MAT 127 Lec 15 Thu 17 Oct

Introduction to ODE's

<u>Def</u>: An ordinary differential equation (ODE) in an equation involving a function $y = y(x)$, one or more of its derivatives, and x.

<u>Ex</u>: $\boxed{y' - 2x = 0}$

A solution is $y = x^2$. We can check it: $y' = 2x$ & so

$$2x - 2x = 0.$$

We can also solve it by integrating:

$$y' = 2x$$

$$\boxed{y = x^2 + C}$$

In fact there are many solutions!

one for each value of C.

<u>Ex</u> $y' + 3y = 6x + 11$

A solution is

$$y = e^{-3x} + 2x + 3.$$

We verify it by differentiation:

$$y' = -3e^{-3x} + 2.$$

$$y' + 3y = 6x + 11$$

$$(\cancel{-3e^{-3x}} + 2) + 3(\cancel{e^{-3x}} + 2x + 3) = 6x + 11$$

$$2 + 6x + 9 = 6x + 11$$

$$6x + 11 = 6x + 11 \quad \checkmark$$

This ODE is more difficult to solve.

<u>Ex</u> $y'' - 3y' + 2y = 24e^{-2x}$

A solution is

$$y = 3e^{x} - 4e^{2x} + 2e^{-2x}.$$

$$y' = 3e^{x} - 8e^{2x} - 4e^{-2x}$$

$$y'' = 3e^{x} - 16e^{2x} + 8e^{-2x}$$

$$y'' - 3y' + 2y = (3e^{x} - 16e^{2x} + 8e^{-2x})$$

$$-3(3e^{x} - 8e^{2x} - 4e^{-2x})$$

$$+2(3e^{x} - 4e^{2x} + 2e^{-2x})$$

$$= \underbrace{(3-9+6)}_{=0}e^{x} + \underbrace{(-16+24-8)}_{=0}e^{2x}$$

$$+ (8+12+4)e^{-2x}$$

$$= 24e^{-2x} \checkmark$$

Def: The <u>order</u> of the ODE is the highest order derivative of y that appears

<u>Ex</u>: (1) $y' - 2x = 0$ order 1.

(2) $y' + 3y = 6x + 11$ order 1

(3) $y'' - 3y' + 2y = 24e^{-2x}$ order 2

(4) $x^2 \boxed{y'''} + 3xy'' + xy' - 3y = \sin x$

order 3

(5) $\frac{4}{x} \boxed{y^{(4)}} + \frac{12}{x^2} y = x^3 + 4x$

order 4

<u>Modeling</u>.

- Population growth.

t = time

$P(t)$ = population at time t

A reasonable model for the population growth of a population of bacteria/animals under ideal conditions

(unlimited environment, perfect nutrition, no predators no diseases, ...)

is that the rate of growth is proportional to the population size:

$$\boxed{\frac{dP}{dt}(t) = \underbrace{\boxed{k}}_{\text{constant}} P(t)}$$

<u>Note</u>:

<u>If $k>0$</u> then the population always grows. (Assuming $P(t)>0$.)

We can in fact solve it!

