SIERPINSKI CARPET HYPERBOLIC COMPONENTS OF DISJOINT TYPE ARE BOUNDED

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ABSTRACT. In this paper, we will prove that Sierpinski carpet hyperbolic components of disjoint type are bounded. Furthermore, we show that for each map f on the closure of the hyperbolic component, there exists a quadratic-like restriction around every non-repelling periodic point. Our methods are applicable for any hyperbolic component of disjoint type. In particular, we describe the post-critical set of any map on the boundary of the hyperbolic component of z^2.

CONTENTS

1. INTRODUCTION

A rational map $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ is called *hyperbolic* if every critical point of f converges to an attracting periodic cycle under iteration. For our purposes, it is convenient to mark all the fixed points, and consider the fixed point marked rational maps $\mathrm{Rat}_{d,\mathrm{fm}}$ and the corresponding moduli space $\mathcal{M}_{d,\mathrm{fm}} = \mathrm{Rat}_{d,\mathrm{fm}} / \mathrm{PSL}_2(\mathbb{C})$ (see §2.1). The set of conjugacy classes of hyperbolic maps form an open and conjecturally dense subset of $\mathcal{M}_{d,\mathrm{fm}}$, and a connected component is called a *(marked) hyperbolic component*.

Let $\mathcal{H} \subseteq \mathcal{M}_{d,\mathrm{fm}}$ be a hyperbolic component. As $[f]$ varies in $\mathcal{H}$, the topological dynamics on the Julia set J_f remains constant, but the geometry of J_f varies. We say $\mathcal{H}$ is a *Sierpinski carpet hyperbolic component* if the Julia set of any map $[f] \in \mathcal{H}$ is a Sierpinski carpet, and it is of *disjoint type* if for any map $[f] \in \mathcal{H}$, all critical points of $[f]$ are in pairwise different periodic cycles of Fatou components. Equivalently, $\mathcal{H}$ is of disjoint type if any map $[f] \in \mathcal{H}$ has exactly $2d-2$ attracting periodic cycles. For disjoint-type hyperbolic components, the multipliers of attracting cycles provide natural identification:

$$\boldsymbol{\rho} : \ \mathcal{H} \ \xrightarrow{\simeq} \ \mathbb{D}^{2d-2} \tag{1.1}$$

Motivated by Thurston's compactness theorem for acylindrical hyperbolic 3-manifold, McMullen conjectured in the early 1990s (see [McM95]) that

Conjecture 1.1. *A Sierpinski carpet hyperbolic component $\mathcal{H}$ is bounded in $\mathcal{M}_{d,fm}$.*

Despite many attempts throughout the decades, the conjecture remains wide open. In this paper, we will prove it in the disjoint type case:

Theorem A. *A Sierpinski carpet hyperbolic component $\mathcal{H} \subseteq \mathcal{M}_{d,fm}$ of disjoint type is bounded in $\mathcal{M}_d$. Moreover,* (1.1) *naturally extends to*

$$\boldsymbol{\rho} : \ \overline{\mathcal{H}} \ \xrightarrow{\simeq} \ \overline{\mathbb{D}^{2d-2}}. \tag{1.2}$$

In particular, $\partial\mathcal{H}$ is locally connected.

Our methods are applicable to any hyperbolic components of disjoint-type. In Theorem C, we will describe the postcritical sets of maps on $\partial\mathcal{H}_{z^2}$, where $\mathcal{H}_{z^2} \subset \mathcal{M}_{2,\mathrm{fm}}$ is the hyperbolic component of $z \mapsto z^2$. Theorem C is the prototype example of Conjecture 1.9 that will be discussed in the follow-up paper.

Our approach also gives uniform bound of the dynamics of maps on $\overline{\mathcal{H}}$:

Theorem B. *Let $\mathcal{H}$ be a Sierpinski hyperbolic component of disjoint type. There exists a constant $\varepsilon > 0$ such that for any map $[f] \in \overline{\mathcal{H}}$ and any non-repelling periodic point x of periodic p, there exists a quadratic-like restriction $f^p : U \longrightarrow V$, with $x \in U \subseteq V$ and $\operatorname{mod}(V - U) \geq \varepsilon$.*

Applying the Douady-Hubbard straightening theorem to all quadratic-like restrictions around non-repelling cycles, we obtain the refinement of (1.2)*:*

$$\mathcal{R}_{\mathrm{ql}} : \ \overline{\mathcal{H}} \ \xrightarrow{\simeq} \ \overline{\Delta}^{2d-2} \ \subset \mathrm{Mand}^{2d-2}, \tag{1.3}$$

where Δ is the main hyperbolic component of the Mandelbrot set.

1.1. **Historical background.** Thurston's hyperbolization theorem is one of the most important development in the study of 3-manifolds. The tools developed along the theorem has revolutionized the theory of Kleinian groups. In the proof of the hyperbolization theorem, two boundedness theorems, the *double limit theorem* and the *Thurston's compactness theorem for acylindrical manifolds*, are the key ingredients (see [Kap10, Thu86]). Based on the Sullivan's dictionary, these two boundedness theorems have natural analogues for rational maps. Since convex cocompact acylindrical Kleinian groups have Sierpinski carpet limit sets, Conjecture 1.1 is the analogue of the Thurston's compactness theorem.

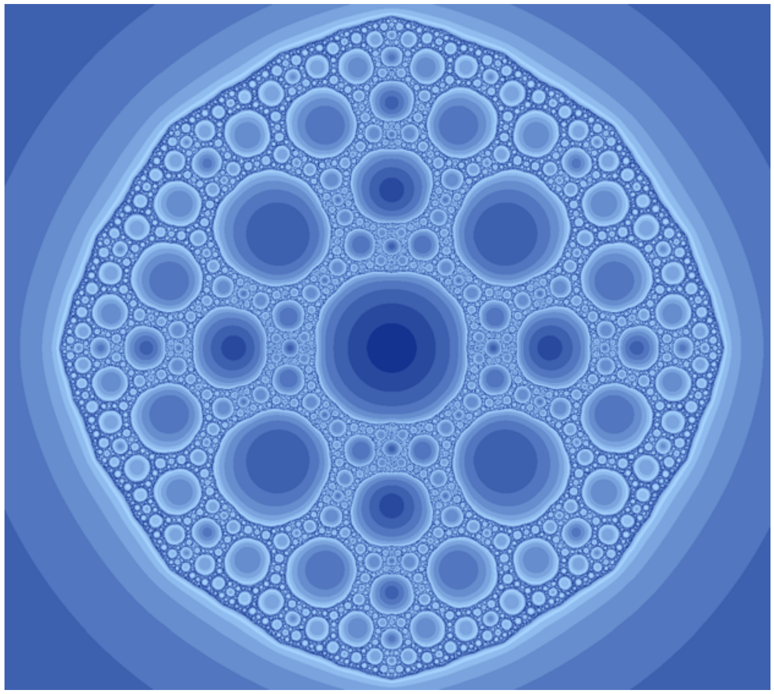

FIGURE 1.1. The Julia set of a Sierpinski carpet hyperbolic rational map.

For Kleinian groups, the proof for both boundedness theorems is by contradiction and can be break down into two steps:

(A) (Geometric part): constructing limiting isometric group actions on $\mathbb{R}$-trees with no global fixed point for any degenerating sequences of Kleinian groups (see [MS84, Bes88, Pau88]);
(B) (Combinatorial/topological part): analyzing possible limiting group actions to get topological decompositions of the underlying 3-manifold (see Rips' theory [Kap10] and Skora's duality theorem [Sko96]).

The contradiction for Thurston's compactness theorem is that acylindrical 3-manifolds do not admit such decomposition constructed in Step (B). The story is similar for the double limit theorem, except the contradiction comes from geometric constraints of the laminations.

It is already suggested in [McM95] that a similar strategy might work for rational maps. There have been many constructions of limiting dynamics on trees for degenerations of rational maps (see [McM09, Kiw15, Luo21b, Luo22a]). These constructions complete Step (A) for rational maps. In this analogy, Step (B) becomes essential for the boundedness results of rational maps.

The boundary of a hyperbolic component $\mathcal{H}$ consists of two types of points: *geometrically finite maps* and *geometrically infinite maps.* To carry out Step (B), we consider these two cases separately.

Geometrically finite maps are the first to be understood and are essentially determined by a finite set of data (somewhat similar to PCF maps). In [Luo22b], the second author showed that in a Sierpinski hyperbolic component $\mathcal{H}$ (of any type) a 'geometrically finite degenerations' $[f_n]$ always land at geometrically finite parameter $[f_\infty] \in \partial\mathcal{H}$; in particular, $[f_n]$ does not diverge in $\mathcal{M}_d$. The main step is the following finiteness statement for the dynamics in the limiting tree.

(b) There is a finite 'core' in the limiting tree if $[f_n]$ diverges. This finiteness induces a decomposition of the rational maps in $\mathcal{H}$ by some *limiting Thurston obstruction.*

Similar to Kleinian groups, the contradiction is that Sierpinski carpet Julia set would prevent the existence of such limiting obstructions.

Geometrically infinite maps are more mysterious. Conjecturally, they all arise as limits of geometrically finite maps. To study such maps, some uniform bound is usually needed. In this paper, we use bounds from renormalization theory developed in [DL22] to prove such a uniform bound for a special class of geometrically infinite maps, called *eventually-golden-mean maps* (see §2.3). The uniform bound allows us to obtain a limiting map on a finite tree of Riemann spheres. Similar as in the geometrically finite case, the finiteness allows us construct a decomposition of the rational map in terms of limiting Thurston obstructions, and we obtain a contradiction here.

Our results are related to the *Thurston's realization problem.* Given a topological branched covering of the sphere S^2, Thurston's realization problem asks when it is equivalent to a rational map. Thurston gives a negative criterion to answer the question for *post-critically finite* branched coverings [DH93]. Recently, Dylan Thurston gives a positive criterion for the realization problem for post-critically finite maps [Thu20] (with non-vacuous Fatou dynamics). For geometrically finite maps, Thurston's realization problem has been studied extensively (see [DH93, CJS04, CJ11, CT11, CT18]). Usually, the realization problems for geometrically finite maps are studied by deformation of hyperbolic maps. Various elementary deformations such as pinching and spinning were constructed and studied (see [Mak00, Tan02, HT04, PT04, CT18]). These operations are generalized in [Luo21a, Luo22b]. From this perspective, Theorem A and Theorem B can be interpreted as a Thurston's realization theorem for geometrically infinite maps. Our method, perhaps for the first time, combines two theories in complex dynamics: bounds from the Thurston theory and the bounds from the renormalization theory.

There have been many previous studies to understand deformations of rational maps and related boundedness problems. Results on unboundedness of hyperbolic components were obtained in [Mak00, Tan02]. In [Eps00], Epstein used algebraic and analytic methods to give the first general boundedness result of hyperbolic component of disjoint type in the quadratic case. This was later generalized in the *bi-critical* setting by Nie and Pilgrim in [NP19]. Related boundedness results in the degree 4 Newton family was also proved in [NP20]. To the best of our knowledge, Theorem A is the first time a boundedness theorem of an entire hyperbolic component in $\mathcal{M}_d$ is proved in degree $d \geq 3$.

Remark 1.2. In Spring 2022, J. Kahn simultaneously presented an independent approach to the boundedness of Sierpinski hyperbolic components of *all types* (not necessarily disjoint); see his MSRI-talks at [Kah22].

1.2. **Estimate on degenerations.** In this subsection, let $\mathcal{H}$ be a hyperbolic component of disjoint type, which may or may not be Sierpinski. Our strategy is to uniformly control the geometry of the Julia set for some special maps on $\partial\mathcal{H}$, called *eventually-golden-mean maps.*

An irrational number $\theta \in (0,1)$ is said to be *eventually-golden-mean* if it has a continuous fraction expansion $\theta = [0; a_1, ..., a_m, ...]$ with $a_m = 1$ for all large m. A map $[f] \in \partial\mathcal{H}$ is called *eventually-golden-mean map* if the multiplier for any of its indifferent periodic cycle is of the form $e^{2\pi i\theta}$, where θ is eventually-golden-mean. In particular, every critical point of $[f]$ is either in an attracting basin or on the boundary of a Siegel disk.

Degenerations of compact Riemann surfaces. To discuss our bound on the geometry, let X be a compact Riemann surface with boundaries. Let γ be a non-peripheral arc connecting ∂X, and let Γ_γ be the family of arcs isotopic to γ. We define the *degeneration* $\mathcal{W}(\gamma)$ for γ of X as the extremal width of the family Γ_γ. The *arc degeneration* for X is

$$\mathcal{W}_{arc}(X) = \sum_{\gamma:\mathcal{W}(\gamma)\geq 2} \mathcal{W}(\gamma).$$

Similarly, let α be a homotopically non-trivial simple closed curve, and let Γ_α be the family of simple closed curves isotopic to α. We define the *degeneration* $\mathcal{W}(\alpha)$ for α of X as the extremal width of Γ_α. We define the *loop degeneration* for X as

$$\mathcal{W}_{loop}(X) = \sum_{\alpha:\mathcal{W}(\alpha)\geq 2} \mathcal{W}(\alpha).$$

By losing ≤ 2 extremal width of the family, we may assume Γ_γ and Γ_α are laminations. We remark that since wide families do not cross, both $\mathcal{W}_{arc}(X)$ and $\mathcal{W}_{loop}(X)$ are in fact finite sums.

Degenerations of eventually-golden-mean maps. Let $[f] \in \partial\mathcal{H}$ be an eventually-golden-mean map. Denote the list of Siegel disks and attracting Fatou components of f by

$$Z_{1,0}, ...Z_{1,p_1-1}, Z_{2,0}, ..., Z_{k_1,p_{k_1}-1} \text{ and } D_{1,0}, ...D_{1,q_1-1}, D_{2,0}, ..., D_{k_2,q_{k_2}-1}.$$

In §2.4 and §3.2, we will define the corresponding *pseudo-Siegel disks* $\widehat{Z}_{i,j}$ and *valuable-attracting domain* $\widehat{D}_{i,j}$. Some important properties are

- $\widehat{Z}_{i,j}$ and $\widehat{D}_{i,j}$ are closed disks, with $\overline{Z_{i,j}} \subseteq \widehat{Z}_{i,j}$ and $\widehat{D}_{i,j} \subseteq D_{i,j}$;
- f is injective on $\widehat{Z}_{i,j}$ and $\widehat{D}_{i,j}$ is forward invariant under f;
- $\widehat{Z}_{i,j}$ and $\widehat{D}_{i,j}$ contain the critical, post-critical points and the non-repelling periodic point in $\overline{Z_{i,j}}$ and $D_{i,j}$ respectively.

To quantify the degeneration of the map $[f]$, we give the following definition.

Definition 1.3. We say $[f]$ has degeneration bounded by K if there exist

- K-quasiconformal pseudo-Siegel disks $\widehat{Z}_{i,j}$,
- K-quasiconformal valuable-attracting domains $\widehat{D}_{i,j}$,

so that the *pseudo-core surface* of $[f]$ (see §3 for more discussions)

$$\widehat{X}_f := \widehat{\mathbb{C}} - \bigcup \mathrm{Int}(\widehat{D}_{i,j}) - \bigcup \mathrm{Int}(\widehat{Z}_{i,j}) \text{ satisfies}$$

- $\mathcal{W}_{arc}(\widehat{X}_f) \leq K$; and
- $\mathcal{W}_{loop}(\widehat{X}_f) \leq K$.

We will prove the following boundedness theorems for eventually-golden-mean maps with uniformly bounded degenerations.

Theorem 1.4. *Let $[f_n] \in \partial\mathcal{H}$ be a sequence of eventually-golden-mean maps. Suppose that $[f_n]$ has degeneration bounded by K. Then after possibly passing to a subsequence, $[f_n] \to [f] \in \mathcal{M}_{d,fm}$, and $[f]$ has $2d-2$ non-repelling cycles.*

The pulled-off constant. We now introduce an important combinatorial constant that controls the degenerations of eventually-golden-mean maps.

Two arcs γ_1 and γ_2 are said to

- *intersect essentially* if for any arcs $(\widetilde{\gamma}_i)_{i\in\{1,2\}}$ homotopic to $(\gamma_i)_{i\in\{1,2\}}$, $\widetilde{\gamma}_1$ intersects $\widetilde{\gamma}_2$; and
- *intersect laminally* if for any arcs $(\widetilde{\gamma}_i)_{i\in\{1,2\}}$ homotopic and disjoint to $(\gamma_i)_{i\in\{1,2\}}$, $\widetilde{\gamma}_1$ intersects $\widetilde{\gamma}_2$.

