1. Symplectic geometry of the cotangent bundle

Given any smooth manifold Q, it has long been known (with roots in the Hamiltonian formulation of classical mechanics) that its cotangent bundle is canonically an exact symplectic manifold (T^*Q, ω), with the primitive 1-form λ to ω given tautologically by $\lambda_{(p,q)}(v) = p(\pi_* v)$, where $q \in Q$, $p \in T^*_q Q$, and $\pi\colon T^*Q \to Q$ is the projection. The symplectic structure is an invariant of the smooth structure on Q. It is natural to ask:

Question 1.1 (Arnol'd, Eliashberg, ...): *If Q is a closed manifold, does the symplectic structure on T^*Q determine the smooth structure on Q, i.e. is (T^*Q, ω) a complete diffeomorphism invariant of Q?*

Remark 1.2: *If Q is allowed to be open, notice that there are infinitely many smooth structures on $\mathbb{R}^4$, but by h-principle their cotangent bundles are symplectomorphic. Thus the restriction to closed Q.*

Notice that in this question, we are viewing T^*Q as an abstract symplectic manifold rather than a bundle. In particular, we are asking whether any symplectomorphism $\varphi\colon T^*Q_1 \to T^*Q_2$ implies a diffeomorphism $Q_1 \cong Q_2$, where φ is not asked to send the zero section $Q_1 \subset T^*Q_1$ to the zero section in T^*Q_2.

The first observation is that the image of the zero section Q_1 under φ in T^*Q_2 is a *Lagrangian*, since Q_1 itself is a Lagrangian in Q_1. However being a Lagrangian in T^*Q_2 itself does not impose a very good topological constraint — for example, in any T^*Q_2, there is an abundance of Lagrangian tori coming from various Darboux charts; in fact, any Lagrangian in the Euclidean space is a Lagrangian in any symplectic manifold of the same dimension!

The key lemma that reduces the study of Question 1.1 to a more tractable question about Lagrangians is the following:

Lemma 1.3 (E.g. Lemma 11.2 in [CE12]): *Any symplectomorphism $\varphi\colon (T^*Q, d\lambda) \to (T^*\tilde{Q}, d\tilde{\lambda})$ is diffeotopic to an* exact symplectomorphism, *i.e. a symplectomorphism ψ such that $\psi^*\tilde{\lambda} - \lambda$ is exact.*

Proof. First notice that if we know $\varphi^*\tilde{\lambda} - \lambda$ is 0 outside a compact set, then the standard Moser argument would apply. In more details, Choose a family of λ_t such that $\lambda_0 = \lambda, \lambda_1 = \varphi^*\tilde{\lambda}$ and coincide outside the same compact set. We seek a family X_t of vector fields on T^*Q which integrates to diffeomorphisms $g_t\colon T^*Q \to T^*Q$ such that

$$g_t^*(d\lambda_t) = d\lambda, \quad g_t^*(\lambda_t) = \lambda + d(\text{function}).$$

Taking the time-derivative of the first equation gives

$$g_t^*(d\dot{\lambda}_t + \mathscr{L}_{X_t} d\lambda_t) = 0$$

Now

$$d\dot{\lambda}_t + \mathscr{L}_{X_t} d\lambda_t = d\dot{\lambda}_t + d\iota_{X_t} d\lambda_t$$

by Cartan's homotopy formula. Therefore we seek a solution to

$$\dot{\lambda}_t + \iota_{X_t} d\lambda_t = 0.$$

E.g. if $\theta = \varphi^*\widetilde{\lambda} - \lambda$, then we can take $\lambda_t = \lambda + t\theta$; since $d\lambda_t$ stays the same, this equation becomes

$$\theta + \iota_X d\lambda = 0. \tag{1.1}$$

But this equation has a unique solution X due to the non-degeneracy of $d\lambda$. Moreover $\dot{\lambda}_t$ vanishes outside a compact set, thus so is X_t, and therefore it indeed integrates to a flow g_t. Now

$$\frac{d}{dt} g_t^* \lambda_t = g_t^*(\dot{\lambda}_t + d\iota_{X_t}\lambda_t + \iota_{X_t} d\lambda_t) = g_t^* d\iota_{X_t}\lambda_t.$$

Therefore this integrates to

$$g_t^*\lambda_t - \lambda = d(\text{function}).$$

and we are done.

