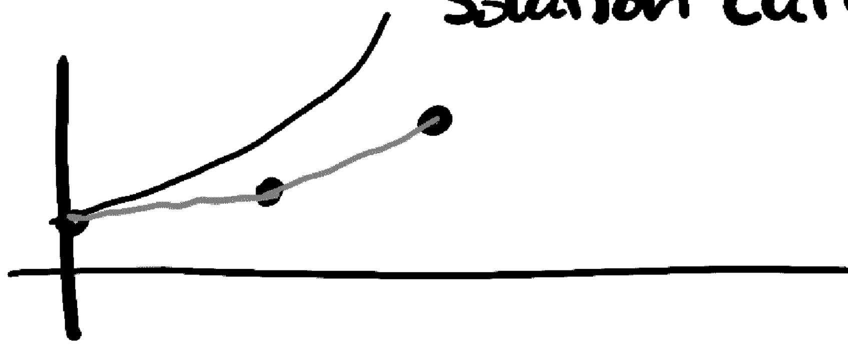
$y' = F(x,y)$



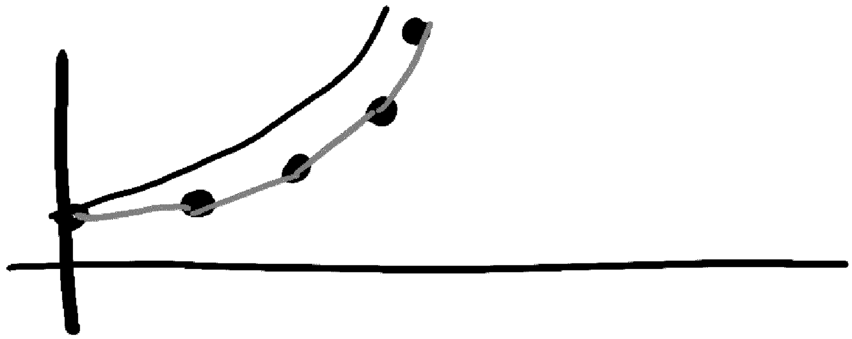
Used to graphically see general features of general solutions

• Euler's method.

Numerically approximate a sol by small straight line steps.



Can use smaller step sizes to get better approximation



Suppose we have a general ODE now:

$$\boxed{y' = F(x,y)}$$

Let (x_0, y_0) be our start point.
Let h be our <u>step size</u>.

$$\begin{cases} x_1 = x_0 + h, \text{ then} \\ y_1 = y_0 + h \cdot F(x_0, y_0) \end{cases}$$

$$\begin{cases} x_2 = x_1 + h \\ y_2 = y_1 + h \cdot F(x_1, y_1) \end{cases}$$

Euler's Method

$$\boxed{\begin{aligned} x_n &= x_{n-1} + h \\ y_n &= y_{n-1} + h \cdot F(x_{n-1}, y_{n-1}) \\ n &= 1, 2, 3, \ldots \end{aligned}}$$