To justify the notations, let $\mathcal{L}_1, \mathcal{L}_2$ be two laminations consisting of homotopic curves. If $\gamma_1 \in \mathcal{L}_1$ intersects laminally $\gamma_2 \in \mathcal{L}_2$, then every curve in $\mathcal{L}_1$ intersects every curve in $\mathcal{L}_2$.

A family of arcs γ_i are said to be *essentially disjoint* (or *laminally disjoint*) if no pairs in the family intersect essentially (or laminally). An essentially (or laminally) disjoint pull back of a map f is an essentially (or laminally) sequence of arcs $\gamma_0, ..., \gamma_n$ so that $f : \gamma_{i+1} \longrightarrow \gamma_i$ is a homeomorphism for each $i = 0, ..., n-1$.

Let $[f] \in \partial\mathcal{H}$ be an eventually-golden-mean map. Let $\gamma \subseteq \widehat{\mathbb{C}} - \bigcup \mathrm{Int}(\widehat{D}_{i,j}) - \bigcup Z_{i,j}$ be a non-peripheral arc connecting boundaries of Siegel disks. The *pulled-off constant* $N(\gamma)$ for γ is the smallest number n so that for any laminally disjoint pull back sequence $\gamma_0 = \gamma, \gamma_1, ..., \gamma_n$, at least one end point of $\partial\gamma_n$ is not on the boundary of a periodic Siegel disk.

Similarly, let $[f_{pcf}] \in \mathcal{H}$ be the post-critically finite center of $\mathcal{H}$, and let $P(f_{pcf})$ be its post-critical set. Let γ be a non-peripheral arc in $\widehat{\mathbb{C}} - P(f_{pcf})$ that connects points in $P(f_{pcf})$. Its pulled-back constant $N(\gamma)$ is the smallest number n so that for any essentially disjoint pull back sequence $\gamma_0 = \gamma, \gamma_1, ..., \gamma_n$, at least one end point of $\partial\gamma_n$ is not in $P(f_{pcf})$.

Definition 1.5 (Pulled-off constant)**.** Let $[f] \in \partial\mathcal{H}$ be an eventually-golden-mean map. The *(Siegel) pulled-off constant* for $[f]$ is

$$N_{Siegel}([f]) := \sup_{\gamma} N(\gamma)$$

where the supreme is over all non-peripheral arcs connecting boundaries of Siegel disks.

Let $[f_{pcf}] \in \mathcal{H}$ be the post-critically finite center of $\mathcal{H}$. The *pulled-off constant* for $[f_{pcf}]$ is

$$N([f_{pcf}]) := \sup_{\gamma} N(\gamma)$$

where the supreme is over all non-peripheral arcs connecting the points in the post-critical set.

In §4, we will prove that

- $N([f_{pcf}]) < \infty$ if and only if $\mathcal{H}$ is Sierpinski.
- $N_{Siegel}([f]) \leq N([f_{pcf}])$ for any eventually-golden-mean map $[f] \in \partial\mathcal{H}$, where $\mathcal{H}$ is a Sierpinski hyperbolic component.

By combining the above two statements, we see that $N_{Siegel}([f])$ is uniformly bounded if $\mathcal{H}$ is a Sierpinski.

The following technical theorem gives the uniform bound for eventually-golden-mean maps, and is the key in our argument.

Theorem 1.6. *Let $\mathcal{H}$ be a hyperbolic component of disjoint type, and let $[f] \in \partial\mathcal{H}$ be an eventually-golden-mean map. Then $[f]$ has degeneration bounded by K, where K depends on*

1. *the hyperbolic component $\mathcal{H}$,*
2. *the pulled-off constant $N_{Siegel}([f])$, and*
3. *the multipliers of the attracting cycles of f.*

For applications, Assumption (2) is the main one in Theorem 1.6. Therefore, the condition "$[f_n]$ has degeneration bounded by K" in Theorem 1.4 can be replaced (for practical purposes) with "$N_{Siegel}([f_n]) \leq M$" for some M.

Remark. We remark that if $\mathcal{H}$ is Sierpinski, then $N_{Siegel}([f]) \leq N([f_{pcf}]) < \infty$. Thus, in this case, the constant K is independent of the pulled-off constant. We also remark that the constant K is independent of the indifferent multipliers of the map $[f]$. This crucial fact allows us to take the limit of those eventually-golden-mean maps.

Sketch of the proof of Theorems A and B. To discuss how Theorem 1.6 allows us to prove Theorems A and B, we start with the following decomposition of the boundary $\partial\mathcal{H}$.

Definition 1.7. Let $\mathcal{H}$ be a hyperbolic component of disjoint type. The boundary

$$\partial\mathcal{H} = \partial_{\text{reg}}\mathcal{H} \sqcup \partial_{\text{exc}}\mathcal{H} \subseteq \mathcal{M}_{d,\text{fm}}$$

splits into the *regular* and *exceptional* parts, where $[f] \in \partial_{\text{reg}}\mathcal{H}$ if $[f]$ has exactly $2d-2$ non-repelling periodic cycles, and $[f] \in \partial_{\text{exc}}\mathcal{H}$ otherwise: at least two non-repelling periodic cycles of $[f]$ collide.

We remark that by transversality, we have that (see Proposition 2.2)

- the natural extension of the multiplier map (1.1) is an embedding on the regular boundary $\boldsymbol{\rho} : \partial_{\text{reg}}\mathcal{H} \hookrightarrow \partial\mathbb{D}^{2d-2}$;
- $\boldsymbol{\rho}(\partial_{\text{reg}}\mathcal{H}) \cap \boldsymbol{\rho}(\partial_{\text{exc}}\mathcal{H}) = \emptyset$.

Thus, to prove Theorem A, it suffices to show that if $\mathcal{H}$ is Sierpinski, then $\boldsymbol{\rho}(\partial_{\text{reg}}\mathcal{H}) = \partial\mathbb{D}^{2d-2}$.

Denote the boundary of eventually-golden-mean maps and geometrically finite maps by $\partial_{\text{egm}}\mathcal{H}$ and $\partial_{\mathbb{Q}}\mathcal{H}$ respectively. If $\mathcal{H}$ is Sierpinski, then $\boldsymbol{\rho}(\partial_{\mathbb{Q}}\mathcal{H})$ is dense in $\partial\mathbb{D}^{2d-2}$ (see [CT18] or [Luo22b]). This allows us to show that $\boldsymbol{\rho}(\partial_{\text{egm}}\mathcal{H})$ is dense in $\partial\mathbb{D}^{2d-2}$ (see Proposition 2.6).

Given any multiplier profile $\rho = (\rho_1, ..., \rho_{2d-2}) \in \partial\mathbb{D}^{2d-2}$, by Theorem 1.4 and Theorem 1.6, we can construct a convergent sequence of eventually-golden-mean maps $[f_n] \to [f]$ with $\boldsymbol{\rho}([f]) = \rho$. Since $[f]$ has $2d-2$ non-repelling cycles, $[f] \in \partial_{\text{reg}}\mathcal{H}$. Thus, $\boldsymbol{\rho}(\partial_{\text{reg}}\mathcal{H}) = \partial\mathbb{D}^{2d-2}$, and Theorem A follows.

To prove Theorem B, we first construct a semiconjugacy between a map $[f] \in \overline{\mathcal{H}}$, and a topological model $\bar{f} : S^2 \longrightarrow S^2$ which is the quotient map of the post-critical finite map $[f_{pcf}] \in \mathcal{H}$ by collapsing Fatou components. This allows us to show there exists a quadratic-like restriction near every non-repelling periodic point. The uniform bound of the modulus in Theorem B then follows from the conclusion of Theorem A that $\overline{\mathcal{H}}$ is compact.

1.3. **Boundaries of hyperbolic components of disjoint type.**

A general hyperbolic components of disjoint type $\mathcal{H}$ may not be bounded in $\mathcal{M}_{d,\text{fm}}$. We give the following definition to parameterize the boundary at infinity.

Definition 1.8. Let $\mathcal{H}$ be a hyperbolic component of disjoint type. We define the *obstructed boundary*

$$\partial^\infty\mathcal{H} = \{\rho \in \partial\mathbb{D}^{2d-2} : \\ \exists [f_n] \in \mathcal{H} \text{ with } [f_n] \to \infty \text{ in } \mathcal{M}_{d,\text{fm}} \text{ and } \boldsymbol{\rho}([f_n]) \to \rho\}.$$

The *rational obstructed boundary*

$$\partial^\infty_{\mathbb{Q}}\mathcal{H} = \partial^\infty\mathcal{H} \cap \partial_{\mathbb{Q}}\mathbb{D}^{2d-2}$$

where $\partial_{\mathbb{Q}}\mathbb{D}^{2d-2}$ consists of tuples $(\rho_1, ..., \rho_{2d-2})$ so that all indifferent multipliers are rational, i.e. of the form $e^{2\pi i p/q}$.

We remark that the rational obstructed boundary $\partial^\infty_{\mathbb{Q}}\mathcal{H}$ can be identified as obstructed geometrically finite maps on the boundary of $\mathcal{H}$, and can be effectively computed. We formulate the following conjecture.

Conjecture 1.9. *Let $\mathcal{H}$ be a hyperbolic component of disjoint type. Then*

$$\partial^\infty\mathcal{H} = \overline{\partial^\infty_{\mathbb{Q}}\mathcal{H}}.$$

In particular, the natural extension of the multiplier map (1.1) *gives an identification for the regular boundary:*

$$\partial_{\text{reg}}\mathcal{H} \cong \partial\mathbb{D}^{2d-2} - \overline{\partial^\infty_{\mathbb{Q}}\mathcal{H}} - \boldsymbol{\rho}(\partial_{\text{exc}}\mathcal{H})$$

Combinatorial bound of $N_{Siegel}([f])$. An important ingredient for Theorem A is that if $\mathcal{H}$ is Sierpinski, the pulled-off constant $N_{Siegel}([f])$ is uniformly bounded for eventually-golden-mean map $[f] \in \partial\mathcal{H}$. It is in general not bounded if $\mathcal{H}$ is not Sierpinski. In the sequel, we plan to bound the pulled-off constant $N_{Siegel}([f])$ in terms of the combinatorial distance between $\boldsymbol{\rho}([f])$ and $\overline{\partial^\infty_{\mathbb{Q}}\mathcal{H}}$, and prove Conjecture 1.9.

1.4. **The example of $\mathcal{H}_{z^2}$.** To illustrate Conjecture 1.9 and the subtlety about the exceptional boundary, consider the hyperbolic component $\mathcal{H}_{z^2}$ in the moduli space quadratic rational maps that contains z^2. The (marked) moduli space $\mathcal{M}_{2,\text{fm}}$ of quadratic rational maps can be parameterized by the multipliers of the three marked fixed points (ρ_1, ρ_2, ρ_3), with the restriction (from the holomorphic index formula) $\rho_1\rho_2\rho_3 - (\rho_1 + \rho_2 + \rho_3) + 2 = 0$. In this coordinate,

$$\mathcal{H}_{z^2} = \{(\rho_1, \rho_2, \rho_3) : |\rho_1|, |\rho_2| < 1, \rho_3 = \frac{2 - \rho_1 - \rho_2}{1 - \rho_1\rho_2}\} \cong \mathbb{D}^2.$$

A simple computation shows that

$$\partial^\infty\mathcal{H}_{z^2} = \{(e^{2\pi i t}, e^{-2\pi i t})\} = \overline{\partial^\infty_{\mathbb{Q}}\mathcal{H}_{z^2}} = \overline{\{(e^{2\pi i p/q}, e^{-2\pi i p/q})\}}.$$

Note that that when $\rho_1 = \rho_2 = 1$, ρ_3 can be an arbitrary number. Thus, it is easy to see that the exceptional boundary contains infinitely many maps and fibers over $(1, 1)$, i.e., $\boldsymbol{\rho}(\partial_{\text{exc}}\mathcal{H}_{z^2}) = \{(1, 1)\}$. Hence, rigidity fails on the exceptional boundary. Depending on how the multipliers converge to $(1, 1)$, the corresponding sequence $[f_n]$ can be either bounded or divergent.

In the case of $\partial_{\text{reg}}\mathcal{H}_{z^2}$, the pulled-off constant $N_{Siegel}([f])$ can be explicitly bounded. Therefore, applying Theorem 1.6, we obtain:

Theorem C. *Consider $f \in \partial_{reg}\mathcal{H}_{z^2}$ with two neutral fixed points. Let c_1, c_2 be its critical points. Consider the associated postcritical sets P_1, P_2. Then*

- *$f \mid P_1$ and $f \mid P_2$ have degree 1;*

- *P_1, P_2 are separated by some annulus whose modulus depends on the distance of $\rho_1, \rho_2 \in S^1$.*

Proof. For any eventually-golden-mean map f on the boundary $\partial\mathcal{H}_{z^2}$ with two neutral fixed points, by definition, $N_{Siegel}([f]) = O\left(\frac{1}{\operatorname{dist}(\rho_1,\rho_2)}\right)$. By taking the limit and applying Theorem 1.6, the corollary follows. □

1.5. **Outline of the proof of Theorem 1.6.** The proof of Theorem 1.6 breaks up into 2 steps. In the first step, we construct K-quasiconformal disks, and show there are no arc degenerations. In the second step, we show there are no loop degenerations. The proof for both steps are summarized as follows.

Step 1: no arc degeneration. For an attracting Fatou component $D_{i,j}$, we define its *valuable domain* $\widehat{D}_{i,j} \subseteq D_{i,j}$ to be the subdisk of $D_{i,j}$ bounded by the equipotential through the unique critical value of the first return map, see §2.4. We fix the multipliers of all attracting cycles; then, the modulus of the annulus $D_{i,j} - \widehat{D}_{i,j}$ is bounded from below by Lemma 2.7.

Denote the pseudo-core surface (see §3) of $[f]$ by

$$X_f := \widehat{\mathbb{C}} - \bigcup \operatorname{Int}(\widehat{D}_{i,j}) - \bigcup Z_{i,j} \text{ and } K_f := \mathcal{W}_{arc}(X_f).$$

We will argue by contradiction and suppose that K_f can be arbitrarily large.

For a Siegel disk $Z_{i,j}$, the dynamics of its first return map $f_{i,j} \colon \partial Z_{i,j} \circlearrowleft$ is conjugate to some rigid irrational rotation on the circle. The conjugacy gives a combinatorial coordinate on $f_{i,j} \colon \partial Z_{i,j} \circlearrowleft$. The renormalization of the irrational rotation gives a level structure on $\partial Z_{i,j}$: a *level m combinatorial interval* is of the form $J = \left[x, f_{i,j}^{q_{m+1}}(x)\right] \subset \partial Z_{i,j}$, where $f_{i,j}^{q_{m+1}}(x)$ is the closest (level m) return of x, see §3.1. We denote the combinatorial length of a level m combinatorial interval by $\mathfrak{l}_m := |J|$. Note that $\mathfrak{l}_m$ satisfies

$$\frac{0.5}{q_{m+1}} < \mathfrak{l}_m < \frac{1}{q_{m+1}}.$$

1.5.1. *Non-uniform Construction of pseudo-Siegel disks.* In Theorem 3.4, using renormalization theory for $\psi^\bullet$-ql maps, we will construct a collection of pseudo-Siegel disks $\widehat{Z}_{i,j} \supset Z_{i,j}$ whose degenerations are bounded in terms of K_f. Roughly, we will show that each Siegel disk $Z_{i,j}$ is contained in a pseudo-Siegel disk $\widehat{Z}_{i,j} \supset Z_{i,j}$ such that

1. $\widehat{Z}_{i,j}$ is a $M = M(K_f)$-quasiconformal disk;
2. for every interval $J \subseteq \partial\widehat{Z}_{i,j}$ ("grounded" rel $\widehat{Z}_{i,j}$) with $\mathfrak{l}_{m+1} < |J| \leq \mathfrak{l}_m$, we have
   (a) $\mathcal{W}^{+,np}(J) = O(K\mathfrak{l}_m + 1)$; and
   (b) $\mathcal{W}^{+,per}_\lambda(J) = O(\sqrt{K\mathfrak{l}_m} + 1)$.

