# An Introduction to Symplectic Cohomology

## 1. Morse homology

Symplectic cohomology and other variants of Floer homology can be thought of as an infinite-dimensional analog of finite-dimensional Morse homology. To motivate our later discussion, we begin with a brief overview of Morse homology. The material in this section can be found in [Hut02].

Let $M$ be a closed smooth manifold equipped with two pieces of data: a smooth function $f : M \to \mathbb{R}$ and a Riemannian metric $g$. We want the function $f$ to be a *Morse function*, i.e., all of its critical points are nondegenerate. In particular, this implies the critical points of $f$ are isolated, hence there are finitely many of them.

The data $(f, g)$ determines a vector field $-\nabla f$ on $M$, the negative gradient of $f$. The maximal integral curves of $-\nabla f$ are called *gradient flow lines*, or just *flow lines*. One can show that for any flow line $\gamma : \mathbb{R} \to M$, the limits $\lim_{s\to\pm\infty} \gamma(s)$ exist and are equal to critical points of $f$.

For $p, q \in \mathrm{Crit}(f)$, let $\mathcal{M}(p, q)$ denote the *moduli space of flow lines* from $p$ to $q$, modulo reparametrization by $\mathbb{R}$. One can show that, for generically chosen $g$, that this moduli space is a smooth manifold of dimension

$$\dim \mathcal{M}(p, q) = \mathrm{ind}(p) - \mathrm{ind}(q) - 1.$$

For instance, this holds if the pair $(f, g)$ is *Morse–Smale*. We will assume this is the case for the remainder of this section.

The moduli space $\mathcal{M}(p, q)$ admits a natural compactification $\overline{\mathcal{M}}(p, q)$ which (again, for generic choice of $g$) is a smooth manifold with corners whose codimension $k$ stratum is the space of *broken flow lines*

$$\overline{\mathcal{M}}(p, q)_k = \bigcup_{r_1,\dots,r_k} \mathcal{M}(p, r_1) \times \mathcal{M}(r_1, r_2) \times \cdots \times \mathcal{M}(r_{k-1}, r_k) \times \mathcal{M}(r_k, q).$$

As a corollary of this fact, we have:

**Corollary 1.1.** If $\mathrm{ind}\, p = \mathrm{ind}\, q + 1$, then $\mathcal{M}(p, q)$ is finite.

*Proof.* The boundary strata of $\overline{\mathcal{M}}(p, q)$ are manifolds of dimension $< 0$, so they are empty. So $\overline{\mathcal{M}}(p, q)$ is a compact 0-dimensional manifold, hence is finite. □

We can now define the Morse complex $(MC_\bullet(f, g), \partial)$. Let $MC_\bullet(f, g)$ be the graded vector space

$$MC_\bullet(f, g) = \bigoplus_{p\in\mathrm{Crit}(f)} \mathbb{Z}_2 \cdot p,$$

where the degree of a critical point $p$ is its index. The differential $\partial : MC_i(f,g) \to MC_{i-1}(f,g)$ is given by counting flow lines, i.e.,

$$\partial(p) = \sum_{\substack{q \in \mathrm{Crit}(f) \\ \mathrm{ind}(q)=i-1}} \#\mathcal{M}(p,q) \cdot q.$$

**Lemma 1.2.** $\partial^2 = 0$.

*Proof.* Let $p$ be a critical point of index $i$. For a critical point $q$ of index $i-2$, the coefficient of $q$ in $\partial^2 p$ is the sum

$$\sum_{\substack{r \in \mathrm{Crit}(f) \\ \mathrm{ind}\, r=i-1}} \#\mathcal{M}(p,r)\#\mathcal{M}(r,q) = \#\overline{\mathcal{M}}(p,q)_1 = \#\overline{\mathcal{M}}(p,q) = 0.$$

The second and third equalities follow from the fact $\overline{\mathcal{M}}(p,q)$ is a compact 1-manifold with boundary. □

The resulting homology $MH_\bullet(f,g)$ is called the *Morse homology* of $(f,g)$.

**Theorem 1.3.** $MH_\bullet(f,g) \cong H_\bullet(M;\mathbb{Z}_2)$. In particular, the Morse homology of $M$ is independent of the data $(f,g)$.