Now, we do not know in general that $\varphi^*\tilde{\lambda} - \lambda$ vanishes outside a compact set. As before define the closed 1-form

$$\theta := \varphi^*\tilde{\lambda} - \lambda.$$

Now we can still solve equation (1.1), but we do not know the global existence of finite-time solution due to non-compactness. However the non-compactness is well-controlled: first notice that the geometry of T^*Q outside a certain compact set looks homogeneous. Specifically, consider the hypersurface $\Sigma \subset T^*Q$ a cosphere bundle (i.e. covectors in T^*Q of some fixed length). Notice that there is an exact symplectic embedding $\Sigma \times [0, \infty) \to T^*Q$. Let $\pi \colon \Sigma \times [0, \infty) \to \Sigma$ be the projection. Then we know that

$$\theta|_{\Sigma \times [0,\infty)} = \pi^*\beta + dF$$

for some $F \in C^\infty(\Sigma \times [0, \infty))$. Now define $G \colon T^*Q \to \mathbb{R}$ by setting it equal to F on $\Sigma \times [1, \infty)$, 0 inside the the complement of $\Sigma \times [0, \infty)$ (the disc bundle), and interpolate in between. Then define

$$\eta := \theta - dG.$$

Now $\eta = \pi^*\beta$ on $\Sigma \times [1, \infty)$ and coincides with θ in the compact domain, so we can solve the (analogue of) equation (1.1)

$$\eta + \iota_X d\lambda = 0.$$

Moreover, since in the cylindrical end $\Sigma \times [1, \infty)$, η looks like $\pi^*\beta$, the vector field X is complete. One can then easily verify the time-1 flow of X gives the desired map. □

The proof illustrates an important point about the symplectic geometry of cotangent bundles: that they have "bounded geometry"; specifically the geometry at infinity is "cylindrical". In general, such open exact symplectic manifolds are called *Liouville domains*. More on this in the following weeks.

This lemma then allows us to say the following: if T^*Q_1 and T^*Q_2 are symplectomorphic, then we can deform the symplectomorphism to an exact one $\varphi \colon T^*Q_1 \to T^*Q_2$, so that $\varphi^*\lambda_2 = \lambda_1 + dF$ for some function F; therefore the image of the zero section Q_1 under φ is an *exact Lagrangian*, i.e. $\lambda_2|_{\varphi(Q_1)}$ is an exact 1-form (if it were a non-exact Lagrangian, we only know that this is a closed 1-form).

This then reduces Question 1.1 to the following question:

QUESTION 1.4: *Is any closed exact Lagrangian in T^*Q diffeomorphic to Q?*

This question is still wide open.

We call any such closed exact Lagrangian $L \subset T^*Q$ a *nearby Lagrangian.* Exactness is a strong condition, which excludes examples like "local" Lagrangians in Darboux charts. For example, in $T^*S^1 \cong S^1 \times \mathbb{R}$, any circle one draws in the cylinder is a Lagrangian, but an exact Lagrangian circle must have the same area above the zero section as that below the zero section.

Finally we mention the famous *nearby Lagrangian conjecture* by Arnol'd, which is a (much) stronger form of Question 1.4:

CONJECTURE 1.5: *Given a closed smooth manifold Q, is any closed exact Lagrangian in T^*Q Hamiltonian diffeomorphic to Q?*

This conjecture basically says any nearby Lagrangian is "indistinguishable" to the zero section in symplectic topology.

2. Cotangent bundles of exotic spheres

Let us now restrict attention to Q being an exotic sphere, i.e. a closed smooth manifold homotopy equivalent ($\iff$ homeomorphic, by Poincaré conjecture) to a sphere.

By now, we know that the question we raised is not "obviously true" or "obviously false".

