Lagrangian Floer Homology and Fukaya Categories

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Let L be a Lagrangian submanifold in a symplectic manifold (M, ω). Suppose that φ_t is a Hamiltonian diffeomorphism generated by some Hamiltonian $H_t: M \to \mathbb{R}$.

Theorem: (Floer) Assume that the symplectic area of any topological disc in M with boundary on L vanishes. Assume moreover that L and $\varphi(L)$ intersect transversely. Then the number of intersection points of L and $\varphi(L)$ satisfies the bound

$$|\varphi(L) \pitchfork L| \geq \sum_i \dim H^i(L; \mathbb{Z}_2).$$

Floer's approach to answering this question was to associate a pair of Lagrangians L_0 and L_1 a chain complex

$$CF(L_0, L_1) = \text{generated by intersection points of } L_0 \text{ and } L_1$$

together with a differential $\partial: CF(L_0, L_1) \longrightarrow CF(L_0, L_1)$ with the properties:

(i) $\partial^2 = 0$ so that Floer cohomology $HF(L_0, L_1)$ is well-defined,

(ii) if L_1 and L_1' are Hamiltonian isotopic, then $HF(L_0, L_1) \cong HF(L_0, L_1')$, and

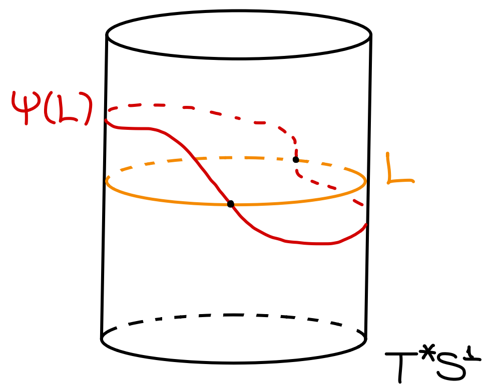
(iii) if L_1 is Hamiltonian isotopic to L_0, then $HF(L_0, L_1) \cong H^*(L_0)$.

Remark: Assuming Floer cohomology can be defined this way, Floer's theorem is trivial since

$$|\varphi(L) \pitchfork L| = \dim CF(\varphi(L), L) \geq \dim HF(\varphi(L), L) = \dim H^*(L)$$

Example: Consider the cylinder $T^*S^1 = \mathbb{R} \times S^1$. The assumption that $\varphi \in Ham(M)$ implies that if L is the zero section, then $\varphi(L)$ can be of the following form. It is then clear that $|\varphi(L) \pitchfork L| \geq 2$ which satisfies the above theorem since

$$\dim H^*(L) = \dim H^*(S^1) = \dim \mathbb{Z}^2 = 2.$$



Note that Floer's Theorem fails if

(i) φ is a symplectomorphism only, or

(ii) we remove the disk assumption.

Lagrangian Floer Cohomology

Let L_0 and L_1 be compact Lagrangians in M such that

(i) L_0 and L_1 intersect transversely, and

(ii) they are equipped with spin structures.

Definition of Floer Cohomology

Def$^{\underline{n}}$: The Novikov field over a base field $\mathbb{K}$ is

$$\Lambda := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \,\middle|\, a_i \in \mathbb{K},\ \lambda_i \in \mathbb{R},\ \lim_{i\to\infty} \lambda_i = \infty \right\}.$$

Def$^{\underline{n}}$: The energy of a map $u: \mathbb{R} \times [0,1] \longrightarrow M$ is defined to be

$$E(u) := \int_{\mathbb{R}\times[0,1]} u^* \omega = \iint \left|\frac{\partial u}{\partial s}\right|^2 ds\, dt$$

Equip M with an ω-compatible almost complex structure J. We can now define the relevant moduli space of J-holomorphic curves.

Def^n: Given a homotopy class $[u] \in \pi_2(M, L_0 \cup L_1)$, we denote

$$\hat{\mathcal{M}}(p,q;[u],J) := \left\{ u: \mathbb{R} \times [0,1] \longrightarrow M \;\middle|\; \begin{array}{l} \bar{\partial} u = 0,\ u(s,0) \in L_0 \text{ and } u(s,1) \in L_1\ \forall s \in \mathbb{R}, \\ \lim\limits_{s \to \infty} u(s,t) = p,\ \lim\limits_{s \to \infty} u(s,t) = q,\ E(u) < \infty \end{array} \right\}$$

There is an obvious $\mathbb{R}$-action on $\hat{\mathcal{M}}(p,q;[u],J)$ given by $a \cdot u(s,t) = u(s-a,t)$. The quotient of $\hat{\mathcal{M}}(p,q;[u],J)$ by this action will be denoted by $\mathcal{M}(p,q;[u],J)$.