**Remark 1.4.** It is possible to define Morse theory with $\mathbb{Z}$ coefficients instead of $\mathbb{Z}_2$ coefficients. This involves orienting the moduli spaces $\mathcal{M}(p,q)$. However, we will mostly ignore issues of orientation.

## 2. Hamiltonian Floer homology

Symplectic cohomology can be thought of as an extension of Hamiltonian Floer homology to open symplectic manifolds (or more precisely, to Liouville domains). Because the construction of symplectic cohomology involves many of the components from the construction of Hamiltonian Floer homology, we will start by discussing the latter (which I believe is a bit more motivated). The material in this section can be found in the book [AD14].

Let $(M,\omega)$ be a closed symplectic manifold. Again, we require two pieces of data: a Hamiltonian function $H : M \to \mathbb{R}$ and an $\omega$-compatible almost complex structure $J$ on $M$. Note that $J$ induces a natural Riemannian metric $g$ on $M$ given by $g = \omega(-, J-)$. We will also make the *very* strong assumption in this section that $M$ is aspherical, i.e., $\pi_2(M) = 0$. This will greatly simplify the definition of the action functional and the grading of the Hamiltonian Floer complex.

## 2.1. The symplectic action functional

Hamiltonian Floer homology can be thought of as the Morse homology of the *symplectic action functional* $\mathcal{A}_H : \mathcal{L}M \to \mathbb{R}$, where $\mathcal{L}M$ is the loop space of $M$. If $\omega = d\lambda$ were exact, then we can define $\mathcal{A}_H$ in the way it is defined in classical mechanics:

$$\mathcal{A}_H(x) = \int_{S^1} (-x^*\lambda + H_t(x(t))\,dt).$$

However, because we have assumed $M$ is closed, $\omega$ cannot be exact. Instead, we define $\mathcal{A}_H$ on the space $\mathcal{L}_0M$ of contractible loops as follows. For $x \in \mathcal{L}_0M$, we choose an extension $u : D^2 \to M$ of $x$ and set

$$\mathcal{A}_H(x) = -\int_{D^2} u^*\omega + \int_{S^1} H_t(x(t))\,dt.$$

Using the asphericality hypothesis and Stokes' theorem, one can check that this definition of $\mathcal{A}_H$ is well-defined.

We want to define the Hamiltonian Floer complex analogously to how we defined the Morse complex. Namely, the chain complex will be generated by the critical points of $\mathcal{A}_H$, and the differential will be defined by counting gradient flow lines between these critical points. Thus, we start by determining the critical points and flow lines of $\mathcal{A}_H$.

**Lemma 2.1.** The critical points of $\mathcal{A}_H$ are precisely the 1-periodic orbits of $H$ which are contractible.

*Proof.* Let $x$ be a critical point of $\mathcal{A}_H$. Consider a family of loops $x_s$ with $x_0 = x$, and let $\xi$ be the vector field along $x$ given by $\xi = \partial_s x_s|_{s=0}$. Let $u : D^2 \to M$ be any extension of $x$ to $D^2$, and choose a family $u_s : D^2 \to M$ such that $u_0 = u$ and $u_s|_{S^1} = x_s$. Note that $\xi$ can be extended to the vector field $\tilde{\xi}$ along $u$ given by $\tilde{\xi} = \partial_s u|_{s=0}$. Then we have

$$\begin{aligned} 0 = \left.\frac{d}{ds}\right|_{s=0} \mathcal{A}_H(x_s) &= -\int_{D^2} u^*(\mathcal{L}_{\tilde{\xi}}\omega) + \int_{S^1} dH_t(\xi)\,dt \\ &= \int_{S^1} (-x^*(\iota_{\tilde{\xi}}\omega) + dH_t(\xi(t))\,dt) = \int_0^1 (\omega(\dot{x}(t), \xi(t)) + dH_t(\xi(t)))\,dt. \end{aligned}$$

Since $\xi$ can be chosen arbitrarily, we see that $dH_t = \iota_{x'}\omega$, which proves $x$ is an orbit of $H$. ◻

