MAT 127 Lec 12 Thu Oct 3

Recall: • Taylor series of $f(x)$ centered at $x=a$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

• Maclaurin series of $f(x)$ is the Taylor series centered at $x=0$.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

• $\log(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \quad |x|<1$

• $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x$$

Ex: $f(x) = \cos x$. Since we know the Maclaurin series for $\sin x$, and $\frac{d}{dx} \sin x = \cos x$, we can differentiate the Maclaurin series of $\sin x$:

$$\frac{d}{dx} \sin x = \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$$

for all x.

<u>Ex</u> $f(x) = \arctan(x)$.

Note $f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$

$$= \sum_{n=0}^{\infty} (-x^2)^n \quad |-x^2| < 1$$

$$\Leftrightarrow |x| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Find f by integrating!

$$\int \frac{1}{1+x^2} dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan(x).$$

for $|x|<1$.

We know $\arctan(0)=0$ so plugging in $x=0$ gives

$$C+0=0.$$

$$\boxed{\arctan(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} \quad |x|<1}$$

<u>Ex</u> Find the Maclaurin series for $x\cos x$. We already know

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

so it suffices to multiply both sides by x.

$$x\cos x = x\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n)!}.$$

Binomial Series

Let k be some real number and try to find the Maclaurin series for $f(x) = (1+x)^k$.

Derivatives & values at $x=0$:

$$f(x) = (1+x)^k, \quad f(0) = 1$$

$$f'(x) = k(1+x)^{k-1}, \quad f'(0) = k$$

$$f''(x) = k(k-1)(1+x)^{k-2}, \quad f''(0) = k(k-1)$$

$$\vdots$$

$$f^{(N)}(x) = k\cdot(k-1)\cdots(k-N+1)(1+x)^{k-N}$$

$$f^{(N)}(0) = k\cdot(k-1)\cdots(k-N+1)$$

So the Maclaurin series for $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k\cdot(k-1)\cdot \dots \cdot(k-n+1)}{n!} x^n.$$

BINOMIAL SERIES

The coefficients are called binomial coefficients

$$\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}$$

$$\boxed{(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n, \quad |x|<1}$$

Ex: $f(x) = \sqrt{1+x} = (1+x)^{1/2}$

$\boxed{k=\frac{1}{2}}$ $\binom{1/2}{0} = 1$

$$\binom{1/2}{1} = \frac{1}{2}, \quad \binom{1/2}{2} = \frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)}{2} = -\frac{1}{8}$$

$$\binom{1/2}{3} = \frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!} = \frac{\left(-\frac{1}{4}\right)\cdot\left(-\frac{3}{2}\right)}{3!} = \frac{1}{16}$$

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots$$

Applications:

Ex: Compute $\lim_{x \to 0} \frac{2\cos x - 2 + x^2}{x^4}$.

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

gives

$$\lim_{x \to 0} \frac{2\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - 2 + x^2}{x^4}$$

$$= \lim_{x \to 0} \frac{\left(2 - x^2 + \frac{2x^4}{4!} - \frac{2x^6}{6!} + \cdots\right) - 2 + x^2}{x^4}$$

$$= \lim_{x \to 0} \frac{\frac{2x^4}{4!} - \frac{2x^6}{6!} + \cdots}{x^4}$$

$$= \lim_{x \to 0} \frac{2}{4!} - \frac{2x^2}{6!} + \frac{2x^4}{8!} - \cdots = \frac{2}{4!} = \frac{1}{12}$$

Ex: $\lim_{x\to 0} \dfrac{\arctan(x) - x}{x^3}$.

$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

$$\lim_{x\to 0} \frac{\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\right) - x}{x^3}$$

$$= \lim_{x\to 0} \frac{-\frac{x^3}{3} + \frac{x^5}{5} - \dots}{x^3} = \lim_{x\to 0} -\frac{1}{3} + \frac{x^2}{5} - \dots$$

$$\boxed{= -\frac{1}{3}}$$

Ex: Find $\int x^2 \sin(x^2)\,dx$ as an infinite series.

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

Therefore

$$\int x^2 \sin(x^2)\,dx = \int x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+4}}{(2n+1)!}\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \int x^{4n+4}\, dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \frac{x^{4n+5}}{4n+5}$$