On the one hand,

PROPOSITION 2.1: *Given any exotic sphere Σ^n, its disc cotangent bundle $D^*\Sigma^n$ is diffeomorphic to that of the standard sphere D^*S^n.*

Proof sketch? [1] This follows from putting together several big results in geometric topology. First, let Σ^n be an exotic sphere, and consider the disc cotangent bundle D^*S^n of the standard sphere. By Adams (might also be in Kervaire-Milnor?), Σ^n and S^n has isomorphic tangent bundles. The Smale-Hirsch immersion theory/h-principle says that the space of immersions is weak homotopy equivalent to the space of fiberwise injective bundle morphisms $T\Sigma^n \to T(T^*S^n)$. Therefore, there is an immersion $\Sigma^n \looparrowright T^*S^n$ in the same class as the zero section. Now by Whitney trick, one can resolve it to an embedding. The normal bundle is isomorphic to the cotangent bundle, so we get an h-cobordism between the unit cotangent bundle of Σ^n and that of S^n. If the dimension is enough, we obtain the desired diffeomorphism by a standard argument from h-cobordism theorem. □

[1] I have not carefully checked whether this argument is correct.

On the other hand, it turns out that

Theorem 2.2 (Abouzaid): *Every nearby Lagrangian homotopy sphere in T^*S^{4k+1} bounds a compact parallelisable manifold.*

There are exotic spheres that do not bound parallelisable manifolds (see Kervaire-Milnor). This shows that symplectic structure does see smooth structures! This is yet another demonstration of the subtle rigidty of symplectic structures. The proof is essentially an ingenious adaptation of an argument already present in Gromov's 1985 pseudo-holomorphic curve paper, and closely resembles Donaldon's proof of diagonalization theorem using moduli of ASD connections.

The assumption of L being a homotopy sphere in the theorem is in fact not necesssary:

Theorem 2.3 (..., Fukaya-Seidel-Smith, Abouzaid, Kragh): *Nearby Lagrangians in T^*Q are homotopy equivalent to Q.*

Understanding the proof of this theorem will be our goal for this semester. The proof uses heavy machinery in Floer-Fukaya theory and categorical homological algebra. For now, we will go into the proof of Theorem 2.2 which only uses analysis of pseudo-holomorphic curves. The remainder of this note is a summary of (a version of) an argument by Gromov that motivates Abouzaid's proof.

3. Non-existence of exact Lagrangians in Euclidean spaces

Theorem 3.1 (Gromov [Gro85]): *For an arbitrary closed C^∞-smooth Lagrange submanifold $W \subset \mathbb{C}^n$, there exists a non-constant holomorphic map $u\colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (\mathbb{C}^n, W)$.*

This theorem comes from a "Fredholm alternative" for the nonlinear Cauchy-Riemann equation:

- Either there is a non-constant solution to $\bar\partial u = 0$;
- Or there is a solution to $\bar\partial u = g$ for any g which is a section of the appropriate bundle.

The geometry of $\mathbb{C}^n$ guarantees the second situation does not occur.

The remainder of this section sketches the proof. But we state a corollary first:

Corollary 3.2: *There is no compact exact Lagrangians in $\mathbb{C}^n$. It follows that any closed Lagrangian in $\mathbb{C}^n$ must have non-trivial $H^1(L;\mathbb{Q})$ (e.g. spheres of any dimension[2]).*

This follows by an energy argument: for any holomorphic map $u\colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (\mathbb{C}^n, W)$, its L^2-energy $\int_{\mathbb{D}^2} |du|^2_J \, d\mathrm{Vol}$ is the same as the "topological energy" $\langle [u], \omega\rangle$ using any ω-tame

[2] Circles are not spheres but tori; Lagrangian tori exists in $\mathbb{C}^n$ for any n.

almost complex structure (Lemma 2.2.1 of [MS12]). But

$$\langle [u], \omega \rangle = \int_{\mathbb{D}^2} u^*\omega = \int_{S^1} u^*\lambda = \int_{S^1} u^* df = 0$$

by exactness of L.

Classical algebraic topology says embedded Lagrangians in $\mathbb{C}^n$ has Euler characteristic 0. This gives a strong constraints on e.g. 2-manifolds, but does not constrain e.g. 3-manifolds. Furthermore Gromov-Lees's h-principle shows that L admits a Lagrangian immersion into $\mathbb{C}^n$ if and only if $TL \otimes \mathbb{C}$ is trivial. In contrast, Theorem 3.1 says e.g. S^3 does not admit Lagrangian embedding into $\mathbb{C}^3$ despite having "sufficient homotopy theoretic data" for doing so.