To define the gradient of $\mathcal{A}_H$, we need a metric on $\mathcal{L}_0M$. The Riemannian metric $g$ induces a natural such metric

$$\langle x, y \rangle_g = \int_{S^1} g(x(t), y(t))\,dt.$$

**Lemma 2.2.** The gradient flow lines of $\mathcal{A}_H$ are precisely the *Floer trajectories*, i.e., the solutions $u : \mathbb{R} \times S^1 \to M$ of Floer's equation

$$\partial_s u + J(u)(\partial_t u - X_{H_t}(u)) = 0.$$

*Proof.* We computed above that

$$d_x \mathcal{A}_H(\xi) = \int_{S^1} \omega(\dot{x} - X_{H_t}, \xi)\, dt = \int_{S^1} g(\xi, J(x(t))(\dot{x} - X_{H_t}))\, dt,$$

hence

$$\nabla \mathcal{A}_H(x) = J(x(t))(\dot{x} - X_{H_t}).$$

Note that paths in $\mathcal{L}M$ are the same as maps $u : \mathbb{R} \times S^1 \to M$. Then $u$ is a gradient flow line iff

$$\partial_s u = -\nabla \mathcal{A}_H(u(s,-)) = -J(u)(\partial_t u - X_{H_t}). \qquad \square$$

## 2.2. Nondegeneracy of orbits

In analogy with the Morse condition in Morse homology, we will require that the orbits of $H$ are nondegenerate in the following sense. Let $\varphi^t$ denote the flow of the Hamiltonian vector field $X_H$. We say an orbit $x$ is *nondegenerate* if the linearized return map

$$d_{x(0)}\varphi^1 : T_{x(0)}M \to T_{x(1)}M$$

does not have 1 as an eigenvalue. One can show that all the orbits of $H$ will be nondegenerate for generic choices of $H$. For the remainder of this section, we will assume that this condition holds.

## 2.3. Energy

For a Floer trajectory $u : \mathbb{R} \times S^1 \to M$, its *energy* is the quantity

$$E(u) = \int_{\mathbb{R}\times S^1} |\partial_s u|^2\, ds\, dt.$$

We will almost always assume that our Floer trajectories have finite energy. The following lemma shows that trajectories with finite energy behave like gradient flow lines from Morse homology, in the sense that they are asymptotic to periodic orbits.

**Lemma 2.3.** Let $u : \mathbb{R} \times S^1 \to M$ be a contractible Floer trajectory with finite energy. Then there exists periodic orbits $x$ and $y$ of $H$ such that

$$\lim_{s\to-\infty} u(s,-) = x, \quad \lim_{s\to+\infty} u(s,-) = y.$$

### 2.4. Grading

To associate an integral index to each contractible 1-periodic orbit $x$ of $H$, we need to choose a canonical trivialization of the symplectic vector bundle $x^*TM$. Let us recall why this may be an issue.

**Lemma 2.4.** The homotopy classes of trivializations of a symplectic vector bundle over $S^1$ are in bijection with $\mathbb{Z}$.

*Proof.* Consider the trivial bundle $S^1 \times \mathbb{R}^{2n}$. A trivialization of $S^1 \times \mathbb{R}^{2n}$ is the same as a loop $S^1 \to \mathrm{Sp}(2n)$. Then the claim follows from $\pi_1(\mathrm{Sp}(2n)) = \pi_1(\mathrm{U}(n)) = \mathbb{Z}$. □

To choose a trivialization of $x^*TM$, we rely on the asphericality hypothesis. Let $u : D^2 \to M$ be an extension of $x$ to $D^2$. Because $D^2$ is contractible, there is a single homotopy class of trivializations of $u^*TM$. This extends to a trivialization of $x^*TM$. By asphericality, one can show that the homotopy class of this trivialization does not depend on the choice of $u$.