Here $\mathcal{W}^{+,np}(J)$ is the extremal width of the family of non-peripheral arcs starting at J, and $\mathcal{W}^{+,per}_\lambda(J)$ is the extremal width of the family of peripheral arcs connecting the interval $J \subseteq \partial \widehat{Z}_{i,j}$ to $\partial \widehat{X}_f - \lambda J$ in $\widehat{X}_f := \widehat{\mathbb{C}} - \bigcup \operatorname{Int}(\widehat{D}_{i,j}) - \bigcup \operatorname{Int}(\widehat{Z}_{i,j})$.

Important Properties of pseudo-Siegel disks are discussed in §3.2 and §3.3 (see, in particular, §3.2.4).

1.5.2. *Pulled-off Argument and Localization.* Let $N_f := N_{Siegel}([f])$ be the pulled-off constant. We show that any wide families of non-peripheral arcs in $\widehat{X}_f$ must intersect some strictly periodic psuedo-Siegel disks of pre-period $\leq N_f$ (see Lemma 5.3). It allows us to localize the degeneration (see Theorem 5.1). More precisely, we show that for every $\epsilon > 0$, if the arc degeneration satisfies $K_f \gg_{\epsilon, N_f, \chi(X_f)} 1$, where $\chi(X_f)$ is the Euler characteristic (i.e., complexity), then there exists some interval I on some periodic pseudo-Siegel disk $\widehat{Z}'$ so that

$$\mathcal{W}^{+,np}(I) + \mathcal{W}^{+,per}_{\lambda}(I) \geq K_f/A \quad \text{and} \quad |I| < \epsilon,$$

for some constant $A \equiv A\big(N_f, \chi(X_f)\big) > 1$ independent of ϵ. We may assume $\mathfrak{l}_{m+1} < |I| \leq \mathfrak{l}_m$.

1.5.3. *Calibration Lemma.* Finally, in Theorem 6.1, we show that we can find an interval $J \subseteq \partial \widehat{Z}'$ (grounded rel $\widehat{Z}'$) such that

$$\mathcal{W}^{+,np}(J) \geq K_f/C \quad \text{and} \quad |J| \leq \mathfrak{l}_{m+1} \leq |I| < \epsilon \tag{1.4}$$

for some constant $C \equiv C\big(N_f, \chi(X_f)\big) > A > 1$ independent of ϵ.

By choosing ϵ sufficiently small, we obtain from Property (2) and Estimate (1.4) that

$$K_f/C \leq \mathcal{W}^{+,np}(J) = O(\mathfrak{l}_{m+1} K_f + 1) = O(\epsilon K_f + 1),$$

which is a contradiction.

Remark 1.10. We can summarize the argument in Step 1 as follows. Theorem 3.4 stated in § 1.5.1 says that the arc degeneration $\mathcal{W}_{arc}(X_f)$ of X_f near Siegel disks $Z_{i,j}$ are uniformly distributed along $\partial Z_{i,j}$. On the other hand, Theorem 5.1 stated in § 1.5.2 says that a substantial part of $\mathcal{W}_{arc}(X_f)$ can be localized on a small interval of some $\partial Z_{i,j}$.

The incompatibility of these two facts almost leads to a contradiction. We note, however, that the estimate in (2a) is not sufficient to rule out degeneration on the "special transition scale" (compare with Remark B.3).

A potential degeneration on the special transition scale is handled in Theorem 6.1 stated in § 1.5.3. Combinatorially, such degeneration obeys certain invariance constraints of $f \mid X_f$ (see Figure 6.1). This leads to a contradiction by producing a bigger than K_f degeneration.

Step 2: no loop degeneration.

1.5.4. *Limiting map on a finite tree.* We will argue by contradiction. Suppose there exists a sequence of eventually golden-mean maps $f_n \in \mathcal{H}$ with $\mathcal{W}_{loop}(\widehat{X}_f) \to \infty$. We prove that, after passing to subsequence if necessary, f_n converges to a non-trivial map on a finite tree of Riemann spheres (see Theorem 8.4).

1.5.5. *Duality to multi-curves.* We show that this limiting finite tree is "dual" to some multi-curves in the complement of periodic Fatou components of f_n for all sufficiently large n (see Proposition 8.16). This step crucially uses the fact that the arc degeneration is uniformly bounded.

1.5.6. *Limiting Thurston obstruction.* The dynamics on the tree is recorded by a Markov matrix M and a degree matrix D. We show that there exists a non-negative vector $\vec{v} \neq \vec{0}$ with $M\vec{v} = D\vec{v}$. Since the limiting tree is dual to multi-curves, for all sufficiently large n, we show that $D^{-1}M$ is no bigger than the Thurston matrix for the corresponding multi-curves of f_n. So the spectral radius of the Thurston's matrix is greater or equal to 1 (see Proposition 8.19). This is a contradiction, and Theorem 1.6 follows.

Structure of the paper. In §2, we give preparations and introduce some notations. Four main ingredients in proving uniformly bounded arc degeneration are introduced in §3.7, §4, §5 and §6, and these ingredients are assembled in §7. The uniformly bounded loop degeneration and Theorem A is proved in §8. Theorem 1.6 is proved combining Theorem 7.1 and Theorem 8.1. Finally, Theorem B is proved in §4.

Notations. In this paper, we will usually fix a hyperbolic component. By a *universal constant*, we mean a constant that depends, potentially, only on the hyperbolic component.

We use $A = O(1)$ to mean there exists a universal constant K so that $A \leq K$. More generally, $A = O_x(1)$ means that there exists a constant K_x depending on x so that $A \leq K_x$. Similarly, we use $A \succeq B$ and $A \succeq_x B$ to mean $B/A = O(1)$ and $B/A = O_x(1)$ respectively.

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2. Background on hyperbolic components

In this section, we summarize some background facts on hyperbolic components, and introduce the notion of *eventually-golden-mean maps* on the boundary of a hyperbolic component in §2.3.

2.1. **Marked hyperbolic component.** Following the terminology in [Mil12], a *fixed point marked rational map* $(f; z_0, z_1, ..., z_d)$ is a rational map $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ of degree $d \geq 2$, together with an ordered list of its $d+1$ (not necessarily distinct) fixed points z_j. Let $\mathrm{Rat}_{d,\mathrm{fm}}$ be the space of all fixed point marked rational maps of degree d. The group of Möbius transformation $\mathrm{PSL}_2(\mathbb{C})$ acts naturally on $\mathrm{Rat}_{d,\mathrm{fm}}$ and we define the *marked moduli space*

$$\mathcal{M}_{d,\mathrm{fm}} = \mathrm{Rat}_{d,\mathrm{fm}} \,/\, \mathrm{PSL}_2(\mathbb{C}).$$

The space of marked hyperbolic maps are open in $\mathcal{M}_{d,\mathrm{fm}}$, and a component is called a (marked) hyperbolic component. To avoid complicated notations, we shall use $[f]$ to denote an element in $\mathcal{M}_{d,\mathrm{fm}}$, and simply refer to it as a (marked) map. We remark that as maps vary in a hyperbolic component $\mathcal{H}$, the topology of the Julia sets remains constant.

Definition 2.1. Let $\mathcal{H} \subseteq \mathcal{M}_{d,\mathrm{fm}}$ be a hyperbolic component.

- It is of *disjoint type* if for any map $[f] \in \mathcal{H}$, each grand orbit of a Fatou component of $[f]$ contains a unique critical orbit.
- It is a *Sierpinski carpet hyperbolic component* if the Julia set of any map $[f] \in \mathcal{H}$ is homeomorphic to a Sierpinski carpet.

We remark that $\mathcal{H}$ is a finite branched covering of $\mathcal{H}$. We choose to work with $\mathcal{H}$ as the markings allows us to have a nice parameterization as follows.

Let $\mathcal{H}$ be a hyperbolic component of disjoint type. There are exactly $2d-2$ attracting periodic cycles for a map $[f] \in \mathcal{H}$. Let $\mathcal{C}_1, ..., \mathcal{C}_{2d-2}$ be a list of attracting periodic cycles and let $\rho_1, ..., \rho_{2d-2}$ be the corresponding multipliers. The marking of the fixed points allows us to consistently label these attracting periodic cycles throughout $\mathcal{H}$ (see [Mil12, Theorem 9.3]), and $\mathcal{H}$ is parameterized by the *multiplier profile*, i.e. the multipliers of these $2d-2$ attracting periodic cycles

$$\boldsymbol{\rho} : \mathcal{H} \xrightarrow{\simeq} \mathbb{D}^{2d-2} = \mathbb{D}_1 \times ... \times \mathbb{D}_{2d-2}.$$

2.2. **Transversality for multipliers.** Recall that the boundary

$$\partial\mathcal{H} = \partial_{\mathrm{reg}}\mathcal{H} \sqcup \partial_{\mathrm{exc}}\mathcal{H} \subseteq \mathcal{M}_{d,\mathrm{fm}}$$

splits into the *regular* and *exceptional* parts, where $[f] \in \partial_{\mathrm{reg}}\mathcal{H}$ if $[f]$ has exactly $2d-2$ non-repelling periodic cycles, and $[f] \in \partial_{\mathrm{exc}}\mathcal{H}$ otherwise.

Let $[f] \in \partial_{\mathrm{reg}}\mathcal{H}$. Let x be a non-repelling periodic point of f with period p. Suppose $[f_n] \in \mathcal{H}$ with $f_n \to f$, and $x_n \to x$ be a sequence of non-repelling periodic points of f_n. We classify the non-repelling periodic point x into three categories:

- Type (1): The multiplier of $[f]$ at x is not 1 and x_n has period p;
- Type (2): The multiplier of $[f]$ at x is not 1 and x_n has period νp, with $\nu \geq 2$;
- Type (3): The multiplier of $[f]$ at x is 1.

Proposition 2.2. *The multiplier map extends to an embedding on the regular boundary* $\boldsymbol{\rho} : \partial_{\text{reg}}\mathcal{H} \hookrightarrow \partial\mathbb{D}^{2d-2}$ *and* $\boldsymbol{\rho}(\partial_{\text{reg}}\mathcal{H}) \cap \boldsymbol{\rho}(\partial_{\text{exc}}\mathcal{H}) = \emptyset$*.*
Here $\boldsymbol{\rho}(\partial_{\text{reg}}\mathcal{H})$ *or* $\boldsymbol{\rho}(\partial_{\text{exc}}\mathcal{H})$ *are understood as the accumulation set of* $\boldsymbol{\rho}([f_n])$ *as* $[f_n] \to \partial_{\text{reg}}\mathcal{H}$ *or* $[f_n] \to \partial_{\text{exc}}\mathcal{H}$ *respectively.*

Proof. Let $[f] \in \partial_{\text{reg}}\mathcal{H}$. Let us first suppose that every non-repelling periodic point of f is Type (1). Let $\mathcal{C}_i$ be a list of non-repelling periodic cycles of $[f]$. Then by implicit function theorem, the cycles $\mathcal{C}_i$ move holomorphically on a neighborhood of f. Thus, we can define a holomorphic map $(\rho_1, ..., \rho_{2d-2}) : U \longrightarrow \mathbb{C}^{2d-2}$ on a neighborhood U of $[f]$, where ρ_i is the multiplier of the cycle $\mathcal{C}_i$. By transversality of the multipliers (see [Lev10, Theorem 6] or [Eps00]), $(\rho_1, ..., \rho_{2d-2})$ gives a local parameterization of the moduli space $\mathcal{M}_{d,\text{fm}}$. Therefore, $\boldsymbol{\rho}$ extends to an embedding near $[f]$, and $\boldsymbol{\rho}^{-1}(\boldsymbol{\rho}([f])) = \{[f]\}$.

If there are Type (2) or Type (3) non-repelling periodic points, the argument is similar, but we need to pass to a branched cover. Indeed, we can consider the space of *n-periodic marked rational maps* consisting of

$$(f; x_0, ..., x_{d^n}) \in \text{Rat}_d \times \widehat{\mathbb{C}}^{d^n-1},$$

where f is a rational map of degree d together with an ordered list of its d^n+1 (not necessarily distinct) periodic points dividing n. Since the iteration map $\text{Rat}_d \longrightarrow \text{Rat}_{d^n}$ is a local immersion (see [Ye15, Proposition 4.1]), by pulling back the local charts for $\text{Rat}_{d^n,\text{fm}}$, we have local holomorphic charts near any n-periodic marked rational map. Note that the forgetful map from n-periodic marked rational maps to fixed point marked rational maps is a branched covering.

If x is a Type (2) point, then two or more periodic points in the same periodic cycle of f_n collide in the limit, as we assume $[f] \in \partial_{\text{reg}}\mathcal{H}$. Denote the period νp and p cycle by $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ respectively. By marking these periodic points, we may assume the $\widetilde{\mathcal{C}}$ and $\mathcal{C}$ move holomorphically on this branched covering, and their multipliers ρ and $\widetilde{\rho}$ are holomorphic functions.

Similarly, if x is a Type (3) point, then there is a period p repelling point $\tilde{x}_n$ of f_n with $\tilde{x}_n \to x$. Denote these two cycles by $\mathcal{C}$ and $\widetilde{\mathcal{C}}$. By marking these periodic points, we may assume that $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ move holomorphically, and their multipliers ρ and $\widetilde{\rho}$ are holomorphic functions.

In this way, there exists a neighborhood U of $[f]$ and a branched cover $\widetilde{U}$ of U so that the multipliers map $(\rho_1, ..., \rho_k, \rho_{k+1}, \widetilde{\rho}_{k+1}, ..., \rho_{2d-2}, \widetilde{\rho}_{2d-2})$ is a holomorphic map on $\widetilde{U}$, where k is the number of Type (1) cycles. By transversality of the multipliers (see [Lev10, Theorem 6] or [Eps00]) and restrict the domain if necessary, the map is a finite branched covering onto its image, and the image of $\widetilde{U}$ under the restricted multiplier map $(\rho_1, ..., \rho_k, \rho_{k+1}, \rho_{k+2}, ..., \rho_{2d-2})$ is open.

It is easy to see that f has at least 3 distinct fixed point. Thus, $\text{Aut}(f) = \{id\}$, where $\text{Aut}(f)$ is the automorphism group of the fixed point marked rational map f. Therefore, the fiber of the branched cover $\widetilde{U} \longrightarrow U$ consists only of the same map with (potentially) a different marking on the periodic points. By fixing a marking of the attracting periodic points in $\mathcal{H}$, we obtain a homeomorphic lift $\widetilde{V} \subseteq \widetilde{U}$ of $V = \mathcal{H} \cap U$. The branched cover $\widetilde{U} \longrightarrow U$ is injective on $\overline{\widetilde{V}}$, and hence a homeomorphism between $\overline{\widetilde{V}}$ and $\overline{V}$. By lifting the map $\boldsymbol{\rho}$ from V to $\widetilde{V}$, it is now easy to see that $\boldsymbol{\rho}$ extends to an embedding of $\partial\mathcal{H}$ near $[f]$, and $\boldsymbol{\rho}^{-1}(\boldsymbol{\rho}([f])) = \{[f]\}$. The proposition now follows. □

2.3. **Eventually-golden-mean maps.** In this subsection, we will fix a hyperbolic component $\mathcal{H}$ of disjoint type.

Let $\theta \in (0, 1)$ be an irrational number, with continued fraction expansion

$$\theta = [0; a_1, ..., a_m, ...] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}.$$

We say θ is of *bounded type* if

$$\sup\{a_m\} < \infty.$$

More generally, we say θ is *Brjuno* if

$$\sum \frac{\log \mathfrak{q}_{m+1}}{\mathfrak{q}_n} < \infty.$$

Note that if θ is of bounded type, then θ is Brjuno.

These arithmetic properties of irrational numbers are relevant to holomorphic dynamics. It is well-known that if f is a holomorphic map defined on $0 \in U$, with $f(0) = 0$ and $f'(0) = e^{2\pi i\theta}$ with θ being Brjuno, then f is conjugate to the rigid rotation $z \mapsto e^{2\pi i\theta} z$ in a neighborhood of 0. If f is a globally defined, then this neighborhood is part of a Siegel disk for f.

If f is a rational map with a fixed point of multiplier $e^{2\pi i\theta}$ with θ of bounded type, then the corresponding Siegel disk has quasi-circle boundary which passes through at least one critical point [Zha11].

Let $\theta = [0; a_1, ..., a_m, ...]$. We say it is *eventually-golden-mean* if there exists m_θ so that $a_n = 1$ for all $n \geq m_\theta$. Note that in this case, θ is automatically of bounded type.