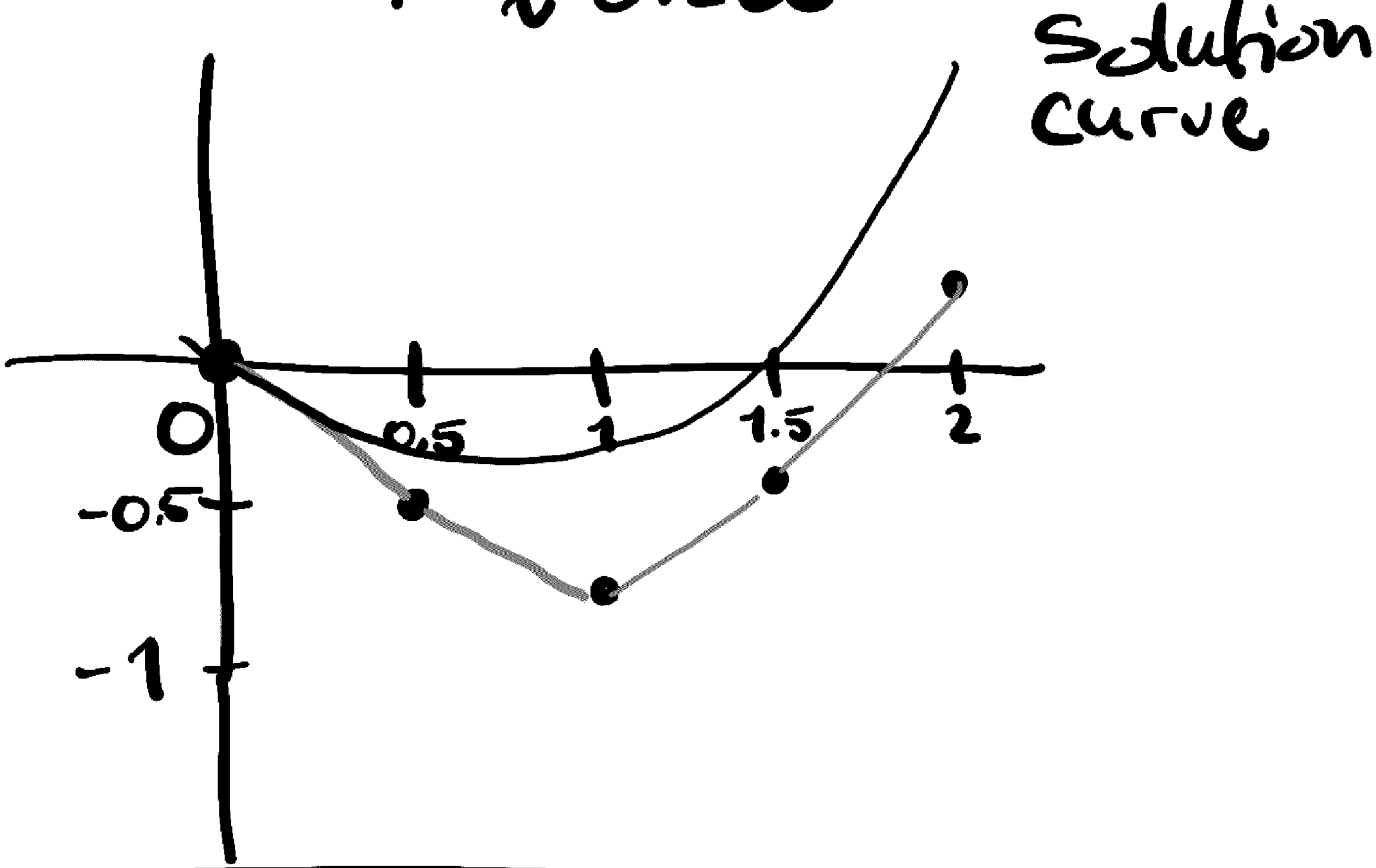
<u>Ex</u>: Consider $y' = x^2 + y^2 - 1$, $y(0) = 0$

Let's create a table of numerical approximations of the sol from $x = 0$ to $x = 2$ w/ step size $h = 0.5$

$(x_0, y_0) = (0, 0)$

x	y
$x_0 = 0$	0
$x_1 = 0.5$	$y_1 = y_0 + h \cdot F(x_0, y_0)$ $= 0 + 0.5 \cdot (0^2 + 0^2 - 1)$ $= -0.5$

$x_2 = 1$	$y_2 = y_1 + h \cdot F(x_1, y_1)$ $= -0.5 + 0.5\left(0.5^2 + (-0.5)^2 - 1\right)$ $= -0.75$
$x_3 = 1.5$	$y_3 = y_2 + h \cdot F(x_2, y_2)$ $= -0.75 + 0.5\left(1^2 + (-0.75)^2 - 1\right)$ $= -0.46875$
$x_4 = 2$	$y_4 = y_3 + h \cdot F(x_3, y_3)$ $= -0.46875$ $+ 0.5\left(1.5^2 + (-0.46875)^2 - 1\right)$ ≈ 0.266



Separable ODEs

We will now learn how to Solve some ODEs.

Def: A separable ODE is a first order ODE that can be written in the form

$$\boxed{\frac{dy}{dx} = g(x)f(y)}$$

Meaning the RHS can be "Separated" into a function that only depends on x and a function that only depends on y.

Ex: • $y' = x^2 y$ is separable

• $x^2 y' = y$ is also separable!

Because its equivalent to

$$y' = \frac{y}{x^2} = \frac{1}{x^2} \cdot y$$

- $y' = x^2 + y^2 - 1$ is <u>not</u> separable, because the RHS is not a product of the form $g(x)f(y)$.

To solve a general separable ODE we use the following trick:

$y' = g(x)f(y)$. If $f(y) \neq 0$ write $h(y) = \frac{1}{f(y)}$ so $\boxed{\frac{dy}{dx} = \frac{g(x)}{h(y)}}$

Rewrite it in the differential form

$$\underbrace{h(y)\,dy}_{\text{only } y} = \underbrace{g(x)\,dx}_{\text{only } x}$$

Then integrate:

$$\int h(y)\,dy = \int g(x)\,dx$$

Ex: $x^2 y' = y \iff \frac{dy}{dx} = \frac{y}{x^2}$

$\iff \frac{1}{y} dy = \frac{1}{x^2} dx \iff \int \frac{1}{y} dy = \int \frac{1}{x^2} dx$

$\iff \ln|y| = -\frac{1}{x} + C$

$\iff y(x) = e^{-\frac{1}{x}+C} = e^C e^{-\frac{1}{x}} = De^{-\frac{1}{x}}$

Ex: Solve the IVP

$y' y^2 = x^2, \quad y(0) = 2.$

The equation is separable, so we first solve it.