Remark: The boundary value problem defining $\mathcal{M}(p,q;[u],J)$ is a Fredholm problem in the sense that the linearization $D_{\bar{\partial}_J,u}$ of $\bar{\partial}_J$ at a given solution u is a Fredholm operator, hence has an index $\operatorname{ind}([u])$.

Thm: The space of solutions $\hat{\mathcal{M}}(p,q;[u],J)$ is a smooth orientable manifold of dimension $\operatorname{ind}([u])$ if $D_{\bar{\partial}_J,u}$ is surjective at each point of $\hat{\mathcal{M}}(p,q;[u],J)$ and L_i is spin.

Def^n: The Floer chain complex is the chain complex

$$CF(L_0,L_1) := \bigoplus_{p \in \mathcal{X}(L_0,L_1)} \Lambda \cdot p$$

equipped with the Floer differential which is Λ-linear and given by

$$\partial p := \sum_{\substack{q \in \mathcal{X}(L_0, L_1) \\ [u] : \mathrm{ind}([u]) = 1}} \left(\# \mathcal{M}(p, q; [u], J) \right) \cdot T^{\omega([u])} \cdot q .$$

Remark: We make two observations.

(i) Gromov's Compactness Theorem ensures that, given any energy bound E_0, there are only finitely many homotopy classes $[u]$ with $\omega([u]) < E_0$ for which the moduli space $\mathcal{M}(p,q;[u],J)$ is nonempty. This is precisely why we use Novikov coefficients and weigh the counts of pseudo-holomorphic strips by symplectic area.

(ii) We consider homotopy classes of $\mathrm{ind}([u]) = 1$ because then

$$\begin{aligned} \dim \mathcal{M}(p,q;[u],J) &= \dim \widehat{\mathcal{M}}(p,q;[u],J) - 1 \\ &= \mathrm{ind}([u]) - 1 = 1 - 1 = 0 \end{aligned}$$

and $\# \mathcal{M}(p,q;[u],J)$ makes sense.

Remark:

(i) Formally, Lagrangian Floer Homology can be considered as an infinite-dimensional analogue of Morse homology for the action functional on the universal cover of the path space $\mathcal{P}(L_0, L_1)$, where

$$\mathcal{A}(\gamma, [\Gamma]) = -\int_\Gamma \omega$$

(ii) Grading on the chain complex is as follows. Consider the $LGr(n)$-bundle $LGr(TM) \longrightarrow M$. Note that $\pi_1(LGr(n)) \cong \mathbb{Z}$. Let $\widetilde{LGr}(TM)$ be the fiberwise universal bundle over M.

Fact: (1) The bundle $\widetilde{LGr}(TM)$ exists if $2c_1(M) = 0$.

(2) There is a canonical short path between any two Lagrangian subspaces in $LGr(n)$.

$$\begin{tikzcd} & \widetilde{LGr}(TM) \arrow[d, "\pi"] \\ L \arrow[r, "s_L"'] \arrow[ur, dashed, "\tilde{s}_L"] & LGr(TM) \end{tikzcd}$$

We have the diagram on the right. The Maslov class is the obstruction to the existence of the lift $L \longrightarrow \widetilde{LGr}(TM)$. Given $p \in \mathcal{X}(L_0, L_1)$, find a path γ between $\tilde{s}_{L_0}(p)$ and $\tilde{s}_{L_1}(p)$. If σ denotes the canonical short path from $s_{L_1}(p)$ to $s_{L_0}(p)$, the grading of p is $\deg(p) = [\sigma \circ \pi(\gamma)] \in \pi_1(LGr(n))$.

Product Operations

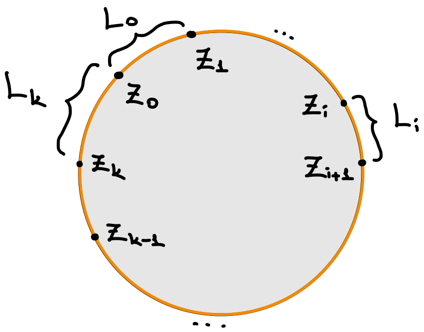
Let

$$\mathcal{M}_{0,k+1} = \{\text{ordered } (k+1)\text{-tuples of points on } S^1\} \big/ \mathrm{Aut}(D)$$

and observe that $\dim \mathcal{M}_{0,k+1} = k-2$.