Let $\mathcal{H}$ be a hyperbolic component of disjoint type, and $[f] \in \partial\mathcal{H}$. Then some attracting periodic cycles must become indifferent. By following the deformations for the corresponding periodic cycles, its multiplier profile $(\rho_1, ..., \rho_{2d-2}) = (\rho_1([f]), ..., \rho_{2d-2}([f]))$ lies on the boundary

$$(\rho_1, ..., \rho_{2d-2}) \in \partial\mathbb{D}^{2d-2}.$$

Definition 2.3. We say a boundary parameter $(\rho_1, ..., \rho_{2d-2}) \in \partial\mathbb{D}^{2d-2}$ is

- *rational* if each ρ_j is either in $\mathbb{D}$ or $\rho_j \in S^1$ and is rational;
- *irrational* if each ρ_j is either in $\mathbb{D}$ or $\rho_j \in S^1$ and is irrational;
- *eventually-golden-mean* if each ρ_j is either in $\mathbb{D}$ or $\rho_j \in S^1$ and is eventually-golden-mean.

We also say $(\rho_1, ..., \rho_{2d-2})$ is *realizable* if there exists $[f] \in \partial\mathcal{H} \subseteq \mathcal{M}_{d,\mathrm{fm}}$ with multiplier profile $(\rho_1, ..., \rho_{2d-2})$.

Let $\mathfrak{S} \subseteq \partial\mathbb{D}^{2d-2}$ be the set of realizable eventually-golden-mean boundary parameter. In this paper, we will focus on the following maps:

Definition 2.4. A map $[f] \in \partial\mathcal{H} \subseteq \mathcal{M}_{d,\mathrm{fm}}$ is called an *eventually-golden-mean map* if its multiplier profile $(\rho_1, ..., \rho_{2d-2}) \in \mathfrak{S}$. We denote by $\partial_{\mathrm{egm}}\mathcal{H} \simeq \mathfrak{S}$ the set of all such maps in $\partial\mathcal{H}$.

We remark that since eventually-golden-mean irrational numbers are of bounded type, any non-repelling cycles of a eventually-golden-mean map $[f]$ are contained either in (super-)attracting Fatou components or Siegel disks. Moreover, since its multiplier profile is on the boundary $\partial\mathbb{D}^{2d-2}$, there is at least one cycle of Siegel disk for $[f]$.

Definition 2.5. Let $(\rho_1, ..., \rho_{2d-2}) \in \partial\mathbb{D}^{2d-2}$. A sequence $(\rho_{1,n}, ..., \rho_{2d-2,n}) \in \partial\mathbb{D}^{2d-2}$ is said to converge to $(\rho_1, ..., \rho_{2d-2})$ *strongly*, denoted by

$$(\rho_{1,n}, ..., \rho_{2d-2,n}) \to_s (\rho_1, ..., \rho_{2d-2}),$$

if $\rho_{j,n} \to \rho_j$ for all j, and $\rho_{j,n} = \rho_j$ when $|\rho_j| < 1$.

If we further assume that the Julia set is a Sierpinski carpet, then we have the following density result for eventually-golden-mean maps.

Proposition 2.6. *Let $\mathcal{H}$ be a Sierpinski carpet hyperbolic component of disjoint type. The set $\mathfrak{S}$ is dense in $\partial\mathbb{D}^{2d-2}$.*

Moreover, for any $(\rho_1, ..., \rho_{2d-2}) \in \partial\mathbb{D}^{2d-2}$, there exists a sequence

$$(\rho_{1,n}, ..., \rho_{2d-2,n}) \in \mathfrak{S}$$

converging to $(\rho_1, ..., \rho_{2d-2})$ strongly.

Proof. It follows from the pinching deformation in [CT18] (see also [Luo22b]) that all rational boundary points are realizable. Let $[f] \in \partial\mathcal{H}$ with rational multiplier profile $(\rho_1, ..., \rho_{2d-2})$. Since $\mathcal{H}$ is Sierpinski, no non-repelling periodic points collide. We may assume $\rho_i \neq 1$ for all i, as other wise, we can pass to a branched cover as in Proposition 2.2. Hence, we can locally parameterized the periodic cycles analytically. Thus, there exists a neighborhood $U \subseteq \mathcal{M}_{d,\mathrm{fm}}$ of $[f]$ so that the multipliers

$$\boldsymbol{\rho}(t) := (\rho_1(t), ..., \rho_{2d-2}(t))$$

is an analytic function on $t \in U$. By transversality of the multipliers (see [Lev10, Theorem 6] or [Eps00]), $\boldsymbol{\rho}^{-1}((\rho_1, ..., \rho_{2d-2})) = \{[f]\}$. Thus by shrinking U if necessary, the image $\boldsymbol{\rho}(U) \subseteq \mathbb{C}^{2d-2}$ is open (see [GR84, p. 107]). Since eventually-golden-mean irrational numbers are dense, we can find an eventually-golden-mean map $[f] \in U$. Since the rational parameters are dense, $\mathfrak{S}$ is dense. The moreover part can be proved in the same way. $\square$

2.4. **Valuable-attracting domains.** Let D be an attracting Fatou component for f of period p. Assume that the multiplier of the attracting periodic point is ρ. Then the first return map $f^p : D \longrightarrow D$ is conjugate to the Blaschke product

$$\begin{aligned} F : \mathbb{D} &\longrightarrow \mathbb{D} \\ z &\mapsto z\frac{z+\rho}{1+\bar{\rho}z}. \end{aligned}$$

Let $\Psi : D \longrightarrow \mathbb{D}$ be the conjugacy map, and let $r := \max\{\frac{1}{2}, |\rho|\}$. We call the closed Jordan disk

$$\widehat{D} = \Psi^{-1}(\overline{B(0,r)}) \subseteq D$$

the *valuable-attracting domain* for D. One can easily verify by our construction that

Lemma 2.7. *Let $\widehat{D}$ be the valuable-attracting domain for D. Then*

- *$\widehat{D}$ is forward invariant under f^p;*
- *$\widehat{D}$ contains the unique critical point of f^p in D;*
- *The annulus $D - \widehat{D}$ has modulus $-\dfrac{1}{2\pi}\log(\max\{\frac{1}{2}, |\rho|\})$.*

3. Core and pseudo-Core Surfaces of maps in $\partial_{\text{egm}}\mathcal{H}$

In this section, we will introduce pseudo-Siegel disks and pseudo-core surfaces. The main construction is in Theorem 3.4; see also §1.5.1.

Let us fix a hyperbolic component $\mathcal{H}$ of disjoint type. Recall from Definition 2.4 that $\partial_{\text{egm}}\mathcal{H}$ denotes the set of eventually-golden-mean maps in $\partial\mathcal{H}$: every neutral periodic cycle of a map in $\partial_{\text{egm}}\mathcal{H}$ is Siegel of the eventually-golden-mean type.

We discuss some combinatorial facts of irrational rotations on a circle in §3.1. In §3.2, we review the notion of almost-invariant pseudo-Siegel disks. They are obtained from regular forward-invariant Siegel disks by filling-in parabolic fjords as illustrated on Figure 3.2; see Definition 3.1.

The *core surface* X_f of $f \in \partial_{\text{egm}}\mathcal{H}$ is the complement to the union of all periodic valuable-attracting domains and Siegel disks; see (3.6). The *pseudo-core surface* $\widehat{X}_f \subset X_f$ is obtained by removing pseudo-Siegel disks instead of Siegel disks; see (3.7). Properties of X_f and $\widehat{X}_f$ are discussed in §3.5. We remark that some terminologies for degeneration of Riemann surfaces are summarised in §A.

3.1. **Combinatorial intervals for Siegel disks.** In this subsection, we introduce the terminologies for dynamics on Siegel disks. We remark that most of the discussions in this section work for any rational map with a Siegel disk with a single critical point on its boundary.

Let $\mathcal{H}$ be a hyperbolic component of disjoint type. Let $[f] \in \partial\mathcal{H}$ be an eventually-golden-mean map. Let Z be a Siegel disk for f of period p

with rotation number θ. Let $h : Z \longrightarrow \mathbb{D}$ be a Riemann mapping with $h(\alpha) = 0$, where α is the fixed point in Z. Since Z is a quasi-disk, h extends continuously to

$$h : \overline{Z} \longrightarrow \overline{\mathbb{D}},\ h(\alpha) = 0$$

which conjugate $f^p|_{\partial Z}$ with the rigid rotation on S^1.

We define the *combinatorial length* of an interval $I \subseteq \partial Z$ as

$$|I| := |h(I)|_{\mathbb{R}/\mathbb{Z}} \in (0, 1).$$

Similarly, we define the *combinatorial distance* between $x, y \in \partial Z$ as

$$\operatorname{dist}(x, y) := \operatorname{dist}_{\mathbb{R}/\mathbb{Z}}(h(x), h(y)) \in [0, 1/2].$$

Let $x \in \partial Z$ and $t \in \mathbb{R}/\mathbb{Z}$, we set

$$x \boxplus t = h^{-1}(h(x) + t),$$

i.e., $x \boxplus t$ is x rotated by angle t. Note that $f^p(x) = x \boxplus \theta$ for all $x \in \partial Z$.

Let $[0; a_1, ..., a_m, ...]$ be the continued fraction expansion for θ. Let

$$\mathfrak{p}_m/\mathfrak{q}_m := \begin{cases} [0; a_1, ..., a_m], & \text{if } a_1 > 1 \\ [0; a_1, ..., a_{m+1}], & \text{if } a_1 = 1 \end{cases}$$

be the sequence of approximations for θ given by the continued fraction. We use the convention and set $\mathfrak{q}_0 = 1$. Then $f^{\mathfrak{q}_0 p} = f^p, f^{\mathfrak{q}_1 p}, ...$ is the sequence of first returns of $f^p|_{\partial Z}$, i.e.,

$$\operatorname{dist}(f^{ip}(x), x) > \operatorname{dist}(f^{\mathfrak{q}_m p}(x), x) =: \mathfrak{l}_m,\ x \in \partial Z \ \text{ for all } i < \mathfrak{q}_m.$$

We define $\theta_m \in (-1/2, 1/2)$ so that

$$f^{\mathfrak{q}_m p}(x) = x \boxplus \theta_m.$$

Note that $\mathfrak{l}_m = |\theta_m|$.

Given two points $x, y \in \partial Z$ with $\operatorname{dist}(x, y) < 1/2$, we let $[x, y]$ be the shortest closed interval of ∂Z between x, y. Let $I \subseteq \partial Z$ be an interval. We define the λ-scaling of I as

$$\lambda I := \{x \in \partial Z : \operatorname{dist}(x, I) \le (\lambda - 1)|I|/2\}.$$

An interval $I \subseteq \partial Z$ is called a *combinatorial interval of level* m, or simply a *level* m *interval* if $|I| = \mathfrak{l}_m$. Note that a level m interval is of the form

$$I = [x, f^{\mathfrak{q}_m p}(x)].$$

Let I be a level m interval. We say the intervals

$$\{f^{ip}(I) : i \in \{0, 1, ..., \mathfrak{q}_{m+1} - 1\}$$

are obtained by *spreading around* I. We enumerate these intervals counterclockwise starting with $I = I_0$

$$I_0 = I, I_1 = f^{i_1 p}(I), ..., I_{\mathfrak{q}_{m+1}-1} = f^{(i_{\mathfrak{q}_{m+1}-1})p}(I),\ i_j \in \{1, 2, ..., \mathfrak{q}_{m+1} - 1\}.$$

Note that the interval I_i is either attached to I_{i+1} or there is a level $m + 1$ combinatorial interval between I_i and I_{i+1}.

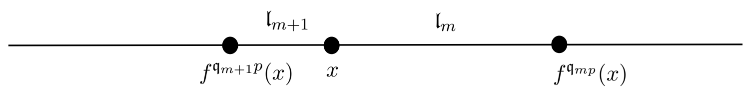

FIGURE 3.1. The first return and combinatorial intervals.

3.1.1. *Diffeo-tiling* $\mathfrak{D}_m$. There is a unique critical point c of f^p on ∂Z. We denote by $\mathrm{CP}_m = \mathrm{CP}_m(Z)$ the set of critical points of $f^{\mathfrak{q}_{m+1}p}$ on ∂Z. We define the *diffeo-tiling* $\mathfrak{D}_m$ of level m as the partition of ∂Z induced by CP_m. Note that there are $\mathfrak{q}_{m+1}$ intervals in $\mathfrak{D}_m$, and each interval has length either $\mathfrak{l}_m$ or $\mathfrak{l}_m + \mathfrak{l}_{m+1}$.

3.2. **Pseudo-Siegel disks.** A pseudo-Siegel disk $\widehat{Z}^m$ is obtained from a Siegel disk $\overline{Z}$ by filling-in all parabolic fjords of levels $\geq m$. The formal definition of $\widehat{Z}^m$ for maps in $\partial_{\mathrm{egm}}\mathcal{H}$ (see §3.2.2) is the same as for quadratic polynomials with the additional requirement that the "territory" $\mathcal{X}(\widehat{Z}^m)$ containing all auxiliary objects of $\widehat{Z}^m$ is peripheral rel $\overline{Z}$; see §3.2.1 and Property (P) in §3.2.2.

3.2.1. *Parabolic fjords and their protections.* As in §3.1, let Z be a periodic Siegel disk of $[f] \in \partial_{\mathrm{egm}}\mathcal{H}$ with period p and rotation number θ.

We say that a disk $D \supset Z$ is *peripheral rel* $\overline{Z}$ if $D \setminus \overline{Z}$ does not intersect the post-critical set of f. More generally, we say that a set $S \subset \widehat{\mathbb{C}}$ is *peripheral rel* $\overline{Z}$ if S is within a peripheral disk D. In other words, S is peripheral rel $\overline{Z}$ if S can be "contracted" rel the postcritical set into $\overline{Z}$.

Consider a diffeo-tiling $\mathfrak{D}_m$ (see §3.1.1) and an interval $I \in \mathcal{D}_m$. Given a peripheral curve $\beta \subset \widehat{\mathbb{C}} \setminus Z$ with endpoints in I, set $\mathfrak{F}_\beta$ to be the closure of the connected component of $\widehat{\mathbb{C}} \setminus (Z \cup \beta)$ enclosed by $\beta \cup I$. If $\mathfrak{F}_\beta$ is peripheral rel $\overline{Z}$, then we call $\mathfrak{F}_\beta$ the *parabolic fjord* bounded by β; see Figure 3.2. We will refer to β as the *dam* of $\mathfrak{F}_\beta$.

Let $\mathcal{X} \subset \widehat{\mathbb{C}} \setminus Z$ be a rectangle with

$$\partial^h \mathcal{X} \subset \mathring{I} \ := \ I \setminus \{\text{ends of } I\}.$$

We denote by $\overset{\star}{\mathcal{X}}$ the union of $\mathcal{X}$ and the closure of the connected component of $\widehat{\mathbb{C}} \setminus (\mathcal{X} \cup I)$ enclosed by $\partial^v \mathcal{X} \cup \mathring{I}$. We say that $\mathcal{X}$ *protects* a fjord $\mathfrak{F}_\beta$ if

- $\mathfrak{F}_\beta \subset \overset{\star}{\mathcal{X}} \setminus \mathcal{X}$;
- $\overset{\star}{\mathcal{X}}$ is peripheral rel $\overline{Z}$.

3.2.2. *Pseudo Siegel disks for rational maps in $\partial_{\mathrm{egm}}\mathcal{H}$.*

Definition 3.1. Following notations from §3.2.1, a *pseudo-Siegel disk* $\widehat{Z}^m$ of $m \geq -1$ and its *territory* $\mathcal{X}(\widehat{Z}^m) \supset \widehat{Z}^m$ are disks inductively constructed as follows (from bigger m to smaller ones):

(1) $\widehat{Z}^m = \overline{Z}$ and $\mathcal{X}(\widehat{Z}^m) = \overline{Z}$ for all sufficiently large $m \gg 0$,

(2) either

$$\widehat{Z}^m := \widehat{Z}^{m+1} \text{ and } \mathcal{X}(\widehat{Z}^m) := \mathcal{X}(\widehat{Z}^{m+1}),$$

or for every interval $I \in \mathfrak{D}_m$ there is

- a parabolic peripheral fjord $\mathfrak{F}_I \equiv \mathfrak{F}_{\beta_I}$ bounded by its dam β_I with endpoints in I; and
- a peripheral rectangle $\mathcal{X}_I$ protecting $\mathfrak{F}_I$

such that

$$\begin{aligned} \widehat{Z}^m &:= \widehat{Z}^{m+1} \bigcup_{I \in \mathfrak{D}_m} \mathfrak{F}_I \\ \mathcal{X}(\widehat{Z}^m) &:= \mathcal{X}(\widehat{Z}^{m+1}) \cup \bigcup_{I \in \mathfrak{D}_m} \overset{\star}{\mathcal{X}}_I \end{aligned} \tag{3.1}$$

and such that $\widehat{Z}^m$ and $\mathcal{X}(\widehat{Z}^m)$ satisfy 7 compatibility condition stated in [DL22, § 5.1] and briefly summarized in §3.2.4.

