MAT 127 Lec 13 Tu Oct 8

<u>Recall</u>:

- Maclaurin series of a function $f(x)$

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}$$

- Binomial series

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \qquad |x|<1$$

$$\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}$$ "Binomial coefficient".

Other Maclaurin series we have seen:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad , \quad |x|<1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{all } x$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{all } x$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad |x|<1$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad |x|<1$$

will not have to memorize these for exams, but they should feel familiar.

<u>Ex</u>: $\lim_{x \to 0} \frac{\sin(x^2) - x^2}{x^6}$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\Rightarrow \sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots$$

$$\lim_{x\to 0} \frac{\left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots\right) - x^2}{x^6}$$

$$= \lim_{x\to 0} \frac{-\frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots}{x^6}$$

$$= \lim_{x\to 0} -\frac{1}{3!} + \frac{x^4}{5!} - \cdots = -\frac{1}{3!} = -\frac{1}{6}.$$

<u>Ex</u> Approximate $\int_0^{1/2} x e^{-x^2}\,dx.$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ so } e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

$$\int_0^{1/2} x e^{-x^2}\,dx = \int_0^{1/2} x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,dx$$

$$= \int_0^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+1}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^{1/2} x^{2n+1}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\frac{x^{2n+2}}{2n+2} \right]_0^{1/2}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{2n+2} \cdot \frac{1}{2^{2n+2}} = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+2)4^n}$$

$$= \frac{1}{4}\left(\frac{1}{1 \cdot 2 \cdot 1} + \frac{-1}{1 \cdot 4 \cdot 4} + \frac{1}{2 \cdot 6 \cdot 16} - \cdots \right)$$

$$= \frac{1}{4}\left(\frac{1}{2} - \frac{1}{16} + \frac{1}{192} - \cdots \right)$$

$$\approx \frac{1}{8} - \frac{1}{48} + \frac{1}{768}$$

$$= \frac{27}{256} \approx 0.1055$$

<u>Bonus exercise</u>: Can compute this integral exactly via u-substitution & compare to our approximated result.

(Exact value is $\frac{1}{2}(\sqrt[4]{e} - 1) \approx 0.1420$)

Ex Approximate $\int_0^1 \sqrt{1+x^4}\,dx$

Recall from earlier

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n, \text{ so}$$

$$\sqrt{1+x^4} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^{4n}.$$

$$\int_0^1 \sqrt{1+x^4}\,dx = \int_0^1 \sum_{n=0}^{\infty} \binom{1/2}{n} x^{4n}\,dx$$

$$= \sum_{n=0}^{\infty} \binom{1/2}{n} \int_0^1 x^{4n}\,dx = \sum_{n=0}^{\infty} \binom{1/2}{n} \left[\frac{x^{4n+1}}{4n+1}\right]_0^1.$$

$$= \sum_{n=0}^{\infty} \binom{1/2}{n} \frac{1}{4n+1} = \underbrace{\binom{1/2}{0}}_{=1} \cdot \frac{1}{1} + \underbrace{\binom{1/2}{1}}_{=1/2} \cdot \frac{1}{5}$$

$$+ \underbrace{\binom{1/2}{2}}_{=-1/8} \cdot \frac{1}{9} + \underbrace{\binom{1/2}{3}}_{=1/16} \cdot \frac{1}{13} + \cdots$$

$$= 1 + \frac{1}{10} - \frac{1}{72} + \frac{1}{208} + \cdots$$

$$\approx 1 + \frac{1}{10} - \frac{1}{72} + \frac{1}{208} = \frac{10211}{9360} \approx 1.0909$$