Def^n: Given a homotopy class $[u] \in \pi_2(M, L_0 \cup \dots \cup L_k)$, we denote

$$\mathcal{M}(p_1,\dots,p_k,q;[u],J) := \left\{ \begin{array}{c} D \setminus \{z_0,\dots,z_k\} \\ \downarrow u \\ M \end{array} \;\middle|\; \right.$$



$$\left. ,\; E(u) < \infty \right\}$$

where we consider each strip up to the action of $\mathrm{Aut}(D^2)$ by reparametrization. Assuming transversality and taking into account the movement of z_i for $i \le k+1$ on $S^1 = \partial D$, the expected dimension of this moduli space is

$$\begin{aligned} \dim \mathcal{M}(p_1,\dots,p_k,q;[u],J) &= \mathrm{ind}([u]) + (k+1) - \dim \mathrm{Aut}(D^2) \\ &= \mathrm{ind}([u]) + k - 2. \end{aligned}$$

Def^{n}: Let $L_0, \dots, L_k$ be Lagrangian submanifolds with spin structures. The operation

$$\mu^k : CF(L_{k-1}, L_k) \otimes \cdots \otimes CF(L_0, L_1) \to CF(L_0, L_k)$$

is the Λ-linear map

$$\mu^k(p_k, \dots, p_1) = \sum_{\substack{q \in \mathcal{X}(L_0, L_k) \\ [u] : \operatorname{ind}([u]) = 2-k}} \Big(\# \mathcal{M}(p_1, \dots, p_k, q ; [u], J) \Big) \cdot T^{\omega([u])} \cdot q$$

Remark: In particular, μ^1 is the Floer differential $\partial : CF(L_0, L_1) \to CF(L_0, L_1)$.

The most important property of the higher product operations μ^k is the following.

Theorem: (A_∞-relations) If $[\omega] \cdot \pi_2(M, L_i) = 0$ for all i, then the operations μ^k satisfy the A_∞-relations

$$\sum_{\ell=1}^{k} \sum_{j=0}^{k-\ell} (-1)^{*} \mu^{k-\ell+1}\big(p_k, \dots, p_{j+\ell+1}, \mu^{\ell}(p_{j+\ell}, \dots, p_{j+1}), p_j, \dots, p_1\big) = 0$$

where $* = j + \deg(p_1) + \cdots + \deg(p_j)$.

Let us comment further on these relations by considering the summands for distinct values of ℓ and j. When $\ell=1$, the product outside is always μ^k and inside is always μ^1, therefore

$$\begin{array}{ccc} j=0 & \Longrightarrow & \mu^k(p_k,\dots,p_2,\mu^1(p_1)) \\ \vdots & \vdots & \vdots \\ j=k-1 & \Longrightarrow & \mu^k(\mu^1(p_k),p_{k-1},\dots,p_1). \end{array}$$

When $\ell=2$, the product outside is always μ^{k-1} and inside is always μ^2, therefore

$$\begin{array}{ccc} j=0 & \Longrightarrow & \mu^{k-1}(p_k,\dots,p_3,\mu^2(p_2,p_1)) \\ \vdots & \vdots & \vdots \\ j=k-2 & \Longrightarrow & \mu^{k-1}(\mu^2(p_k,p_{k-1}),p_{k-1},\dots,p_1) \end{array}$$

Continuing in this manner, when $\ell=k$ (and j can only be 0), the product outside is μ^1 and inside is μ^k, therefore

$$j=0 \quad\Longrightarrow\quad \mu^1(\mu^k(p_k,\dots,p_1)) = \partial(\mu^k(p_k,\dots,p_1))$$

Hence, one can see the A_∞-relations as a certain compatibility condition of the higher products μ^k.

Corollary: (Floer product) There is a product

$$\cdot : CF(L_2, L_1) \otimes CF(L_1, L_0) \longrightarrow CF(L_2, L_0)$$

satisfying the Leibniz-type formula

$$\partial(p_2 \cdot p_1) = \mp (\partial p_2) \cdot p_1 + p_2 \cdot (\partial p_1).$$

In particular, this product induces a well-defined product

$$HF(L_2, L_1) \otimes HF(L_1, L_0) \longrightarrow HF(L_2, L_0)$$

which is independent of the chosen almost complex structure and Hamiltonian perturbations and is associative.