We remark that in addition to 7 compatibility conditions from [DL22, § 5.1], a pseudo-Siegel disk satisfies the following additional property:

(P) $\mathcal{X}(\widehat{Z}^m)$ is peripheral rel $\overline{Z}$.

If $\widehat{Z}^m \neq \widehat{Z}^{m+1}$, then we say $\widehat{Z}^m$ is a *regularization* of $\widehat{Z}^{m+1}$ at level m. We denote $\widehat{Z} = \widehat{Z}^{-1}$, and call it the pseudo-Siegel disk. We remark that Definition 3.1 allows us to potentially take $\widehat{Z}^n = \overline{Z}$ and $\mathcal{X}(\widehat{Z}^n) = \overline{Z}$ for all n, which will satisfy all the compatible conditions. Thus, $\overline{Z}$ is trivially a pseudo-Siegel disk (of any level). Similarly, any level m pseudo-Siegel disk $\widehat{Z}^m$ can be extended to lower levels by setting $\widehat{Z}^n = \widehat{Z}^m$ for all $n \leq m$. With this in mind, when we introduce definitions or state theorems for pseudo-Siegel disks, they apply to regular Siegel disks as well.

3.2.3. *Regular intervals.* A point $x \in \partial\widehat{Z}^m$ is *regular* if $x \in \partial\widehat{Z}^m \cap \partial Z$. By construction, if a point $x \in \partial\widehat{Z}^m$ is regular, then x is regular on $\partial\widehat{Z}^k$ for all $k \geq m$. A *regular interval* $I \subseteq \partial\widehat{Z}^m$ is an interval with regular endpoints.

The projection of a regular interval $I \subseteq \partial\widehat{Z}^m$ onto ∂Z is the interval $I^\bullet \subseteq \partial Z$ with the same endpoints and the same orientation as I. We define the combinatorial length of I by $|I| := |I^\bullet|$. Similarly, we can define the projection I^k of a regular interval $I \subseteq \partial\widehat{Z}^m$ onto $\partial\widehat{Z}^k$ for $k > m$.

For an interval $I \subseteq \partial Z$, the projection I^m onto ∂Z^m is the smallest regular interval whose projection onto ∂Z contains I. Similarly, we can define the projection of an interval $I \subseteq \partial\widehat{Z}^m$ onto $\partial\widehat{Z}^n$ for $n < m$.

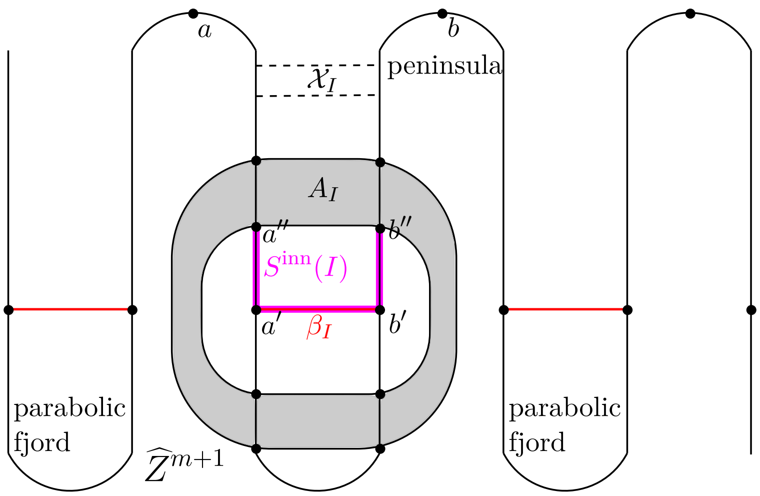

FIGURE 3.2. An illustration of psuedo-Siegel disk. The intersection patterns of the protecting annulus A_I, the inner buffer $S^{\text{inn}}(I)$, extra outer protection $\mathcal{X}_I$ are indicated on the graph.

Let $I \subseteq \partial\widehat{Z}^m$ be a regular interval. Abusing the notations, we denote $\lambda I \subseteq \partial\widehat{Z}^m$ as the projection of $\lambda I^\bullet$ on $\partial\widehat{Z}^m$ where $I^\bullet \subseteq \partial Z$ is the projection of I onto ∂Z.

3.2.4. *Compatibility conditions between $\widehat{Z}^m$ and $\overline{Z}$.* The 7 compatibility conditions stated in [DL22, § 5.1] are designed to ensure the following key-properties of $\widehat{Z}^m$:

(A) $\widehat{Z}^m$ is almost invariant under f^i for $|i| \le \mathfrak{q}_{m+1}$;
(B) the "slight" shrinking

$$\widehat{\mathbb{C}} \setminus \overline{Z} \ \rightsquigarrow\ \widehat{\mathbb{C}} \setminus \widehat{Z}^m$$

has small affect on the width of rectangles in $\widehat{\mathbb{C}} \setminus \overline{Z}$ that have vertices in $\widehat{\mathbb{C}} \setminus \mathcal{X}(\widehat{Z}^m)$; see Lemma 3.2

Below we will recall the main aspects of the axiomatization of $\widehat{Z}^m$ from [DL22, § 5.1]. Various minor technical conditions will be omitted. We remark that $\widehat{Z}^m$ can be defined explicitly using explicite hyperbolic geodesics in the complement of $\overline{Z}$; see Appendix B.5.

Property (B) follows the requirement that all $\mathcal{W}(\mathcal{X}_I)$ are sufficiently wide and will be discussed in §3.2.5.

Let us now discuss (A). It follows from (3.1) that

(C) that critical points CP_m of $f^{\mathfrak{q}_{m+1}p} \cap \partial Z$ are regular points of $\widehat{Z}^n$ for any $n \ge m$.

In particular, the projections I^m of $I \in \mathfrak{D}_m$ induce a well-defined diffeo-tiling of $\partial \widehat{Z}^n$.

As illustrated on Figure 3.2, for all $I \subset \mathfrak{D}_m$, we require the existence of annuli A_I around the β_I with

$$\text{mod } (A_I) \geq \delta > 0, \qquad \text{where } \delta > 0 \text{ is small but fixed} \tag{3.2}$$

such that for all $|i| \leq q_{m+1}$ the annuli $(A_I)_{I\in\mathfrak{D}_m}$ control the difference between $f^i(\widehat{Z}^m)$ and $\widehat{Z}^m$ in the following sense.

(D) Assume $f^i \colon \overline{Z} \to \overline{Z}$ maps $J \in \mathfrak{D}_m$ into most of the $J \in \mathfrak{D}_m$; i.e., the difference $f^i(J) \setminus I$ is either empty or consists of an interval in $\mathfrak{D}_{m+1}$. Then we require that A_I also surrounds $f^i(\beta_J)$.

We remark that A_I was denote by $A^{\text{out}}(\beta_I)$ in [DL22].

Write $I = [a, b] \subset \partial\overline{Z}$ and denote by a', b' the endpoints of β_I as shown on Figure 3.2. Denote by a'', b'' the intersection of the inner boundary of A with $[a, a']^{m+1}, [b', b'']^{m+1} \subset I^{m+1}$, where the superscript indicates the projections of the intervals onto $\widehat{Z}^{m+1}$. The *inner buffer* is defined by

$$S^{\text{inn}}(I) := \; [a'', b'']^m \; = \; [a'', a']^{m+1} \cup \beta_I \cup [\beta', \beta'']^{m+1} \; \subset \partial\widehat{Z}^m.$$

We also define

$$S^{\text{inn}}(\widehat{Z}^m) := \bigcup_{n\geq m,\ I} S^{\text{inn}}(I) \subset \partial\widehat{Z}^m,$$

where the union is taken over all $I \in \mathfrak{D}_n$ and $n \geq m$.

It is required that there is an annulus A_I^{inn} separating $\{a'',\ b''\}$ from β_I with mod $(A^{\text{inn}}) \geq \delta$ such that $(A_I^{\text{inn}})_{I\in\mathfrak{D}_m}$ also control the difference between $f^i(\widehat{Z}^m)$ and $\widehat{Z}^m$ as in (D) above. In short:

- the $A_I \equiv A_I^{\text{out}}$ guarantee that wide families typically submerge into $\widehat{Z}^m$ through "grounded intervals," see §3.2.6;
- the A_I^{inn} guarantee that $f^i \mid \widehat{Z}^m$ is "geometrically close" to the standard rigid rotation; conseequently, $\partial\widehat{Z}^m$ has inner geometry similar to that of a Siegel disk; see [DL22, Theorem 5.12].

3.2.5. *Robustness of the outer geometry under* $\widehat{\mathbb{C}} \setminus \overline{Z} \rightsquigarrow \widehat{\mathbb{C}} \setminus \widehat{Z}^m$*.* In [DL22, Remark 5.11], it is assumed that

$$\mathcal{W}(\mathcal{X}_I) \geq \Delta, \qquad \text{where } \Delta > 0 \text{ is sufficiently big but fixed}$$

for all $I \in \mathfrak{D}_n$ and all $n \geq m$.

Consider a rectangle $\mathcal{R} \subset \widehat{\mathbb{C}} \setminus Z$. Assume that:

- vertices $V_\mathcal{R}$ of $\mathcal{R}$ are outside of $\text{int}(\widehat{Z}^m)$; and
- $\mathcal{R} \setminus \widehat{Z}^m$ has a connected component $\mathcal{R}'$ such that $V_\mathcal{R} \subset \partial\mathcal{R}'$.

Then $\mathcal{R}'$ is Jordan domain, and we view its closure $\mathcal{R}^m := \overline{\mathcal{R}}'$ as a rectangle with vertex set $V_\mathcal{R}$ and the same orientation of sides as $\mathcal{R}$. We call $\mathcal{R}^m$ the *restriction* of $\mathcal{R}$ to $\widehat{\mathbb{C}} \setminus \text{int}(\widehat{Z}^m)$.

Lemma 3.2 ([DL22, (5.12) in §5.2.4])**.** *If $\mathcal{R}^m$ is the restriction of a rectnalge $\mathcal{R}$ to $\widehat{\mathbb{C}} \setminus \operatorname{int}(\widehat{Z}^m)$ as above and if the vertex set $V_{\mathcal{R}}$ of $\mathcal{R}$ is outside of $\operatorname{int} \mathcal{X}(\mathcal{Z}^m)$, then*

$$1 - \varepsilon_\Delta < \frac{\mathcal{W}(\mathcal{R}^m)}{\mathcal{W}(\mathcal{R})} \leq 1 + \varepsilon_\Delta,$$

where $\varepsilon_\Delta \to 0$ as $\Delta \to \infty$.

3.2.6. *Grounded intervals.* We will be usually working with a special type of regular intervals called *grounded intervals*. An interval $I \subseteq \partial \widehat{Z}^m$ is a grounded interval if the end points ∂I are in $\partial \widehat{Z}^m - S_{\text{inn}}(\widehat{Z}^m)$.

Lemma 3.2 implies the following fact about grounded intervals. For a pair $I, J \subset \partial \widehat{Z}^m$ of disjoint grounded intervals, consider a rectangle

$$\mathcal{R} \subset \widehat{\mathbb{C}} \setminus Z \qquad \text{with } \partial^{h,0}\mathcal{R} = I, \quad \partial^{h,1}\mathcal{R} = J.$$

Since I, J are grounded, $\mathcal{R}$ restrict to a rectangle $\mathcal{R}^m$ in $\widehat{\mathbb{C}} \setminus \widehat{Z}^m$ with $\partial^{h,0}\mathcal{R}^m = I^m$ and $\partial^{h,1}\mathcal{R}^m = J^m$.

Then the argument in [DL22, Lemma 5.10] implies that

$$\mathcal{W}(\mathcal{R}) - O_\delta(1) < \mathcal{W}(\mathcal{R}^m) < (1 + \varepsilon_\Delta)\mathcal{W}(\mathcal{R}) + O_\delta(1). \tag{3.3}$$

In the paper, we will usually replace "$1 + \varepsilon$" with "2"; see for example Proposition 3.3.

3.3. **Stability of $\widehat{Z}^m$ and pseudo-bubbles.** Recall that Z has period p. Given a peripheral closed disk $D \supset \overline{Z}$ and an iteration f^{np}, set $\widetilde{D}$ to be the closure of the connected component of $f^{-np}(\operatorname{Int}(D))$ containing Z. If $\widetilde{D} \xrightarrow{f^{np}} D$ has degree 1 (i.e., it is a homeomorphism), then we call

$$D(-np) \ := \ \widetilde{D}$$

the *pullback* of D under f^{np} (rel $\overline{Z}$).

Observe that $D(-np)$ is well defined if and only if $\partial(D \setminus \overline{Z})$ does not contain any critical value of f^{np}. Since D is peripheral, every critical value of f^{np} in D are necessary on ∂Z.

We say that a pseudo-Siegel disk $\widehat{Z}^m$ is k-stable if for every $n \leq k q_{m+1}$ the pullback of $\mathcal{X}(\widehat{Z}^m)$ under f^{np} is well defined. It follows then

$$[\widehat{Z}^m(-n), \mathcal{X}(\widehat{Z}^m)(-n)] := (f^n)^*[\widehat{Z}^m, \mathcal{X}(\widehat{Z}^m)]$$

is a well-defined pseudo-Siegel disk together with its territory.

For every $I = [a, b] \in \mathfrak{D}_m$, set

$$k_I := \frac{\operatorname{dist}(\partial^h \mathcal{X}_I, \ \{a, b\})}{\mathfrak{l}_{m+1}} - 2, \qquad k_I^+ := \max\{k_I, 0\},$$

and $k_m := \min\limits_{n \geq m} \ \min\limits_{I \in \mathfrak{D}_n} \{k_I^+\}$. Then it follows from the above discussion that $\widehat{Z}^m$ is k_m-stable.

Observe that if $\widehat{Z}^m$ is T-stable, then so is any $\widehat{Z}^n$ for $n \geq m$. In particular, if $\widehat{Z} = \widehat{Z}^{-1}$ is T-stable, then $\widehat{Z}^m$ is T-stable for all m. We remark that this T can be chosen arbitrarily large, (see Remark 3.5 and § B.5.3).

3.3.1. *Pseudo-bubbles.* A *bubble* B is a closed strictly preperiodic Siegel disk; i.e., it is the closure of a connected component of $f^{-k}(Z)\backslash Z$. The *generation* of B is the minimal k such that $f^k(B) = \overline{Z}$; i.e. $f^k\colon B \to \overline{Z}$ is the first landing. Given a pseudo-Siegel disk $\widehat{Z}^m$, the *pseudo-bubble* $\widehat{B}$ is the closure of the connected component of $f^{-k}\big(\operatorname{int}\widehat{Z}^m\big)$ containing $\operatorname{int} B$. In other words, $\widehat{B}$ is obtained from B by adding the lifts of all reclaimed fjords (components of $\widehat{Z}^m \setminus \overline{Z}$) along $f^k\colon B \to \overline{Z}$.

Dams β_I, collars A_I, extra protections $\mathcal{X}_I$ are defined for $\widehat{B}$ as pullbacks of the corresponding objects along $f^k\colon \widehat{B} \to \widehat{Z}^m$. For instance, $\mathcal{X}(\widehat{Z}_\ell)$ is the pullback of $\mathcal{X}(\widehat{Z}^m)$ under f^k. The length of an interval $I \subset \partial\widehat{B}$ is the length of its image $f^k(I) \subset \partial\widehat{Z}^m$. Properties of pseudo-Siegel disks can also be obtained for pseudo-bubbles by pulling back using the dynamics.