(3a) Inhomogeneous Cauchy-Riemann equations: We shall present the argument in a form that is close to Abouzaid's proof. I learned the argument in this form from [Fuk06] and an earlier version of this is in [Oh97]. Fix a compact exact Lagrangian in $\mathbb{C}^n$.

We study the solutions to a perturbed Cauchy-Riemann equation, depending on one parameter $\lambda \in [0, \infty)$

$$u\colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (\mathbb{C}^n, L), \quad \bar{\partial} u = g_\lambda$$

where g_λ is a section of some appropriate bundle, and compare the space of solutions at $\lambda = 0$ (when $g_\lambda = 0$, i.e. the homogeneous Cauchy-Riemann equation), and that at $\lambda \to \infty$.

We choose our perturbation in the following way: pick a family of cutoff functions $\chi_\lambda\colon \mathbb{R} \to [0,1]$ for each $\lambda \in [0, \infty)$ by

$$\chi_\lambda(t) = \begin{cases} 1, & |t| < R-1 \\ 0, & |t| > R \end{cases}$$

and with bounded C^k-norm. Pick an arbitrary direction $\vec{v}$ and consider the Hamiltonian $H\colon \mathbb{C}^n \to \mathbb{R}$ by

$$H(\vec{z}) = \langle \vec{z}, \vec{v} \rangle.$$

Then the Hamiltonian flow $\phi_t := \exp_t(X_H)$ from integrating the Hamiltonian vector field X_H displaces L from itself by compactness: by rescaling we can assume

$$\phi_1(L) \cap L = \emptyset.$$

In fact all that is needed for the argument is this "*Hamiltonian displaceability*" condition so the precise choice of H does not matter.

For each fixed $\lambda \in [0, \infty)$, we then study the equation

$$\tilde{u}\colon \mathbb{R}_s \times [0,1]_t \to \mathbb{C}^n, \quad \frac{\partial \tilde{u}}{\partial s} + J\left(\frac{\partial \tilde{u}}{\partial t} - \chi_\lambda(s) X_H(\tilde{u})\right) = 0,$$

with boundary condition

$$\tilde{u}(s,0), \tilde{u}(s,1) \in L$$

and finite energy

$$\int_{\mathbb{R}\times[0,1]} \tilde{u}^*\omega < \infty.$$

Remark 3.3: *This "topological" energy is related to the "analytic" L^2-energy as follows:*

$$\int_{\mathbb{R}\times[0,1]} \left|\frac{\partial \tilde{u}}{\partial s}\right|^2 ds\,dt = \int_{\mathbb{R}\times[0,1]} \left|\frac{\partial \tilde{u}}{\partial t} - X_H(u)\right|^2 \le \int_{\mathbb{R}\times[0,1]} \tilde{u}^*\omega + \|H\|$$

where $\|H\|$ is some constant (the "Hofer norm") depending on H; see [Oh97].

At $|s| \gg 0$, the equation reduces to the homogeneous Cauchy-Riemann equation $\bar{\partial}\tilde{u} = 0$, and by the finite energy condition, we can apply the *removable singularity theorem* by Gromov and Oh to extend $\tilde{u}$ to a map

$$u\colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (\mathbb{C}^n, L)$$

where we view $\mathbb{R}\times[0,1]$ as $\mathbb{D}^2 \setminus \{\pm 1\}$.

(3b) Behavior in the limits of λ: We now consider solutions to the equation as we take $\lambda = 0$ and $\lambda \to \infty$.

At $\lambda = 0$, the equation reduces to the homogeneous Cauchy-Riemann equation $\bar{\partial}u = 0$, and by exactness the only solutions are the constant maps.