Proof: Letting $p_2 \cdot p_1 := \mu^2(p_2, p_1)$, the A_∞-relations imply that

$$\partial(p_2 \cdot p_1) = \mu^1(\mu^2(p_2, p_1)) = \mp \mu^2(\mu^1(p_2), p_1) \mp \mu^2(p_2, \mu^1(p_1))$$

$$= \mp (\partial p_2) \cdot p_1 \mp p_2 \cdot (\partial p_1)$$

as desired. ∎

Wrapped Fukaya Category

Def^{n}: The Liouville vector field on an exact symplectic manifold $(M, \omega = d\theta)$ is the unique vector field Z satisfying $\iota_Z \omega = \theta$, or equivalently by Cartan's formula, $\mathcal{L}_Z \omega = \omega$.

Def^{n}: A Liouville manifold is an exact symplectic manifold $(M, \omega = d\theta)$ such that the Liouville vector field Z is complete and outward pointing at infinity.

↳ More precisely, we require that there is a compact domain M^{in} with boundary ∂M on which $\alpha = \theta|_{\partial M}$ is a contact form. Moreover, Z is positively transverse to ∂M and has no zeros outside of M^{in}.

⇓

Then, the flow of Z can be used to identify $M \setminus M^{in}$ with the symplectization $(1, \infty) \times \partial M$ equipped with the symplectic form $\omega = d(r\alpha)$ and Liouville vector field $Z = r\,\partial/\partial r$.

Def$^{\underline{n}}$: An exact Lagrangian in $(M, d\theta)$ is a Lagrangian L such that there is a function $f: L \to \mathbb{R}$ with the property $\theta|_L = df$.

We restrict our attention to exact Lagrangian submanifolds L in M which are canonical at infinity, i.e. if L is noncompact, then at infinity, it must coincide with the cone $(1, \infty) \times \partial L$ over some Legendrian submanifold ∂L of ∂M.

Def$^{\underline{n}}$: An A_∞-category is a category C such that

(i) for all objects $X, Y \in Ob(C)$ the morphisms $Hom_C(X,Y)$ is a finite dimensional chain complex of $\mathbb{Z}$-graded modules,

(ii) for all objects $X_0, \dots, X_n \in Ob(C)$, there is a family of linear composition maps (higher products)

$$m_n: Hom_C(X_0, X_1) \otimes \cdots \otimes Hom_C(X_{n-1}, X_n) \longrightarrow Hom(X_0, X_n)$$

(iii) m_1 is the differential on the chain complex $Hom_C(X,Y)$, and

(iv) m_n satisfy the A_∞-relations.

Def^{n}: Given two Lagrangians L_0, L_1, the wrapped Floer complex, denoted by $CW(L_0, L_1; H)$, is generated by points of $\Phi_H^1(L_0) \pitchfork L_1$ over $\mathbb{K}$. The differential counts solutions to Floer's equation, as before.

Remark:

(i) We only consider Hamiltonians $H: M \to \mathbb{R}$ which, outside a compact set, satisfy $H = r^2$ where $r \in (1, \infty)$ is the radial coordinate.

(ii) It turns out that the naturally defined product map would take values in $CW(L_0, L_2; 2H)$. There is a rescaling trick solving this issue.

(iii) Using the rescaling trick, the higher products can also be defined

$$\mu^k : CW(L_{k-1}, L_k, H) \otimes \cdots \otimes CW(L_0, L_1; H) \longrightarrow CW(L_0, L_k; H)$$

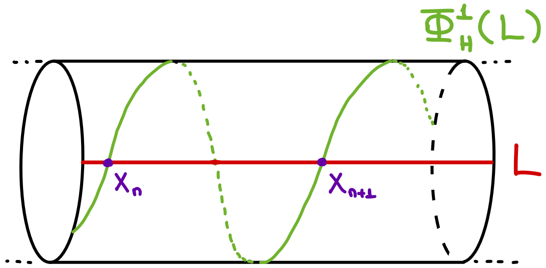
which makes the wrapped Fukaya category, denoted $\mathcal{W}(M)$, an A_∞-category, whose objects are exact Lagrangians that are canonical at infinity and $\mathrm{Hom}_{\mathcal{W}(M)}(L_0, L_1) := CW(L_0, L_1)$.

Example: Wrapped Floer Complex in $\mathbb{R}\times S^1$

Let $M = T^*S^1 = \mathbb{R}\times S^1$ be equipped with the standard Liouville form $r\,d\theta$ and the wrapping Hamiltonian $H = r^2$. Consider the exact Lagrangian $L = \mathbb{R}\times\{pt\}$.