With a trivialization $x^*TM$ fixed, we can associate to $x$ a path $\Psi : [0, 1] \to \mathrm{Sp}(2n)$ of symplectic matrices, where $\Psi_t$ is given by the composition

$$\mathbb{R}^{2n} \xrightarrow{\cong} T_{x(0)}M \xrightarrow{d_{x(0)}\varphi^t} T_{x(t)}M \xrightarrow{\cong} \mathbb{R}^{2n},$$

where $\varphi^t$ is the flow of the Hamiltonian vector field $X_H$. Because $x$ is nondegenerate, the matrix $\Psi_1$ does not have 1 as an eigenvalue. For such paths of symplectic matrices, one can associate an integer called the *Conley–Zehnder index*, denoted $\mathrm{CZ}(\Psi)$. The construction of the Conley–Zehnder index is quite involved, so we will omit it.

We define the *degree* of the Hamiltonian orbit $x$ to be

$$\deg(x) = \mathrm{CZ}(\Psi).$$

This notion of degree will allow us to define a grading on the Hamiltonian Floer complex.

### 2.5. Moduli spaces of trajectories

Given contractible 1-periodic orbits $x$ and $y$ of $H$, we let $\mathcal{M}(x, y)$ denote the *moduli space of Floer trajectories* from $x$ to $y$, modulo reparametrization by $\mathbb{R}$. As in the case of Morse homology, we require results concerning the transversality and compactness of $\mathcal{M}(x, y)$.

**Theorem 2.5.** For generic choices of almost complex structure $J$, the moduli space $\mathcal{M}(x, y)$ is a smooth manifold of dimension $\deg(x) - \deg(y) + 1$.

**Theorem 2.6.** There is a natural compactification $\overline{\mathcal{M}}(x, y)$ of $\mathcal{M}(x, y)$, called the *Gromov–Floer compactification*, given by the union

$$\overline{\mathcal{M}}(x, y) = \bigcup_{z_1,\dots,z_r} \mathcal{M}(x, z_1) \times \mathcal{M}(z_1, z_2) \times \cdots \times \mathcal{M}(z_{r-1}, z_r) \times \mathcal{M}(z_r, y).$$

The analog of Corollary 1.1 holds.

**Corollary 2.7.** For generic $J$ and orbits $x, y$ satisfying $\deg x = \deg y + 1$, the moduli space $\mathcal{M}(x, y)$ is finite.

One difficulty of Hamiltonian Floer homology is that $\overline{\mathcal{M}}(x, y)$ is not obviously a manifold with corners for generic $J$. It certainly may be the case, but proving this would require a sophisticated gluing argument. As a special case, we have the following result which we will need to prove $\partial^2 = 0$.

**Theorem 2.8.** For generic $J$ and orbits $x, y$ satisfying $\deg x = \deg y + 2$, the moduli space $\overline{\mathcal{M}}(x, y)$ is a 1-dimensional manifold with boundary

$$\partial\overline{\mathcal{M}}(x, y) = \bigcup_{\deg z = \deg y + 1} \mathcal{M}(x, z) \times \mathcal{M}(z, y).$$

### 2.6. The Hamiltonian Floer complex

We are now able to define the Hamiltonian Floer complex $FC_\bullet(H, J)$. Let $\mathcal{O}(H)$ be the set of contractible 1-periodic orbits of $H$. Let $FC_\bullet(H, J)$ be the graded vector space

$$FC_\bullet(H, J) = \bigoplus_{x \in \mathcal{O}(H)} \mathbb{Z}_2 \cdot x,$$

where the degree of $x \in \mathcal{O}(H)$ is as defined above using the Conley–Zehnder index. The differential $\partial$ is given by counting Floer trajectories, i.e.,

$$\partial(x) = \sum_{\substack{y \in \mathcal{O}(H) \\ \operatorname{ind}(y) = \operatorname{ind}(x) - 1}} \#\mathcal{M}(x, y) \cdot y.$$

The proof that $\partial^2 = 0$ is identical to the proof in Morse homology once one has Theorem 2.8. The resulting homology $FH_\bullet(H, J)$ is called the *Hamiltonian Floer homology* of $(H, J)$.

### 2.7. Invariance

In this subsection, we outline the proof that $FH_\bullet(H, J)$ is independent of the choice of the pair $(H, J)$. For two pairs $(H^-, J^-)$ and $(H^+, J^+)$ for which Floer homology is defined, one can define a *continuation map* $\Phi : FH_\bullet(H^-, J^-) \to FH_\bullet(H^+, J^+)$ between Floer homology groups. These continuation maps will satisfy the following properties:

(i) If $(H^-, J^-) = (H^+, J^+)$, then $\Phi$ is the identity map.