3.4. **Convention for valuable-attracting domains and pseudo-Siegel disks.** We assume the following convention throughout the paper. Let $\mathcal{C}$ be a cycle of attracting Fatou components of $[f]$ with period p, and let $D \in \mathcal{C}$ be the unique Fatou component that contains the critical point. Then

$$\widehat{f^j(D)} = f^j(\widehat{D}) \text{ for all } j = 1, ..., p-1. \tag{3.4}$$

Similar to valuable-attracting domains, we shall use the following convention for pseudo-Siegel disks throughout the paper. Let $\mathcal{C}$ be a cycle of Siegel disks of $[f]$ with period p, and let $Z \in \mathcal{C}$ be the unique Fatou component that contains the critical point on its boundary. Then

$$\widehat{f^j(Z)}^m = f^j(\widehat{Z}^m) \text{ for all } m \text{ and for all } j = 1, ..., p-1. \tag{3.5}$$

3.5. **Core and pseudo-Core Surfaces.** Let $\mathcal{H}$ be a hyperbolic component of disjoint type. Let $[f] \in \partial\mathcal{H}$ be an eventually-golden-mean map. Let

$$Z_{1,0}, ...Z_{1,p_1-1}, Z_{2,0}, ..., Z_{k_1,p_{k_1}-1} \text{ and } D_{1,0}, ...D_{1,q_1-1}, D_{2,0}, ..., D_{k_2,q_{k_2}-1}$$

be the list of Siegel disks and attracting Fatou components of f. Denote the corresponding pseudo-Siegel disks and valuable-attracting domains by $\widehat{Z}^m_{i,j}$ and $\widehat{D}_{i,j}$ respectively. As usual, we define $\widehat{Z}_{i,j} := \widehat{Z}^{-1}_{i,j} = \bigcup_m \widehat{Z}^m_{i,j}$.

The indices are chosen so that for all

$$\widehat{Z}^m_{i,j+1} = f(\widehat{Z}^m_{i,j}) \text{ for all } m \in \mathbb{N}, i = 1, ..., k_1, j = 0, 1, ..., p_{k_i} - 2;$$

and

$$\widehat{D}_{i,j+1} = f(\widehat{D}_{i,j}) \text{ for all } i = 1, ..., k_2, j = 0, 1, ..., p_{k_i} - 2.$$

By construction, $\widehat{D}_{i,j}$ and $Z_{i,j}$ are all Jordan domains with pairwise disjoint closures. Thus, we define the *core surface* as the Riemann surface with boundary by

$$X_f := \widehat{\mathbb{C}} - \bigcup_{i,j} \mathrm{Int}(\widehat{D}_{i,j}) - \bigcup_{i,j} Z_{i,j}, \tag{3.6}$$

and the *pseudo-core surface* as

$$\widehat{X}_f := \widehat{\mathbb{C}} - \bigcup_{i,j} \mathrm{Int}(\widehat{D}_{i,j}) - \bigcup_{i,j} \mathrm{Int}(\widehat{Z}_{i,j}). \tag{3.7}$$

Since we will construct pseudo-Siegel disks for a single cycle while fixing other cycles of pseudo-Siegel disks and valuable-attracting domains, it is useful to introduce the following notations.

Let $Z_k := Z_{k,0}$. We define the *level m pseudo-core surface* for k-th cylce of Siegel disks Z_k as

$$\widehat{X}^m_f(Z_k) := \widehat{\mathbb{C}} - \bigcup_{i,j} \mathrm{Int}(\widehat{D}_{i,j}) - \bigcup_{i\neq k,j} \mathrm{Int}(\widehat{Z}_{i,j}) - \bigcup_{j=0}^{p_k-1} \mathrm{Int}(\widehat{Z}^m_{k,j}).$$

Note that under this notation, $\widehat{X}_f = \widehat{X}^{-1}_f(Z_k)$ for any k. We also define

$$\widehat{X}^\infty_f(Z_k) = \widehat{\mathbb{C}} - \bigcup_{i,j} \mathrm{Int}(\widehat{D}_{i,j}) - \bigcup_{i\neq k,j} \mathrm{Int}(\widehat{Z}_{i,j}) - \bigcup_{j=0}^{p_k-1} \mathrm{Int}(\overline{Z}_{k,j}),$$

as $\widehat{Z}^m_{k,j} = \overline{Z}_{k,j}$ for all sufficiently large m.

To avoid too many subindices, we shall simplify the notation as $X^m_f := X^m_f(Z_k)$ if the underlying cycle of Siegel disks is not ambiguous.

3.5.1. *Pullback of pseudo-Siegel disks and pseudo-core surfaces.* Note that pseudo-Siegel disks are not necessarily forward invariant. Thus, it is important to introduce notations for the pullbacks. Let us assume that all pseudo-Siegel disks are T-stable.

For each iterate $n \leq T$, the preimage $f^{-n}(\bigcup_{i,j} \mathrm{Int}(\widehat{Z}^m_{i,j}))$ is union of disks, each mapped conformally to some component $\mathrm{Int}(\widehat{Z}^m_{i,j})$. We denote by $\widehat{Z}^m_{i,j}(-n)$ the closure of the unique component of $f^{-n}(\bigcup_{i,j} \mathrm{Int}(\widehat{Z}^m_{i,j}))$ that contains $Z_{i,j}$.

Generalizing such notations for pseudo-core surfaces, we define

$$\widehat{X}_f(-n) := \widehat{\mathbb{C}} - \bigcup_{i,j} \mathrm{Int}(\widehat{D}_{i,j}) - \bigcup_{i,j} \mathrm{Int}(\widehat{Z}_{i,j}(-n)),$$

and

$$\widehat{X}^m_f(-n) := \widehat{\mathbb{C}} - \bigcup_{i,j} \mathrm{Int}(\widehat{D}_{i,j}) - \bigcup_{i\neq k,j} \mathrm{Int}(\widehat{Z}_{i,j}(-n)) - \bigcup_{j=0}^{p_k-1} \mathrm{Int}(\widehat{Z}^m_{k,j}(-n)).$$

We also define

$$\widehat{Y}_f(-n) := f^{-n}(\widehat{X}_f),$$

and

$$\widehat{Y}_f^m(-n) := f^{-n}(\widehat{X}_f^m).$$

We summarize the relations between these spaces in the following diagram.

$$\begin{tikzcd}
\widehat{Y}_f(-n) \arrow[d, hook] \arrow[drr, "f^n"] & & \\
\widehat{X}_f(-n) \arrow[dr, hook] & & \widehat{X}_f \arrow[dl, hook'] \\
& X_f &
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{Y}_f^m(-n) \arrow[d, hook] \arrow[drr, "f^n"] & & \\
\widehat{X}_f^m(-n) \arrow[dr, hook] & & \widehat{X}_f^m \arrow[dl, hook'] \\
& X_f^m &
\end{tikzcd}$$

Here the hooked arrows represent inclusions and $f^n : \widehat{Y}_f(-n) \longrightarrow \widehat{X}_f$ or $f^n : \widehat{Y}_f^m(-n) \longrightarrow \widehat{X}_f^m$ are covering maps.

3.6. **Local degeneration on pseudo-core surfaces.** In this subsection, using the dynamics of f on the boundaries of the Siegel disks, we introduce some special families of curves for the pseudo-core surfaces. The extremal widths for such families are crucial in our analysis.

Set $Z := Z_{1,0}$ with period $p = p_1$. Consider the level m pseudo-core surface for Z

$$\widehat{X}_f^m := \widehat{\mathbb{C}} - \bigcup_{i \neq 1} \mathrm{Int}(\widehat{Z}_{i,j}) - \bigcup \mathrm{Int}(\widehat{D}_{i,j}) - \bigcup_{j=0}^{p-1} \mathrm{Int}(\widehat{Z}_{1,j}^m).$$

Let $I \subseteq \partial \widehat{Z}^m$ be a regular interval. Denote

$$B = B(I, \lambda) := \partial \widehat{X}_f^m - \lambda I.$$

We use the notations

$$\mathcal{F}_\lambda^+(I) = \mathcal{F}_{\lambda, \widehat{Z}^m}^+(I) \text{ and } \mathcal{F}_\lambda(I) = \mathcal{F}_{\lambda, \widehat{Z}^m}(I)$$

to denote that curves families connecting I with B in $\widehat{X}_f^m$ and in $\widehat{\mathbb{C}}$ respectively. As we will be working with pseudo-Siegel disks of Z of different levels simultaneously, the subindex $\widehat{Z}^m$ is sometimes added to clarify which pseudo-Siegel disk we are considering. We denote

$$\mathcal{W}_\lambda^+(I) = \mathcal{W}_{\lambda,m}^+(I) = \mathcal{W}_{\lambda, \widehat{Z}^m}^+(I) := \mathcal{W}_{\widehat{X}_f^m}(I, B(I, \lambda)) = \mathcal{W}(\mathcal{F}_\lambda^+(I)) \tag{3.8}$$

$$\mathcal{W}_\lambda(I) = \mathcal{W}_{\lambda,m}(I) = \mathcal{W}_{\lambda, \widehat{Z}^m}(I) := \mathcal{W}_{\widehat{\mathbb{C}}}(I, B(I, \lambda)) = \mathcal{W}(\mathcal{F}_\lambda(I)). \tag{3.9}$$

We shall refer to the quantities $\mathcal{W}_\lambda^+(I)$ as *local degenerations*. We remark that here local means that we have localized one end of the arcs to be in the interval I. The arcs are not necessarily restricted in a local part of $\widehat{X}_f^m$.

The family $\mathcal{F}_\lambda^+(I)$ can be decomposed into the peripheral and non-peripheral parts, and we denote the corresponding families by

$$\mathcal{F}_\lambda^{+,per}(I) \text{ and } \mathcal{F}_\lambda^{+,np}(I).$$

Their widths are denoted by

$$\mathcal{W}_{\lambda}^{+,per}(I) \text{ and } \mathcal{W}_{\lambda}^{+,np}(I)$$

We remark that λ is chosen to be a large constant. Thus, there is a large combinatorial distance between the end points of any arc in $\mathcal{F}_{\lambda}^{+}(I)$.

We also use the notation $\mathcal{W}_{m}^{+,np}(I) := \mathcal{W}_{0,m}^{+,np}(I)$, i.e. the width of non-peripheral arcs in $\widehat{X}_f^m$ starting on the interval I.

3.6.1. *Comparing local degenerations.* One of the most important properties of grounded intervals is that local degenerations behave nicely as we pass from Siegel disks to Pseudo-Sigel disks. Moreover, for non-peripheral degenerations, we can simply replace an interval by a grounded interval with some uniform control on the correction.

Proposition 3.3. *Let $I \subseteq \partial \widehat{Z}^m$ be a grounded interval, and let $I^\bullet \subseteq \partial Z$ be the projection of I onto ∂Z. Suppose that $\lambda \geq 10$. Then*

$$\mathcal{W}_{\lambda,Z}^{+}(I^\bullet) - O(1) \leq \mathcal{W}_{\lambda,\widehat{Z}^m}^{+}(I) \leq 2\mathcal{W}_{\lambda,Z}^{+}(I^\bullet) + O(1).$$

For any interval $I \subset \partial Z$, let $I^{GRND}, I^{grnd} \subseteq \partial Z$ be the smallest grounded interval of level m that contains I and the largest grounded interval of level m that is contained in I respectively. Then

$$\mathcal{W}_{Z}^{+,np}(I^{GRND}) - O(1) \leq \mathcal{W}_{Z}^{+,np}(I) \leq \mathcal{W}_{Z}^{+,np}(I^{GRND}),$$
$$\mathcal{W}_{Z}^{+,np}(I^{grnd}) \leq \mathcal{W}_{Z}^{+,np}(I) \leq \mathcal{W}_{Z}^{+,np}(I^{grnd}) + O(1).$$

Proof. The Thin-Thick Decomposition (see [Lyu, Theorem 7.25] allows us, up to $O(1)$, to replace $\mathcal{F}_{\lambda,Z}^{+}(I^\bullet)$ with a union of finitely many rectangles in $\mathcal{F}_{\lambda,Z}^{+}(I^\bullet)$. Therefore, (3.3) implies the first statement.

The second statement follows from the observation that $I^{\mathrm{GRND}} \setminus I$ is within a union of at most two intervals, each being surrounded by an annulus $A^{\mathrm{out}}(.)$ with modulus $\geq \varepsilon$. Therefore, the width of curves in $\mathcal{F}_{Z}^{+,np}(I^{\mathrm{GRND}}) \setminus \mathcal{F}_{Z}^{+,np}(I)$ is bounded by $\dfrac{2}{\varepsilon}$. A similar argument holds for I and I^{grnd}. □

We remark that it is possible to replace 2 by by $1+\delta$, where $\delta = \delta(\Delta)$ can be arbitrary small if the protection Δ for the Pseudo-Siegel disk is sufficiently big.

3.7. **Non-uniform construction of pseudo-Siegel disks.** In this subsection, we construct pseudo-Siegel disks so that non-peripheral degeneration dominates peripheral degeneration – see (2a) and (2a) below. More precisely, with the notations introduced in §3.5 and §3.6, we will prove

Theorem 3.4. *Let $\mathcal{H}$ be a hyperbolic component of disjoint type. Let $[f] \in \partial\mathcal{H}$ be an eventually-golden-mean map with the pseudo-core surface X_f. Let $K := \mathcal{W}_{arc}(X_f)$ be the arc degeneration of X_f. There exist a constant $M = M(K)$ depending on K, pseudo-Siegel disks $\widehat{Z}_{i,j}^m$ for all i, j so that for all $\lambda \geq 10$,*

(1) $\widehat{Z}_{i,j} = \widehat{Z}_{i,j}^{-1}$ *is an M quasiconformal disk;*
(2) *for every grounded interval $J \subseteq \partial Z_{i,j}$ rel $\widehat{Z}_{i,j}^m$ with $\mathfrak{l}_{m+1} < |J| \le \mathfrak{l}_m$, we have*
   (a) $\mathcal{W}_m^{+,np}(J^m) = O(K\mathfrak{l}_m + 1)$*; and*
   (b) $\mathcal{W}_{\lambda,m}^{+,per}(J^m) = O(\sqrt{K\mathfrak{l}_m} + 1)$*.*

We remark that (2a) and (2b) can be improved for all m unless m is the "special transition" level; see refined versions in Appendix B, Theorem B.2. See also Remark B.3 for an explanation of the estimates.

Proof. The construction is by induction on the cycles of Siegel disks. Suppose that we have constructed pseudo-Siegel disks $\widehat{Z}_{i,j}^m$ for $i < k$, and we want to construct the pseudo-Siegel disks for the k-th cycle. Abusing the notations, denote the pseudo-core surface by $\widehat{X}_f$. Let $Z := Z_{k,1}$ be a Siegel disk for f. Note that Z is a boundary component of $\widehat{X}_f$. Since we can construct $\widehat{Z}_{k,j}$ by $f^{j-1}(\widehat{Z}_{k,1})$, it suffices to construct the pseudo-Siegel disk $\widehat{Z} = \widehat{Z}_{k,1}$. After passing to an iterate, we may assume Z is fixed by f.

The idea is to construct a $\psi^\bullet$-ql (pseudo-bullet-quadratic-like) map (see §B.1 for the definition). By the construction in §B.2, we can associate a $\psi^\bullet$-ql map
$$F = (f^p, \iota)\colon U \rightrightarrows V$$
with
$$\mathcal{W}^\bullet(F) = 2\mathcal{W}_{arc}(Z) + O(1).$$
Since the vertical (or non-vertical) degeneration for F corresponds to, up to a width of $O(1)$, the non-peripheral (or peripheral) degeneration of $\widehat{X}_f$ with endpoints on Z (see §A.5 and §B.2), the statements for intervals on ∂Z now follow from Theorem B.2; more specifically from Eqation B.6.

By Proposition 3.3, replacing the Siegel disk Z by a pseudo-Siegel disk only changes the the degeneration by a bounded error, we conclude that statements for intervals on $\partial Z_{i,j}$ for $i < k$ still holds. Since there are only finitely many cycles of Siegel disks, we conclude the theorem. □

Remark 3.5. We remark that for any given $T > 1$, we can construct the pseudo-Siegel disk that are T-stable. This parameter T affects only the constant M and constants representing the "$O(\ )$" in (2) (see § B.5.3).

In this paper, we will select T to be sufficiently big to dominate the pulled-off constant N in §4 and the constant $\mathbf{a}$ in Theorem 5.1. See §7.1 for the choice of these constants. This selection will be used in:

- the proof of Localization of arc degenerations in Theorem 5.1;
- the proof of the Calibration lemma on shallow levels in Theorem 6.1.