We can label the intersection points by integers:

$$\mathcal{X}(L,L) = \{x_i : i\in\mathbb{Z}\}$$



Recall that the differential counts rigid pseudoholomorphic strips with boundary on L and $\Phi^1_H(L)$. It is clear from the diagram that no such strip exists. Hence $\partial = 0$ and

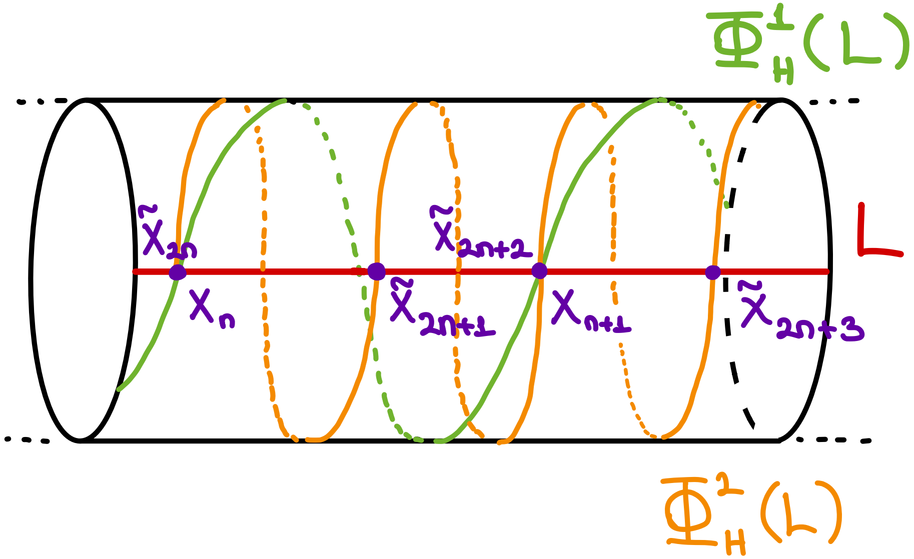
$$HW(L,L) = CW(L,L) = \operatorname{span}\{x_i : i\in\mathbb{Z}\}$$

Remark: Since L is invariant under the Liouville flow, the rescaling trick from before simply amounts to identifying

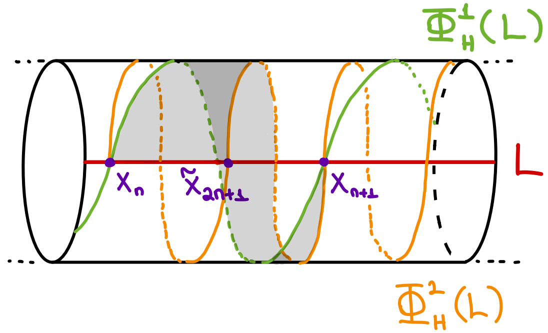
$$\mathcal{X}(L,L;2H) = \Phi^2_H(L)\cap L \qquad \& \qquad \mathcal{X}(L,L;H) = \Phi^1_H(L)\cap L$$

via the radial rescaling $r\mapsto 2r$. In other words, the intersection point of $\Phi^2_H(L)$ and L lying between x_n and x_{n+1} is $\tilde{x}_{2n+1}$ and is identified with x_{2n+1};

and the intersection point of $\Phi_H^2(L)$ and L lying at x_n is $\tilde{x}_{2n}$ and is identified with x_{2n}.



After this identification, we see that $x_n \cdot x_{n+1} = \tilde{x}_{2n+1} = x_{2n+1}$. This further generalizes to $x_i \cdot x_j = x_{i+j}$.



Theorem: (Wrapped Floer Complex of T^*S^1) There is an A_∞-algebra isomorphism $CW(L,L) \cong \mathbb{K}[x, x^{-1}]$.

Cotangent Bundles

The above theorem is a simple case of a more general result.

Theorem. (Abouzaid) Let N be a compact spin manifold. Let $L = T_q^*N$ be the cotangent fiber at some point $q \in N$. Then there is a quasi-isomorphism

$$CW^*(L,L) \simeq C_{-*}(\Omega_q N)$$

of A_∞-algebras, where the right hand side is the chains on the based loop space.

Remember the conjecture by Arnold that Shuhao told us about in the first two weeks.

Conjecture: (Arnold) Let N be a compact closed manifold. Then any compact closed exact Lagrangian submanifold of T^*N is Hamiltonian isotopic to the zero section.

This conjecture remains out of reach of current technology, however we have:

Theorem: (Fukaya-Seidel-Smith, Nadler-Zaslow, Abouzaid, Kragh) Let L be a compact connected exact Lagrangian submanifold of T^*N. Then, as an object of $W(T^*N)$, L is quasi-isomorphic to the zero section and the restriction $\pi|_L : L \to N$ is a homotopy equivalence.