(ii) Given pairs $(H^0, J^0), (H^1, J^1), (H^2, J^2)$ and the respective continuation maps $\Phi_{01}, \Phi_{02}, \Phi_{12}$, we have

$$\Phi_{12} \circ \Phi_{01} = \Phi_{02}.$$

It follows that $\Phi$ is an isomorphism for all pairs $(H^-, J^-)$ and $(H^+, J^+)$, thus proving invariance.

The continuation map $\Phi : FH_\bullet(H^-, J^-) \to FH_\bullet(H^+, J^+)$ is constructed in the following way. Let $(H^s, J^s)$ for $s \in \mathbb{R}$ be a homotopy between $(H^-, J^-)$ such that $(H^s, J^s) = (H^-, J^-)$ for $s \ll 0$ and $(H^s, J^s) = (H^+, J^+)$ for $s \gg 0$. We consider a modified version of Floer's equation

$$\partial_s u + J^s(u)(\partial_t u + X_{H_t^s}) = 0$$

given by interpolating between Floer's equation for the pairs $(H^-, J^-)$ and $(H^+, J^+)$. Given orbits $x^- \in \mathcal{O}(H^-)$ and $x^+ \in \mathcal{O}(H^+)$ of the same degree, let $\mathcal{K}(x^-, x^+)$ denote the moduli space of solutions $u$ to the above equation which converge to the orbits $x^-$ and $x^+$. Note that we are not quotienting by $\mathbb{R}$ in the definition of $\mathcal{K}(x^-, x^+)$ since solutions to the above equation are no longer translation invariant. One can show that, for a generic homotopy $(H^s, J^s)$, the moduli space $\mathcal{K}(x^-, x^+)$ is a compact 0-dimensional manifold. Thus, we can define $\Phi$ by the formula

$$\Phi(x^-) = \sum_{\substack{x^+ \in \mathcal{O}(H^+) \\ \deg(x^-) = \deg(x^+)}} \#\mathcal{K}(x^-, x^+) \cdot x^+.$$

One can show that $\Phi$ is a chain map. Moreover, if $\Phi'$ is the chain map induced by a different homotopy between $(H^-, J^-)$ and $(H^+, J^+)$, then one can show that the chain maps $\Phi$ and $\Phi'$ are chain homotopic. Thus, there is a well-defined continuation map $\Phi : FH_\bullet(H^-, J^-) \to FH_\bullet(H^+, J^+)$ on homology, as desired.

### 2.8. Relation to Morse homology

One can prove that if $H$ is a $C^2$-small Morse function on $M$, that $J$ can be chosen so that the pair $(H, g)$ is Morse–Smale and $FC_\bullet(H, J)$ is equal to the Morse complex $MC_\bullet(H, g)$. Thus,

$$FH_\bullet(M) = MH_\bullet(M) = H_\bullet(M; \mathbb{Z}_2).$$

A well-known corollary of this fact is Arnold's conjecture.

## 3. Symplectic cohomology

The following material can be found in [Abo14], [Oan04], and [Rit13].

### 3.1. Liouville domains

Let $(M, \omega = d\lambda)$ be a compact exact symplectic manifold, possibly with boundary. A vector field $V$ on $M$ is called a *Liouville vector field* if $\mathcal{L}_V \omega = 0$. Note that the 1-form $\lambda$ determines a canonical Liouville vector field $V$ defined by

$$\omega(V, -) = \lambda.$$

The manifold $M$ is called a *Liouville domain* if $V$ is strictly outward pointing on the boundary $\partial M$.

**Lemma 3.1.** If $(M, \omega = d\lambda)$ is a Liouville domain, then the form $\alpha = \lambda|_{\partial M}$ is a contact form on $\partial M$.