At $\lambda \to \infty$, we claim that displaceability implies there are no solutions. A rough idea of the proof is as follows: supposing that solutions exist for all $\lambda \gg 0$. Take a sequence u_k of solutions with parameter $\lambda_k \to \infty$. Then the proportion of the strip $\mathbb{R}_s \times [0,1]_t$ that follows the inhomogeneous Floer equation becomes larger and larger, and we can extract a limit which is a finite-energy Floer strip, i.e. a map

$$u_\infty\colon (\mathbb{R}\times[0,1], \mathbb{R}\times\{0,1\}) \to (\mathbb{C}^n, L), \quad \frac{\partial u_\infty}{\partial s} + J\left(\frac{\partial u_\infty}{\partial t} - X_H(u_\infty)\right) = 0$$

with finite energy

$$E(u_\infty) = \int_{\mathbb{R}\times[0,1]} \left|\frac{\partial u_\infty}{\partial s}\right|^2 ds\,dt = \int_{\mathbb{R}\times[0,1]} \left|\frac{\partial u_\infty}{\partial t} - X_H(u_\infty)\right| ds\,dt < \infty,$$

which should limit to Hamiltonian chords on L (i.e. a path following the Hamiltonian vector field with starting and ending points on L) as $s \to \pm\infty$, following a by now standard argument in Floer theory. Hamiltonian chords on L corresponds to intersection points of L with $\phi_1(L)$, and this is the contradiction. For slightly more details see Proposition 3.3 of [Fuk06] or Lemma 2.2 of [Oh97].

Remark 3.4: *In the argument, finiteness of energy of the limit Floer strip follows from an a priori estimate of energies of u_k, and this is where the bounds of the derivatives of the cutoff function χ_λ are needed.*

(3c) Moduli space for the family: We form the moduli space $\mathcal{N}$, consisting of pairs $(\lambda, \tilde{u})$ of $\lambda \in [0,\infty)$ and

$$\tilde{u}\colon (\mathbb{R}\times[0,1], \mathbb{R}\times\{0,1\}) \to (\mathbb{C}^n, L), \quad \partial_s\tilde{u} + J\big(\partial_t\tilde{u} - \chi_\lambda(s)X_H(\tilde{u})\big) = 0$$

with finite energy, and such that the the unique extension $u\colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (\mathbb{C}^n, L)$ is trivial as a homotopy class in $\pi_2(\mathbb{C}^n, L)$. Supposing transversality, this is a smooth manifold of dimension

$$\dim \mathcal{N} = n + 1$$

from standard index theory (relative Riemann-Roch formula; see Theorem C.1.10 in Appendix C of [MS12]).

We now consider the topology of this moduli space. Define the projection $\pi\colon \mathcal{N} \to [0,\infty)$ to the first coordinate λ and denote by $\mathcal{N}(\lambda) := \pi^{-1}(\lambda)$; then $\mathcal{N}(0) = \pi^{-1}(0) \cong L$ is a topological boundary of $\mathcal{N}$. There are no other topological boundaries, and the possible non-compactness comes from the equation.

There are 4 possible sources of non-compactness (best understood with pictures):

(1) $\lambda \to \infty$, which is impossible because we know there are no solutions at large λ;
(2) Energy concentration at an interior point of $\mathbb{R} \times [0,1]$, which results in non-constant a holomorphic sphere by a standard rescaling argument plus removable singularity theorem; this is impossible since the symplectic structure on $\mathbb{C}^n$ is exact;
(3) Energy concentration at some boundary point of $\mathbb{R} \times [0,1]$, which results in a non-constant holomorphic disc with boundary on L by similar arguments, which is impossible by exactness of L;
(4) Energy goes off to $\pm\infty$ in the $\mathbb{R}$ direction, which results in a non-constant holomorphic disc with boundary on L by similar arguments, which is impossible by exactness of L.

Therefore we conclude that $\mathcal{N}$ is a compact manifold-with-boundary $\mathcal{N}(0) \cong L$. This then has a fundamental chain $[\mathcal{N}] \in H_{n+1}(\mathcal{N}, \partial\mathcal{N})$.

(3d) Concluding the proof: Fixing the point $1 \in \mathbb{D}^2$ (the point at $+\infty$ from the viewpoint of $\mathbb{R} \times [0,1]$), we get an evaluation map

$$\mathrm{ev}\colon \mathcal{N} \to L; \quad (\lambda, \tilde{u}) \mapsto u(1).$$

Restricting to $\mathcal{N}(0)$, the map

$$\mathrm{ev}|_{\mathcal{N}(0)}\colon \mathcal{N}(0) \cong L \to L$$

is the identity map and therefore the induced map on homology $(\mathrm{ev}|_{\mathcal{N}(0)})_*$ pushes forward the fundamental class of $\mathcal{N}(0)$ to the fundamental class $[L] \in H_n(L)$. However, since the map extends to a null-cobordism of $\mathcal{N}(0)$, the pushforward of the fundamental chain $[\mathcal{N}] \in H_{n+1}(\mathcal{N}, \partial\mathcal{N})$ has $[L] \in H_n(L)$ as its boundary, and therefore $[L]$ is null-homologous, a contradiction!