$$\boxed{P(t) = Ce^{kt}}$$

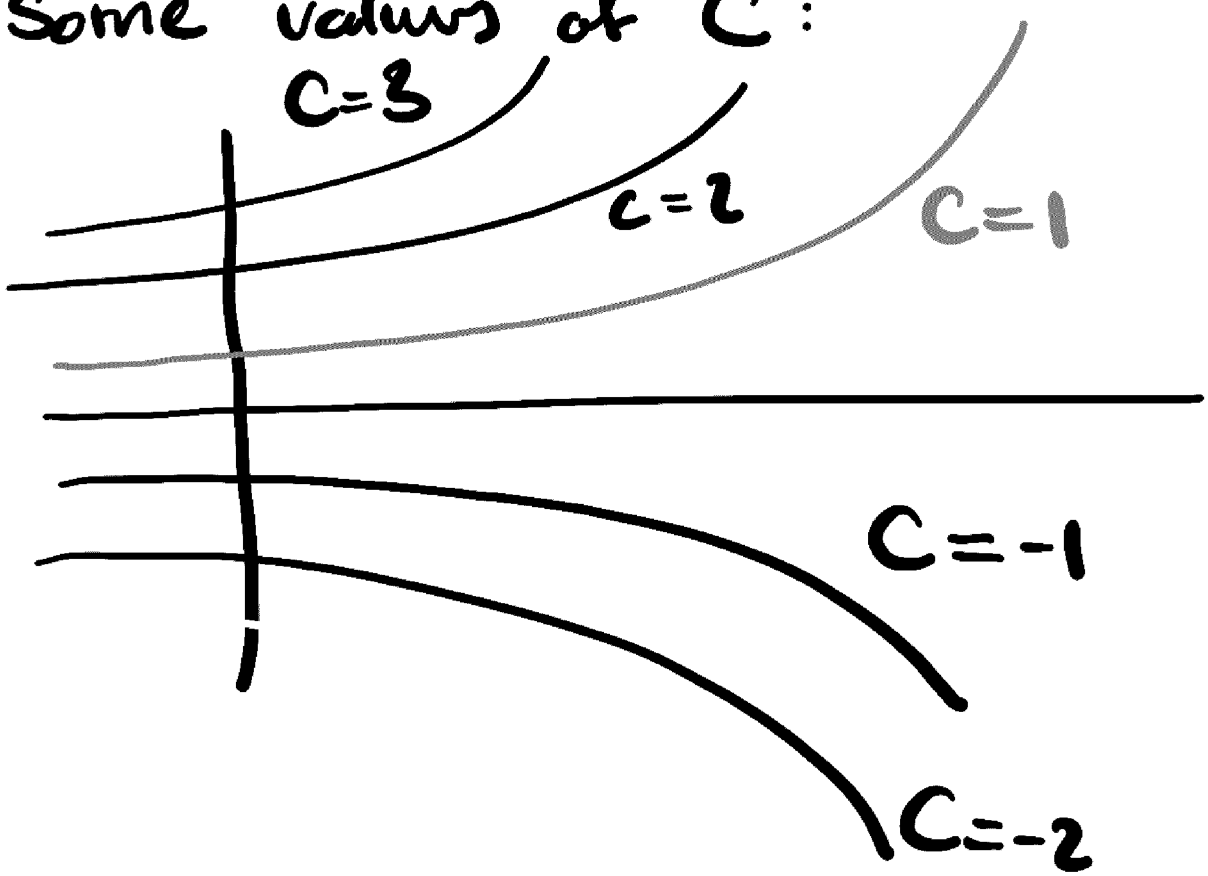
Can check that this indeed solves the ODE: $P'(t) = k(Ce^{kt})$
$= k\,P(t)$.

Any value of the const C gives a solution. Note

$$P(0) = Ce^{k\cdot 0} = C,$$ so

C = population at time 0

Graphs of some solutions for some values of C:



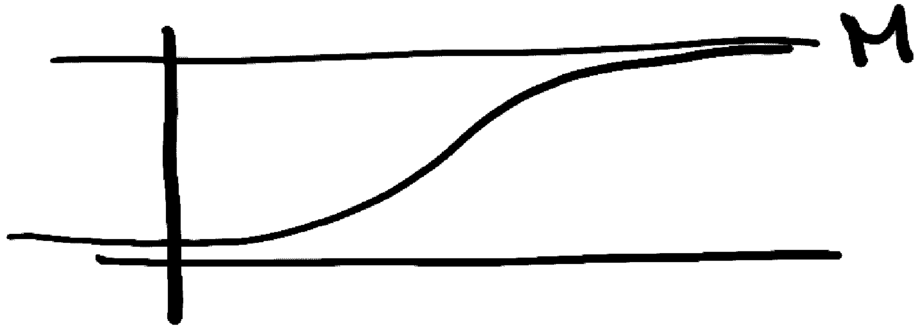
Of course we would not consider in our model because the initial population can not be negative.

<u>More realistic model</u>:

In less ideal situations, the amount of resources (e.g. food) are <u>limited</u>.