*Proof.* Recall that, by definition, $\alpha$ is a contact form iff $d\alpha|_{\ker \alpha} = \omega|_{\ker \alpha}$ is nondegenerate. Fix $p \in \partial M$. Since $T_p\partial M$ is odd-dimensional, there exists a vector $R \in T_p\partial M \cap (T_p\partial M)^{\omega}$. Then $\omega(V(p), R) \neq 0$ since $\omega$ is nondegenerate. Now, note that $\ker \alpha$ is precisely the symplectic complement of $\mathbb{R}\{V(p), R\}$, hence $\omega|_{\ker \alpha}$ is nondegenerate. $\square$

**Example 3.2.** Consider $\mathbb{C}^n$ with the standard symplectic form $\omega_{\mathrm{std}}$. Note that $\omega_{\mathrm{std}} = d\lambda$, where $\lambda$ is the 1-form

$$\lambda = \frac{1}{2} \sum_{i=1}^{n} (x_i \, dy_i - y_i \, dx_i).$$

The associated Liouville vector field $V$ is the radial vector field

$$V = \frac{1}{2} \sum_{i=1}^{n} \left( x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} \right).$$

Note that $V$ is transverse to the unit sphere $S^{2n-1} \subseteq \mathbb{C}^n$. This gives the unit ball $B^{2n-1}$ the structure of a Liouville domain. In particular, $\alpha = \lambda|_{S^{2n-1}}$ is a contact form on the sphere.

Every Liouville domain can be completed in the following way. Note that the flow of $V$ exists for small times $\rho \in (-\varepsilon, 0]$, which defines a smooth embedding $\Phi : (-\varepsilon, 0]_\rho \times \partial M \to M$. One can show that

$$\Phi^* \omega = d(e^\rho \lambda).$$

Hence, we can attach an infinite cylindrical end $[0,\infty)_\rho \times \partial M$ to $M$, where this cylindrical end has the symplectic form $d(e^\rho\lambda)$. The resulting manifold $\widehat{M}$ is called the *completion* of the Liouville domain $M$.

A subtle but important remark: instead of using the coordinate $\rho$ to parametrize the cylindrical end of $\widehat{M}$, it is convention to use the coordinate $r = e^\rho$. Thus, $\widehat{M}$ should be thought of as the union

$$\widehat{M} = M \cup_{\partial M} ([1,\infty)_r \times \partial M).$$

This distinction will be important when we define the notion of a Hamiltonian linear at infinity in the next subsection.

**Example 3.3.** Let $V$ be the Liouville vector field on $\mathbb{C}^n$ from the previous example. The flow of $V$ is given by

$$\varphi^\rho : \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n, \quad \varphi^\rho(z) = e^{\rho/2}z.$$

Restricting the flow to $S^{2n-1}$ gives a parametrization of $\mathbb{C}^n - \{0\}$ where the radial coordinate $\rho$ is equal to $2\log|z|$. Hence, the coordinate $r = e^\rho$ is equal to $|z|^2$.

### 3.2. Symplectic cohomology

The symplectic cohomology of a Liouville domain $M$ as follows. First, for any admissible pair $(H, J)$ of a Hamiltonian and an almost complex structure, we define the symplectic cohomology $SH^\bullet(H, J)$ in the same way we defined Hamiltonian Floer homology $HF_\bullet(H, J)$, but with the differentials reversed. However, invariance of the data $(H, J)$ no longer holds, so we instead define symplectic cohomology $SH^\bullet(M)$ of the manifold $M$ to be the direct limit

$$SH^\bullet(M) = \varinjlim SH^\bullet(H, J).$$

Since the construction is mostly the same as that of Hamiltonian Floer homology, in this section we will mostly focus on the parts of the definition that must be changed.

First, we define the admissible pairs $(H, J)$ we will consider. We say a Hamiltonian $H$ is *linear* if it satisfies

$$H|_{[1,\infty)\times\partial M} = br$$

for some slope $b \in \mathbb{R}$. We define a preorder $\preceq$ on the set of linear Hamiltonians where $H \preceq H'$ iff the slope of $H$ is $\leq$ the slope of $H'$.

**Lemma 3.4.** If $H$ is linear with slope $b$, then its Hamiltonian orbits on the cylindrical end are precisely the $b$-periodic Reeb orbits of $\partial M$.