4. Some remarks

We will follow Abouzaid's summary of the argument in section 2 of [Abo12] in the talk; however we will make some brief remarks about features of the proof.

(4a) Buhovsky's displaceability trick: It will be nice if we can use this argument for exact Lagrangian homotopy spheres in cotangent bundles! But exact Lagrangians cannot be Hamiltonian displaced, as Gromov's argument shows.

The key point that starts the proof is the following trick (first mentioned in [ALP94] and used for obstructing Lagrangian embeddings by [Buh04]) that embeds T^*S^n symplectically into a bigger symplectic manifold in which Lagrangians are Hamiltonian displaceable. First recall that by construction (as a symplectic reduction), the Fubini-Study symplectic form on $\mathbb{C}\mathrm{P}^{n-1}$ is characterized by

$$\pi^*\omega_{\mathbb{C}\mathrm{P}^{n-1}} = \omega_{\mathbb{C}^n}|_{S^{2n-1}},$$

where $\pi\colon S^{2n-1} \to \mathbb{C}\mathrm{P}^{n-1}$ is the Hopf fibration. Therefore the map

$$S^{2n-1} \hookrightarrow \mathbb{C}^n \times \mathbb{C}\mathrm{P}^{n-1},$$

where the first factor is inclusion and the second factor is the Hopf map, is a Lagrangian embedding. Therefore, by Darboux-Weinstein theorem, a neighborhood of the zero section in T^*S^{2n-1} embeds symplectically in $\mathbb{C}^n \times \mathbb{C}\mathrm{P}^{n-1}$.

Therefore, any Lagrangian L in T^*S^{2n-1} embeds as a Lagrangian in $\mathbb{C}^n \times \mathbb{C}\mathrm{P}^{n-1}$. Moreover, due to the $\mathbb{C}^n$-factor, L is Hamiltonian displaceable. It then makes sense to repeat the argument by constructing the moduli space of solutions to the deformed Cauchy-Riemann equation. However, as Gromov's argument shows, this manifold cannot be compact and we have to deal with the non-compactness coming from sphere and disc bubblings.

(4b) The bounding manifold: Abouzaid's proof of Theorem 2.2 is to build a parallelizable manifold with boundary diffeomorphic to the nearby Lagrangian homotopy sphere Σ^n in T^*S^n using moduli spaces of curves. It is not too hard to show that the moduli space given by 1-parameter family of perturbations of the Cauchy-Riemann equation is parallelizable (if the appropriate transversality results are in place), but exactly because of Gromov's argument we know that this moduli space cannot be compact, and the non-compactness can be due to either sphere or disc bubblings. Therefore some careful sculpture-work on the moduli space needs to be performed. Very briefly, this is possible because the codimension-1 stratum of disc bubbling can be "closed up" by constructing another moduli space of disc bubble configurations whose boundary is our codimension-1 stratum (thanks to the simply-connectivity of Σ^n), and the codimension-2 stratum of sphere bubbling can be dealt with because we have explicit knowledge about what that stratum (and its neighborhood) looks like (since the sphere bubbles live in $\mathbb{C}\mathrm{P}^{n-1}$).

(4c) A related result: A construction in the same spirit is used by Ekholm-Smith ([ES16, ES14]) to prove rigidity result beyond homotopy type for Lagrangian immersions (with fixed number of double points!) in $\mathbb{C}^n$:

THEOREM 4.1 (Ekholm-Smith): *In even dimension n, a homotopy sphere Σ^n admits an exact Lagrangian immersion $\Sigma^n \looparrowright \mathbb{C}^n$ with exactly one transverse double point (and no other self-intersections) if and only if Σ^n is diffeomorphic to the standard S^n.*

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