There's a "roof" or "carrying capacity" M such that

- $\frac{dP}{dt} \approx kP$ if P is small
- $\frac{dP}{dt} < 0$ if $P > M$



The logistic equation

$$\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)}$$

If P small compared to M then $1 - \frac{P}{M} \approx 1$ so $\frac{dP}{dt} \approx kP$

If $P > M$ then $1 - \frac{P}{M} < 0$ so assuming $k > 0$ we get

$\frac{dP}{dt} < 0.$

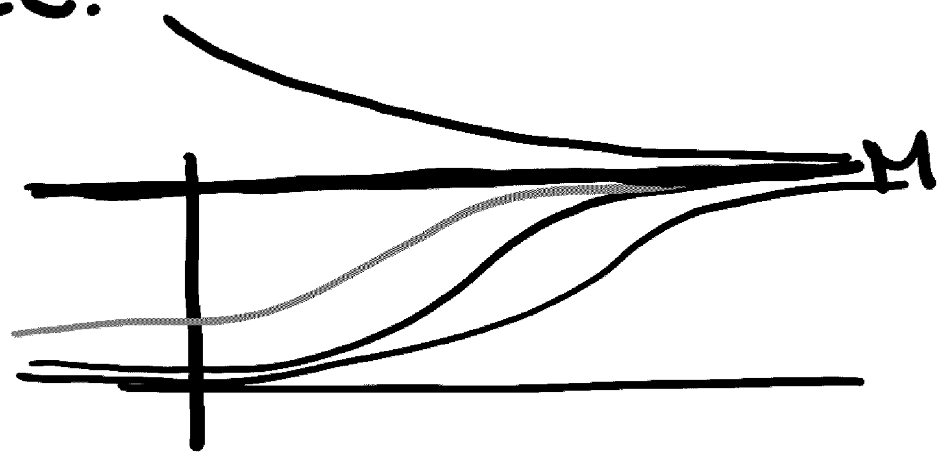
If $P \to M$ then $1 - \frac{P}{M} \to 0$

so $\frac{dP}{dt} \to 0$ meaning population levels off as the population reaches the carrying capacity.

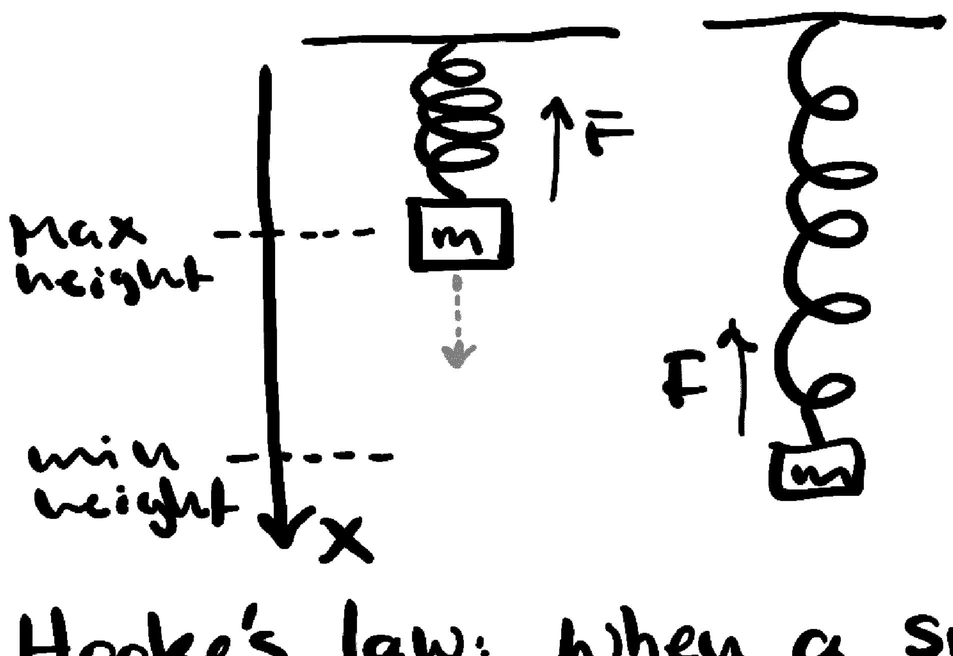
There are two special solutions:

$P(t) = 0$ and $P(t) = M.$

called equilibrium solutions.
These make sense: if population reaches 0 or M it stays at that size.



Motion of a spring:



Hooke's law: When a spring is stretched, the force it exerts on the weight is proportional to the height.

$$\boxed{\text{Force} = -kx}$$

$$F = ma = m\frac{d^2x}{dt^2} = -kx$$

$$\boxed{x''(t) = -\frac{k}{m}x}$$

In fact a solution is

$$\boxed{x(t) = \sin\left(\sqrt{\frac{k}{m}}\,t\right)}$$