$\frac{dy}{dx} y^2 = x^2 \iff y^2\, dy = x^2\, dx$

$\iff \int y^2 dy = \int x^2 dx$

$\iff \frac{y^3}{3} = \frac{x^3}{3} + C$

$\Leftrightarrow \quad y^3 = x^3 + D$

$\Leftrightarrow \quad \boxed{y = \sqrt[3]{x^3+D}}$ GENERAL SOLUTION

To find specific solution we look at the initial condition.

$y(0) = 2 \Leftrightarrow \sqrt[3]{0+D} = 2$

$\Leftrightarrow D = 2^3 = 8$

Solution to the IVP is therefore

$\boxed{y(x) = \sqrt[3]{x^3+8}}$

Ex: Find a general solution to

$\frac{dy}{dx} = \frac{6x^2}{2y+\cos y}$.

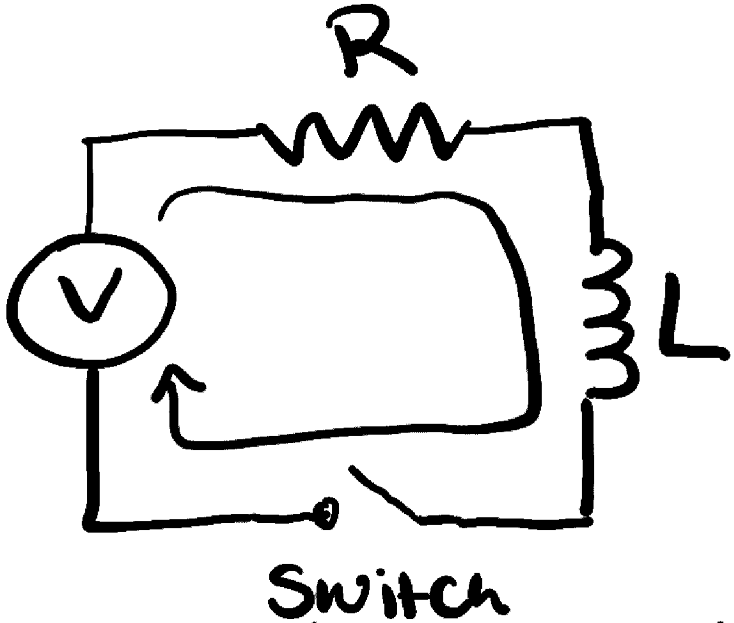
$2y + \cos y \, dy = 6x^2 dx$

$\int 2y + \cos y \, dy = \int 6x^2 dx$

$$\Leftrightarrow \boxed{y^2 + \sin y = 2x^3 + C}$$

Can not solve for y, so this equation defines y implicitly as a function of x.

<u>Ex</u>: Simple electric circuit:



- Resistor w/ resistence R
- Inductor w/ inductance L

Want to model the <u>current</u> I through the circuit as a function of time when switch is closed.

The voltage "drops" as it passes through the two components

The Voltage drop over the resistor is given by Ohm's law

$$V = R \cdot I$$

The Voltage drop over the inductor is $V(t) = L\frac{dI}{dt}$

Kircholl's laws says that the total voltage drop is equal to the total supplied voltage

$$\rightsquigarrow V = L\frac{dI}{dt} + RI(t)$$

Now let

$$\left| \begin{array}{l} R = 12\ \Omega \\ L = 4\ H \\ V = 60\ V \end{array} \right.$$

$$60 = 4\frac{dI}{dt} + 12\,I(t)$$

$$\Leftrightarrow 15 = \frac{dI}{dt} + 3I(t)$$

$$\Leftrightarrow \frac{dI}{dt} = 15 - 3I$$

$$\Leftrightarrow \frac{1}{15-3I}\, dI = dt$$

$$\Leftrightarrow \int \frac{1}{15-3I}\, dI = \int dt$$

$$\Leftrightarrow -\frac{1}{3} \ln|15-3I| = t + C$$

$$\Leftrightarrow \ln|15-3I| = -3t + D$$

$$\Leftrightarrow 15 - 3I = e^{-3t+D} = E e^{-3t}$$

$$\Leftrightarrow 3I = 15 - E e^{-3t}$$

$$\boxed{I = 5 - F \cdot e^{-3t}}$$

The specific solution with $I(0) = 0$ is given by:

$$I(0) = 5 - E \cdot e^{0} = 5 - F = 0$$

$\Leftrightarrow F = 5$ meaning

$I(t) = 5 - 5e^{-3t}$.