MAT 127 Lec 14 Thu Oct 10

Absolute and conditional convergence (§8.4 in Stewart)

__Recall:__ $\sum_{n=1}^{\infty} a_n$ converges if

$$\lim_{N\to\infty} \underbrace{\sum_{n=1}^{N} a_n}_{\text{partial sum}} = \lim_{N\to\infty} S_N \text{ exists.}$$

__Def:__ $\sum_{n=1}^{\infty} a_n$ is called __absolutely convergent__ if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

__Note:__ If the terms a_n are positive then $|a_n| = a_n$ so convergence and absolute convergence is the same for those series

Ex: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, then $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$

p-series w/ $p=2>1$ so it's conv.

Therefore $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent.

Ex $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent by the alternating series test

$\left(\sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n = \frac{1}{n} \text{ and } a_n \text{ is decreasing } \& \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0 \right)$

However $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is div

So $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent, but not absolutely convergent.

Def: $\sum_{n=1}^{\infty} a_n$ is called conditionally convergent if it's conv but not absolutely convergent

Previously we have used the alt. series test to deal with alternating series, but sometimes we can use absolute convergence.

<u>Theorem</u>. If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, it is convergent.

<u>Note</u>: The reverse implication convergence $\Rightarrow$ absolute convergence is <u>false</u> as we saw above with the alternating harmonic series.

<u>Ex</u>: Determine whether $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges or not.

<u>Note</u> $-1 \leq \cos n \leq 1$ & series "looks like" $\sum_{n=1}^{\infty} \frac{1}{n^2}$ p-series w/ $p = 2 > 1$ which converges.

So we <u>would</u> like to use the Comparison test, but we can't since $\cos n$ is sometimes negative.

Instead $\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$

$|\cos n| \leq 1$ and now it's a positive series. Comparison test:

$$\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2} \leq \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{CONV}}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2} \text{ convergent}$$

means $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ — " — .

Recall ratio test, $\sum_{n=1}^{\infty} a_n$

$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$.

- $\rho > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ <u>div</u>

Now if $\rho < 1$, the ratio test actually tells us that $\sum a_n$ is <u>absolutely</u> <u>convergent</u>.

($\rho = 1$ is still inconclusive).

<u>Ex</u>: Is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ absolutely conv? conv?

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ p-series w/ $p = \frac{1}{2} < 1$

So this series diverges

$\rightsquigarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is <u>not</u> abs. conv.

Is it convergent? Yes by the alt. series test.

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n, \quad \boxed{a_n = \frac{1}{\sqrt{n}}}$$

decreasing: $\frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}}$ ✓

$\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$ ✓

$\Rightarrow$ Convergent by alternating series test.

<u>Ex</u>: Determine if $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$ is absolutely convergent.

$$\sum_{n=1}^{\infty} \left| \frac{(-2)^n}{n^2} \right| = \sum_{n=1}^{\infty} \left| \frac{(-1)^n 2^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{2^n}{n^2}$$

<u>div test</u>: $\lim_{n \to \infty} \frac{2^n}{n^2} = \infty$

$\Rightarrow \sum_{n=1}^{\infty} \frac{2^n}{n^2}$ is divergent, so $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$ is not absolutely conv.

Is it conv? No!

Div test is already conclusive

$\lim_{n\to\infty} \frac{(-1)^n 2^n}{n^2}$ does not exist, so

$\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$ is divergent.