4. The pulled-off constant and expanding model

Let $\mathcal{H}$ be a hyperbolic component of disjoint type. Let $[f] \in \partial\mathcal{H}$ be an eventually-golden-mean map with the pseudo-core surface, and $[f_{pcf}] \in \mathcal{H}$

be the post-critically finite center. Let $N_{Siegel}([f])$ and $N([f_{pcf}])$ be the pulled-off constant as in Definition 1.5.

In this section, we will show that a pulled-off constant is uniformly bounded for a Sierpinski carpet hyperbolic component. This is one of the key reasons why a Sierpinski carpet hyperbolic component is bounded.

Theorem 4.1 (Pulled-off Principle)**.** *Let $\mathcal{H}$ be a Sierpinski carpet hyperbolic component of disjoint type. Then there exists a constant* $\mathbf{N}$ *so that for any eventually-golden-mean map $[f] \in \partial\mathcal{H}$, $N([f]) \leq \mathbf{N}$.*

We will deduce this theorem by justifying that the expanding model of maps in $\mathcal{H}$ persists for eventually-golden-mean maps on $\partial\mathcal{H}$. Then, assuming Theorem 1.6, we will show that the expanding model persists for all maps $\partial\mathcal{H}$ implying Theorem B.

4.1. **Characterization of Sierpinski carpet hyperbolic component.**

Theorem 4.2. *Let $\mathcal{H}$ be a hyperbolic component, and let $[f_{pcf}] \in \mathcal{H}$ be the post-critically finite center. Then $\mathcal{H}$ is Sierpinski if and only if $N([f_{pcf}]) < \infty$.*

Proof. By [Pil94, Corollary 5.18], the map f_{pcf} has Sierpinski carpet Julia set if and only if there is no periodic Levy arc. Here a Levy arc is a non-peripheral simple curve γ with endpoints in the post-critical set $P(f_{pcf})$ so that $f^n_{pcf}(\gamma)$ is isotopic rel $P(f_{pcf})$ to γ for some n.

If there is a periodic Levy arc, then, up to isotopy, it can be realized as a concatenation of two internal rays and, hence, $N([f_{pcf}]) = \infty$.

Conversely, suppose that $N([f_{pcf}]) = \infty$. Then there exist arbitrarily long essentially disjoint pull back sequence $\gamma_0, ..., \gamma_n$. Note that the number of essentially disjoint isotopic classes of arcs is bounded by the topological complexity of $\widehat{\mathbb{C}} - P(f_{pcf})$. Thus, for all large n, some pairs in $\gamma_0, ..., \gamma_n$ are isotopic. Therefore, there exists a periodic Levy arc. □

4.2. **Semiconjugacy to an expanding model.**

Let $\mathcal{H}$ be a Sierpinski carpet hyperbolic component of disjoint type. In this subsection, we will show that an eventually-golden-mean map $[f] \in \partial\mathcal{H}$ is semiconjugate to a topologically expanding map.

Let $[f_{pcf}] \in \mathcal{H}$ be the center of $\mathcal{H}$, i.e., the unique post-critical finite map in $\mathcal{H}$. We define $\bar{f} : S^2 \longrightarrow S^2$ as the quotient map of f_{pcf} by collapsing each Fatou component to a point. Note that $\bar{f}$ is topologically expanding, as f_{pcf} has Sierpinski carpet Julia set.

Let $[f] \in \partial\mathcal{H}$ be an eventually-golden-mean map. Using renormalization theory on Siegel disks, we will prove

Theorem 4.3 (Expanding model for $\partial_{\mathrm{egm}}\mathcal{H}$)**.** *Let $[f] \in \partial\mathcal{H}$ be an eventually-golden-mean map. Then there exists a topological semiconjugacy*

$$h : \widehat{\mathbb{C}} \longrightarrow S^2 \text{ with } \bar{f} \circ h = h \circ f.$$

In particular, $N_{Siegel}([f]) \leq N([f_{pcf}])$.

Proof. Denote the multiplier profile for $[f]$ as $(\rho_1, ..., \rho_{2d-2})$. Let

$$(\rho_{1,n}, ..., \rho_{2d-2,n})$$

be rational parameters converging strongly to $(\rho_1, ..., \rho_{2d-2})$, with corresponding maps $[f_n] \in \partial\mathcal{H}$. By approximating each $[f_n]$ with hyperbolic maps in $\mathcal{H}$ radially, there exists semiconjugacy (see [CT18, Theorem 1.5])

$$h_n : \widehat{\mathbb{C}} \longrightarrow S^2 \text{ with } \bar{f} \circ h_n = h_n \circ f_n.$$

We assume the representatives are chosen so that $f_n \to f$ as rational maps.

Since any orbit on the boundary of a Siegel disk is dense on the boundary, it is easy to see that the Siegel disks and valuable attracting domains have disjoint closures. By [DLS20, Theorem 6.9], for sufficiently large n, we can find *parabolic valuable flowers* $L_{i,j,n}$ approximating the Siegel disks $Z_{i,j}$. Therefore, we can find disjoint small neighborhoods $U_{i,j}$ of $Z_{i,j}$ and $W_{i,j}$ of $\widehat{D}_{i,j}$ so that for sufficiently large n, we have $L_{i,j,n} \subseteq U_{i,j}$ and $\widehat{D}_{i,j,n} \subseteq W_{i,j}$. For sufficiently large n, we can find a small perturbation h^0 of h_n so that $h^0(U_{i,j})$ and $h^0(W_{i,j})$ are points. Note that the union $U = \bigcup U_{i,j} \cup \bigcup W_{i,j}$ contains the union of parabolic valuable flowers and valuable attracing domains of f_n, so U also contains the post-critical set of f_n. Then we can pull back h^0 and get

$$h^1_n : \widehat{\mathbb{C}} \longrightarrow S^2 \text{ with } \bar{f} \circ h^0 = h^1_n \circ f_n,$$
$$h^1 : \widehat{\mathbb{C}} \longrightarrow S^2 \text{ with } \bar{f} \circ h^0 = h^1 \circ f.$$

Since h_n is a semiconjugacy between f_n and $\bar{f}$, $h^1_n \sim h^0$ on X_f for sufficiently large n. Since $f_n \to f$, $h^1 \sim h^0$ on X_f as well.

Since $\mathrm{Int}(X_f)$ contains no post-critical point and $\bar{f}$ is topologically expanding, a standard pull-back argument gives the semiconjugacy.

Note that any laminally disjoint pull-back sequence for an eventually-golden-mean map $[f]$ gives a laminally disjoint pull-back sequence for $\bar{f}$. If γ for $[f]$ connects boundaries of Siegel disks, then the corresponding arc δ for $\bar{f}$ connects points in critical periodic cycles. Since each laminally disjoint pull-back sequence is essentially disjoint, and $\bar{f}$ is homotopically equivalent to f_{pcf}, we have that $N_{Siegel}([f]) \leq N([f_{pcf}])$. □

Proof of Theorem 4.1. By Theorem 4.3, $N_{Siegel}([f]) \leq N([f_{pcf}])$. By Theorem 4.2, $N([f_{pcf}]) < \infty$. Therefore, $N_{Siegel}([f])$ is uniformly bounded. □

4.3. **Proof of Theorem B.** Recall that $\bar{f} : S^2 \longrightarrow S^2$ is the topologically expanding map obtained from collapsing Fatou components of the center $[f_{pcf}] \in \mathcal{H}$.

Theorem 4.4 (Expanding model for maps in $\partial\mathcal{H}$)**.** *Let $[f] \in \overline{\mathcal{H}}$. There exists a topological semiconjugacy*

$$h : \widehat{\mathbb{C}} \longrightarrow S^2 \textit{ with } \bar{f} \circ h = h \circ f.$$

Proof. Let $[f] \in \partial\mathcal{H}$ with multiplier profile $(\rho_1, ..., \rho_{2d-2})$. Let $[f_n] \in \partial\mathcal{H}$ be a sequence of eventually-golden-mean maps with

- $[f_n] \to [f]$; and
- its multiplier profile $(\rho_{1,n}, ..., \rho_{2d-2,n}) \to_s (\rho_1, ..., \rho_{2d-2})$.

Assume that the representatives are chosen so that $f_n \to f$.

By Theorem 1.6, after passing to a subsequence, we may assume pseudo-Siegel disks $\widehat{Z}_{i,j,n}$ and valuable-attracting domains $\widehat{D}_{i,j,n}$ converge in Hausdorff topology to $\widehat{Z}_{i,j}$ and $\widehat{D}_{i,j}$. Note that $\widehat{Z}_{i,j}$ and $\widehat{D}_{i,j}$ contain the post-critical set of f. Denote the Riemann surface

$$\widehat{X}_f := \widehat{\mathbb{C}} - \bigcup \operatorname{Int}(\widehat{Z}_{i,j}) - \bigcup \operatorname{Int}(\widehat{D}_{i,j}).$$

By Theorem 4.3, there exists semiconjugacies

$$h_n : \widehat{\mathbb{C}} \longrightarrow S^2 \text{ with } \bar{f} \circ h = h \circ f_n.$$

We can construct a similar purtabation h^0 of h_n for some sufficiently large n. Let h^1 be the pull back of h^0 under f. A similar argument as in Theorem 4.3 gives that $h^0 \sim h^1$ on $\widehat{X}_f$ with

$$\bar{f} \circ h^0 = h^1 \circ f$$

Since $\widehat{X}_f$ is disjoint from the post-critical set and $\bar{f}$ is topologically expanding, a standard pull back argument gives the semiconjugacy h. $\square$

Theorem B now follows immediately from Theorem 4.4.

Proof of Theorem B. Let $\tilde{x} = h(x) \in S^2$, where h is the semiconjugacy in Theorem 4.4. Let $\tilde{x} \in \tilde{U}$ be a small neighborhood so that $\tilde{U} \Subset \bar{f}^p(\tilde{U})$. Let $U = h^{-1}(\tilde{U})$. Then $f^p : U \longrightarrow V = f^p(U)$ gives the quadratic-like restriction. Since $\overline{\mathcal{H}}$ is compact, the modulus of $V - \overline{U}$ is uniformly bounded. This proves the theorem. $\square$

5. Localization of arc degeneration

In this section, we will prove that if the arc degeneration $\mathcal{W}_{arc}(\widehat{X}_f)$ of the pseudo-core surface $\widehat{X}_f$ is sufficiently big, then there exists some small grounded interval I whose local degeneration $\mathcal{W}^+_\lambda(I)$ is at least comparable to $\mathcal{W}_{arc}(\widehat{X}_f)$; compare with §1.5.2. More precisely, we will prove

Theorem 5.1 (Localization of arc degeneration)**.** *Let $\mathcal{H}$ be a hyperbolic component of disjoint type. Let $[f] \in \partial\mathcal{H}$ be an eventually-golden-mean map with pulled-off constant $N = N_{Siegel}([f])$. There exist*

- *a constant $\mathbf{a} > 1$ that depends on N,*
- *and a threshold constant $\mathbf{\Lambda} \gg 1$*

such that for every $\lambda \geq \mathbf{\Lambda}$ and for every $0 < \epsilon < 1/2\lambda$, there exists a threshold constant $\mathbf{K}_{\epsilon,\lambda,N} \gg 1$ depending on ϵ, λ, N and the multipliers of attracting cycles of f with the following properties.

Suppose that the Riemann surface $\widehat{X}_f := \widehat{\mathbb{C}} - \bigcup \mathrm{Int}(\widehat{Z}_{i,j}) - \bigcup \mathrm{Int}(\widehat{D}_{i,j})$ *has*

$$\mathcal{W}_{arc}(\widehat{X}_f) := K \geq \mathbf{K}_{\epsilon,\lambda,N}.$$

Suppose that all pseudo-Siegel disks are at least N-stable. Then there exists a pseudo-Siegel disk $\widehat{Z} = \widehat{Z}_{i,j}$ and a grounded interval $I \subseteq \partial\widehat{Z}$ with $|I| \leq \epsilon$ such that

$$\mathcal{W}^{+,np}(I) + \mathcal{W}^{+,per}_{\lambda}(I) \geq K/\mathbf{a}.$$

Remark 5.2. We remark that if $\mathcal{H}$ is a Sierpinski carpet hyperbolic component, then by Theorem 4.1, the pulled-off constant is uniformly bounded. In this case, the constant $\mathbf{a}$ can be chosen to be universal, and the constant $\mathbf{K}_{\epsilon,\lambda,N}$ depends only on ϵ, λ and the multipliers of the attracting cycles.

Pulled-off argument. We will follow the notations introduced in §3.5. It follows from Proposition 3.3 and the fact that the Siegel disks are N-stable that for sufficiently large arc degeneration $K = \mathcal{W}_{arc}(\widehat{X}_f)$, we have

$$\frac{1}{8}\mathcal{W}_{arc}(\widehat{X}_f) \leq \mathcal{W}_{arc}(\widehat{X}_f(-N)) \leq 8\mathcal{W}_{arc}(\widehat{X}_f).$$

Note that there are only finitely many homotopy classes of non-peripheral arcs $\gamma \subseteq \widehat{X}_f(-N)$ with $\mathcal{W}(\gamma) \geq 2$. This number is bounded by a constant M, which depends only on the number of boundary components of $\widehat{X}_f(-N)$. Let $\gamma \subseteq \widehat{X}_f(-N)$ be a non-peripheral arc with

$$\mathcal{W}(\gamma) \geq \frac{\mathcal{W}_{arc}(\widehat{X}_f(-N))}{2M} \geq \frac{\mathcal{W}_{arc}(\widehat{X}_f)}{16M} = \frac{K}{16M}.$$

We may realize such wide families by a rectangle R_γ whose vertical arcs are homotopic to γ and

$$\mathcal{W}(\mathcal{F}_\gamma) = \mathcal{W}(\gamma) - O(1),$$

where $\mathcal{F}_\gamma$ is the family of vertical arcs in R_γ.

By our construction, the modulus of the annulus $D_{i,j} - \widehat{D}_{i,j}$ is bounded below in terms of the multipliers of the attracting cycles of f. By making the threshold $\mathbf{K}_{\epsilon,\lambda,N}$ larger if necessary, we may assume γ connects two pseudo-Siegel disks $\widehat{Z}(-N)$ and $\widehat{Z}'(-N)$. Note that $\widehat{Z}(-N)$ may equal to $\widehat{Z}'(-N)$.

Let U be a component of $f^{-N}(\bigcup Z_{i,j})$. Denote $\widehat{U}(-N)$ as the corresponding pseudo-Siegel disks, i.e., $\widehat{U}(-N)$ is the closure of the component of $\widehat{\mathbb{C}} - \widehat{Y}_f(-N)$ that contains U.

Lemma 5.3 (Submergence into U)**.** *There exists*

- *a constant $\mathbf{a}_1$ depending on N,*
- *a strictly pre-periodic Siegel disk $U \subseteq f^{-\mathbf{N}}(\bigcup Z_{i,j})$ so that ∂Z and ∂U are in different connected components of $\partial Y_f(-N)$,*
- *a family $\mathcal{G}$ of homotopically equivalent non-peripheral arcs connecting $\widehat{Z}$ and $\widehat{U}$, and*

- *a subfamily* $\mathcal{F}_1 \subseteq \mathcal{F}_\gamma$

so that

- $\mathcal{W}(\mathcal{F}_1) \geq K/\mathbf{a}_1$*;*
- *the family* $\mathcal{F}_1$ *overflows* $\mathcal{G}$*.*

Proof. Note that $\gamma \subseteq \widehat{X}_f(-N)$ determines a unique homotopy class of non-peripheral arc $\widetilde{\gamma} \subseteq \widetilde{X}_f$. We first claim after removing some buffers, we may assume any arc in $\mathcal{F}_\gamma$ intersects some strictly pre-periodic component of $f^{-N}(\bigcup Z_{i,j} \cup \bigcup \text{Int}(\widehat{D}_{i,j}))$. Indeed, otherwise, we get N disjoint rectangles $R_0 \subseteq R_\gamma, R_1 = f(R_0), ..., R_N = f^N(R)$. All vertical arcs of each of the rectangles are homotopic to some non-peripheral arc in X_f. This produces a disjoint pull back sequence $\gamma_0, ..., \gamma_N$. This is impossible by the definition of pulled-off constant N.

Since the modulus of the annulus $D_{i,j} - \widehat{D}_{i,j}$ is bounded below, by making the threshold $\mathbf{K}_{\epsilon,\lambda,N}$ larger if necessary, we may assume there exists a subfamily $\mathcal{F}' \subseteq \mathcal{F}_\gamma$ so that

- $\mathcal{W}(\mathcal{F}') \geq \mathcal{W}(\mathcal{F}_\gamma)/2$; and
- no curve in $\mathcal{F}'$ intersects the preimages of valuable-attracting domains $f^{-N}(\bigcup \text{Int}(\widehat{D}_{i,j}))$.