*Proof.* Recall that the Reeb vector field of $\partial M$ with the contact form $\alpha$ is the unique vector field $R$ such that $\alpha(R) = 1$ and $d\alpha(R, -) = 0$. A straightforward computation shows that $X_H = bR$ on the cylindrical end, proving the lemma. □

We say $H$ is *admissible* if it is linear with slope $b$ which is not a Reeb period $\partial M$. In particular, this implies $H$ has no 1-periodic orbits on the cylindrical end. We say an almost complex structure $J$ is *admissible* if along the cylindrical end $[1, \infty) \times \partial M$ it satisfies $J(\partial/\partial r) = R$.

**Lemma 3.5.** If $(H, J)$ is admissible, then the Floer trajectories in $\widehat{M}$ are all contained in the compact region $M$.

*Proof idea.* If a Floer trajectory $u$ intersects the cylindrical end, it attains a maximum value of $r$. This contradicts the maximum principle. □

As a consequence of this lemma, the moduli spaces $\mathcal{M}(x, y)$ (defined in the same way as the previous section) admit natural compactifications. This allows us to define the symplectic cohomology $SH^\bullet(H, J)$. Let $SC^\bullet(H, J)$ be the vector space

$$SC^\bullet(H, J) = \bigoplus_{x \in \mathcal{O}(H)} \mathbb{Z}_2 \cdot x.$$

The differential $d$ is defined by

$$dy = \sum_{x \in \mathcal{O}(H)} \#\mathcal{M}_0(x, y) \cdot x,$$

where $\mathcal{M}_0(x, y)$ is the space of isolated trajectories from $x$ to $y$. The proof that $d^2 = 0$ is the same as before. The *symplectic cohomology* $SH^\bullet(H, J)$ is the homology of this complex. Note that the differential is reversed in comparison to the definition of Hamiltonian Floer homology. Thus, we recover a cohomology theory instead of a homology theory.

**Remark 3.6.** We have pushed under the rug the issue of grading (as well as some related issues). Although we no longer have the asphericality hypothesis, one can obtain a grading by using the Conley–Zehnder index if $c_1(TM, J) = 0$.

If $(H^-, J^-)$ and $(H^+, J^+)$ are admissible pairs such that $H^- \preceq H^+$, we can define a continuation map $SH^\bullet(H^-, J^-) \to SH^\bullet(H^+, J^+)$ in the same way as before. Again, one needs to make sure that elements of the moduli space $\mathcal{K}(x_-, x_+)$ are contained in the compact region $M$. This will be true as long as the homotopy $(H^s, J^s)$ is chosen so that $H^s$ is linear with slope $b_s$ such that $b_s$ is descreasing with $s$.

### 3.3. A computation

We end with a computation of the symplectic cohomology of the ball $B^{2n} \subseteq \mathbb{C}^n$. Consider the cofinal family of Hamiltonians $H_b(z) = b|z|^2$ for $b$ which is not a Reeb period of $S^{2n-1}$. Thus, the only Hamiltonian orbit of $H_b$ is the critical point at 0. One can show that the Conley–Zehnder index of the critical point 0 is

$$n(2\lfloor b/\pi \rfloor + 1).$$

Thus, $SH^\bullet(H_b, J_{\mathrm{std}})$ consists of a single copy of $\mathbb{Z}_2$ in degree $n(2\lfloor b/\pi \rfloor + 1)$. Taking a direct limit, this copy of $\mathbb{Z}_2$ escapes to infinity, hence $SH^\bullet(B^{2n}) = 0$.

## References


[Abo14] Mohammed Abouzaid. Symplectic cohomology and viterbo's theorem, 2014.

[AD14] Michèle Audin and Mihai Damian. *Morse theory and Floer homology*. Springer, 2014.

[Hut02] Michael Hutchings. Lecture notes on Morse homology (with an eye towards Floer theory and pseudoholomorphic curves), 2002.

[Oan04] Alexandru Oancea. A survey of floer homology for manifolds with contact type boundary or symplectic homology. *Ensaios matemáticos*, 2004.

[Rit13] Alexander Ritter. Topological quantum field theory structure on symplectic cohomology. *Journal of Topology*, 6, 2013.