Thus, any arc in $\mathcal{F}'$ must intersect some strictly pre-periodic Siegel disk in $f^{-N}(\bigcup Z_{i,j}) \subseteq f^{-N}(\bigcup \text{Int}(\widehat{Z}_{i,j}))$.

Note that there are a bounded number (depending on N) of strictly pre-periodic Siegel disks $U \subseteq f^{-N}(\bigcup Z_{i,j})$. So there are a bounded number of homotopy classes of wide non-peripheral arcs in $Y_f(-N)$. Thus there exists a constant $\mathbf{a}_1$ depending on N, some family $\mathcal{G}$ of homotopically equivalent arcs connecting $\widehat{Z}$ and a strictly pre-periodic pseudo-Siegel disk $\widehat{U}(-N)$ so that the arcs in $\mathcal{F}'$ overflowing $\mathcal{G}$ has width at least $K/\mathbf{a}_1$. Let $\mathcal{F}_1 \subseteq \mathcal{F}'$ be this collection of arcs and we conclude the lemma. $\square$

Let $\mathcal{F}_1$ be the family of arcs in Lemma 5.3. Consider an arc $\gamma : [0,1] \longrightarrow \widehat{X}_f(-N)$ in $\mathcal{F}_1$ with $\gamma(0) \in \partial\widehat{Z}(-N)$ and $\gamma(1) \in \partial\widehat{Z}'(-N)$. Let $t_0 > 0$ be the first time that $\gamma(t_0) \in \partial\widehat{Y}_f(-N)$. Let $\gamma' = \gamma|_{[0,t_0]}$. By our construction, $\gamma' \in \mathcal{G}$. Thus $\gamma' \subseteq \widehat{Y}_f(-N)$ is an arc connecting $\widehat{Z}(-N)$ and $\widehat{U}(-N)$.

Lemma 5.4 (Localization of the submergence as $I' \subset \partial\widehat{U}(-N)$)**.** *There exists a threshold constant* $\mathbf{K}_{\epsilon,\lambda,N} \gg 1$ *so that if*

$$K = \mathcal{W}_{arc}(\widehat{X}_f) \geq \mathbf{K}_{\epsilon,\lambda,N},$$

then there exist a constant $\mathbf{a}_2$ *depending on* N *and a grounded interval* $I' \subseteq \partial\widehat{U}(-N)$ *with* $|I'| \leq \epsilon$ *so that the collection* $\mathcal{F}_2 \subseteq \mathcal{F}_1$ *of arcs passing through* I' *has width* $\mathcal{W}(\mathcal{F}_2) \geq K/\mathbf{a}_2$*.*

Proof. Let $\gamma_1, \gamma_2 \in \mathcal{F}_1$ be the left and right most arcs in $\mathcal{F}_1$. Let x_i be an intersection point of γ_i with $\partial\widehat{U}(-N)$. Since the end points of γ_i are outside

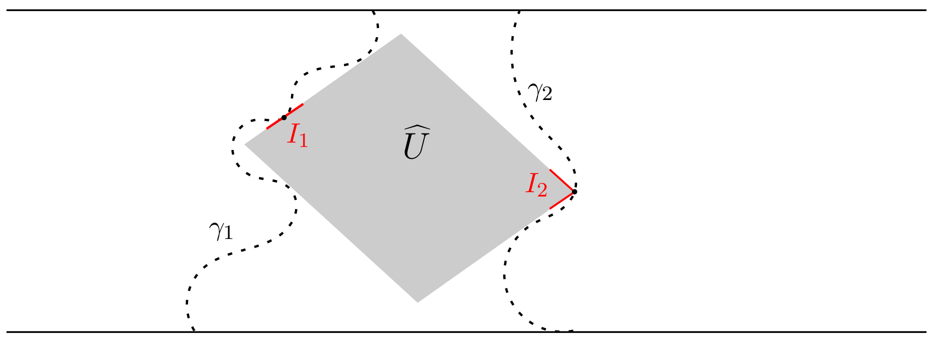

FIGURE 5.1. The curve γ_1 and γ_2 are the left and right most arcs in $\mathcal{F}_1$. Most of the arcs in $\mathcal{F}_1$ passes through I_1 or I_2.

of $\mathcal{X}(\widehat{U}(-N))$ and the extra outer protection $\mathcal{X}_I$ has width bounded below, by removing a collection of arcs of bounded width, we may assume that x_i is away from the extra outer protections $\mathcal{X}(\widehat{U}(-N))$.

Let $x_i \in I_i \subseteq \partial\widehat{U}(-N)$ be a grounded interval with $|I_i| \leq \epsilon$. Let $\mathcal{F}' \subseteq \mathcal{F}_1$ be the family of arcs that is disjoint from $I_1 \cup I_2$. Let $\mathcal{F}''$ be the collection of arcs $\delta \subseteq \widehat{U}(-N)$ connecting the two components of $\partial U - I_1 - I_2$. Then any arc $\mathcal{F}'$ must overflow an arc in $\mathcal{F}''$. However, the width $\mathcal{W}(\mathcal{F}'') \preceq |\log\epsilon|$. Thus, $\mathcal{W}(\mathcal{F}') \preceq |\log\epsilon|$.

By choosing the threshold $\mathbf{K}_{\epsilon,\lambda,N}$ larger if necessary, we may assume $K/\mathbf{a}_1 \gg |\log\epsilon|$. Thus the collection of arcs passing through $I_1 \cup I_2$ has width $\geq K/2\mathbf{a}_1$. Without loss of generality, we assume the arcs of at least half of the width pass through I_1. The lemma now follows by letting $I' = I_1$. $\square$

We are ready to prove Theorem 5.1. In the proof, we first push forward by f^N the wide family from Lemma 5.4 passing through I' to obtain a wide family $\mathcal{F}'$ based at $I'' = f^N(I') \subset \partial\widehat{Z}$. We then lift the appropriate restriction of $\mathcal{F}'$ to the associated $\psi^\bullet$-map $\mathbf{g}$ around $\widehat{Z}$. Applying Lemma B.4, we construct an appropriate interval $\widetilde{I}$ in the dynamical plane of $\mathbf{g}$. Using natural properties of the Thin-Thick decomposition, the projection of $\widetilde{I}$ back to $\partial\widehat{Z}$ gives a required interval I.

Proof of Theorem 5.1. Consider an arc $\gamma : [0,1] \longrightarrow \widehat{X}_f(-N)$ in $\mathcal{F}_2$ with $\gamma(0) \in \partial\widehat{Z}(-N)$ and $\gamma(1) \in \partial\widehat{Z}'(-N)$. Let $t_0 > 0$ be the first time that $\gamma(t_0) \in I'$. Denote the truncation $\gamma|_{[0,t_0]}$ by γ'. Let $\mathcal{F}_2'$ be the collection of

such truncations
$$\mathcal{F}_2' := \{\gamma' : \gamma \in \mathcal{F}_2\}.$$
Let $I'' = f^N(I')$. Then I'' is a grounded interval on some periodic pseudo-Siegel disk $\widehat{Z}''$. Since ∂Z and ∂U are in different connected components of $\partial Y_f(-N)$, we conclude that $\alpha := f^N(\gamma')$ is a non-peripheral arc starting at I'', i.e., α is an arc in $\widehat{\mathbb{C}}$ so that at least one component of $\alpha \cap \widehat{X}_f$ is non-peripheral. Let α' be the smallest sub-arc of α starting at I'' so that $\alpha' \cap \widehat{X}_f$ contains a non-peripheral component, and let $\mathcal{F}_3$ be the collection of such truncations. Note that $\mathcal{W}(\mathcal{F}_3) \geq K/\mathbf{a}_3$ for some constant $\mathbf{a}_3$ depending only on N. Let $\mathbf{g}$ be the associated $\psi^\bullet$-map around $\widehat{Z}$. Then the arcs in $\mathcal{F}_3$ can be lifted to vertical arcs for the $\psi^\bullet$-map (see §B.2 and §B.3). Let $\widetilde{I}$ be the lift of the interval I''. By the Thin-Thick Decomposition (see §A.5), we obtain a rectangle $\mathcal{R}$ of width $K/\mathbf{a}_4$ consisting of vertical arcs that start at $\widetilde{I}$, for some constant $\mathbf{a}_4$ depending only on N. We now apply Lemma B.5 with $\lambda' = \max\{\lambda, \dfrac{1}{\epsilon}\}$. We choose the threshold $\mathbf{K}_{\epsilon,\lambda,N}$ large enough so that there exists some constant $\mathbf{a}_5$ depending only on N so that there exists either

- a subrectangle $\mathcal{R}_1$ of $\mathcal{R}$ with $\mathcal{W}(\mathcal{R}_1) \geq K/\mathbf{a}_5$ such that $\mathcal{R}_1$ is outside of int $\widetilde{\widehat{Z}}$; or
- a grounded interval $J \subset \partial\widetilde{\widehat{Z}}$ such that $\mathcal{W}^{+,per}_{\lambda}(J) \geq \mathcal{W}^{+,per}_{\lambda'}(J) \geq K/\mathbf{a}_5$.

Note that in the second case, since $\mathcal{W}^{+,per}_{\lambda'}(J) > 0$, we have $|J| < \dfrac{1}{\lambda'} \leq \epsilon$. We project the wide lamination down to the dynamical plane of f. In the first case, we obtain some grounded interval $I \subseteq I''$ with
$$\mathcal{W}^{+,np}(I) \geq K/\mathbf{a}.$$
for some constant $\mathbf{a}$ depending on N. In the second case, we obtain some grounded interval $I \subseteq \widehat{Z}$ with $|I| < \epsilon$ so that
$$\mathcal{W}^{+,per}_{\lambda}(I) \geq K/\mathbf{a}.$$
This proves the theorem. □

6. Calibration lemma on shallow levels for $\mathcal{W}^{+,np}_m(I)$

In this section, we will prove a calibration lemma for non-peripheral arc degenerations. Roughly speaking, we will show that if there exists a grounded interval I with sufficiently large non-peripheral arc degeneration $\mathcal{W}^{+,np}_m(I)$, then there is a grounded interval on a deeper level with comparable local degeneration; see also §1.5.3.

Theorem 6.1 (Calibration lemma on shallow levels)**.** *Let $\mathcal{H}$ be a hyperbolic component of disjoint type. Let $[f] \in \partial\mathcal{H}$ be an eventually-golden-mean map with pulled-off constant $N = N([f])$. Let Z be a Siegel disk of period p and $\widehat{Z} = \widehat{Z}^m$ be a pseudo-Siegel disk of level m.*

For every $a > N$, there is a constant $\boldsymbol{\chi}_a > 1$ and a threshold constant $\mathbf{K}_a > 1$ with the following property.

Suppose that $\mathcal{W}_{arc}(\widehat{X}_f) \geq \mathbf{K}_a$, that all pseudo-Siegel disks are at least $4apN$-stable and that I is a grounded interval with $\mathfrak{l}_{m+1} < |I| \leq \mathfrak{l}_m$, $\mathfrak{l}_m > 1/4a$ such that

$$K := \mathcal{W}^{+,np}_m(I) \geq \mathcal{W}_{arc}(\widehat{X}_f)/a.$$

Then there is a grounded interval $J \subseteq \partial \widehat{Z}$ with $|J| \leq \mathfrak{l}_{m+1}$ such that

$$\mathcal{W}^{+,np}_{m+1}(J) \geq K/\boldsymbol{\chi}_a \geq \mathcal{W}_{arc}(\widehat{X}_f)/\boldsymbol{\chi}_a a.$$

We remark that the shallow level refers to that $\mathfrak{l}_m$ (and hence m) is bounded from below.

Bounds on $\mathfrak{q}_{m+1}$. One important observation is the following lemma, which bounds the iterations of f to consider.

Lemma 6.2. *Suppose that $\mathfrak{l}_m \geq 1/4a$. Then*

$$\mathfrak{q}_{m+1} \leq 4a.$$

Proof. Note that by spreading around a level m interval I, we get $\mathfrak{q}_{m+1}$ disjoint intervals of length $\mathfrak{l}_m$:

$$I_0 = I, I_1 = f^{i_1 p}(I), ..., I_{\mathfrak{q}_{m+1}-1} = f^{(i_{\mathfrak{q}_{m+1}-1})p}(I),\ i_j \in \{1, 2, ..., \mathfrak{q}_{m+1} - 1\}.$$

So $\mathfrak{q}_{m+1}\mathfrak{l}_m \leq 1$. Thus, if $\mathfrak{l}_m \geq 1/4a$, $\mathfrak{q}_{m+1} \leq 1/\mathfrak{l}_m \leq 4a$. □

Proof of the calibration lemma. Following the definitions in §3.5, let $\widehat{Z}_{i,j}(-n)$ be the closure of the component of $f^{-n}(\mathrm{Int}(\bigcup \widehat{Z}_{i,j}))$ that contains $Z_{i,j}$. Since $\mathfrak{q}_{m+1}p \leq 4ap$ by Lemma 6.2 and all pseudo-Siegel disks are $4apN$-stable, we define

$$\widehat{X}_f(-N\mathfrak{q}_{m+1}p) := \widehat{\mathbb{C}} - \bigcup \mathrm{Int}(\widehat{D}_{i,j}) - \bigcup \mathrm{Int}(\widehat{Z}_{i,j}(-N\mathfrak{q}_{m+1}p)),$$

and

$$\widehat{Y}_f(-N\mathfrak{q}_{m+1}p) := f^{-N\mathfrak{q}_{m+1}p}(\widehat{X}_f).$$

Since $\mathfrak{q}_{m+1}p \leq 4ap$ and $N < a$, the topological complexity, i.e., the number of boundary components of $\widehat{Y}_f(-N\mathfrak{q}_{m+1}p)$ is bounded in terms of a.

Note that since the interval I is grounded, by Proposition 3.3, the wide families for I of $\widehat{X}_f$ and $\widehat{X}_f(-N\mathfrak{q}_{m+1}p)$ have compatible width, as they are both compatible to the corresponding family in X_f. Denote the corresponding family of $\widehat{X}_f(-N\mathfrak{q}_{m+1}p)$ for $\mathcal{W}^{+,np}_m(I)$ by $\mathcal{F}$. Let γ be a directed arc in $\widehat{X}_f(-N\mathfrak{q}_{m+1}p) \subseteq \widehat{Y}_f(-N\mathfrak{q}_{m+1}p)$. The *initial segment* δ of γ in $\widehat{Y}_f(-N\mathfrak{q}_{m+1}p)$ is the first segment of the union of arcs $\gamma \cap Y_f(-N\mathfrak{q}_{m+1}p)$. We say two initial segments δ_1, δ_2 are *homotopic* if they are homotopic in $\widehat{Y}_f(-N\mathfrak{q}_{m+1}p)$ and they both connect ∂U with ∂V where U, V are component of $\widehat{\mathbb{C}} - \widehat{Y}_f(-N\mathfrak{q}_{m+1}p)$. We remark that the homotopy condition does not imply the second condition as $\partial U \cup \partial V$ may be connected.

Since the topological complexity of $\widehat{Y}_f(-N\mathfrak{q}_{m+1}p)$ is bounded in terms of a, there exists a constant $C_1 = C_1(a)$ depending on a and a subfamily $\mathcal{F}_1 \subseteq \mathcal{F}$ with

- $\mathcal{W}(\mathcal{F}_1) \geq \mathcal{W}(\mathcal{F})/C_1 = K/C_1$; and
- all arcs in $\mathcal{F}_1$ have homotopic initial segments.

Note that we may assume $\mathcal{F}_1$ forms the vertical foliations of a rectangle connecting $I_1 \subseteq I$ and $L_1 \subseteq \partial\widehat{X}_f(-N\mathfrak{q}_{m+1}p)$. Since the non-peripheral arc degeneration for I_1 is large, by Proposition 3.3, we may assume that I_1 is grounded. Since $|I_1| \leq \mathfrak{l}_m$, there are at most N critical points of $f^{N\mathfrak{q}_{m+1}p}$ in I_1. Subdividing I_1 into $N+1$ subintervals if necessary, we may also assume that there are no critical points of $f^{N\mathfrak{q}_{m+1}p}$ on I_1.

We are now ready to prove Theorem 6.1.

Proof of Theorem 6.1. Suppose for contradiction that any grounded interval $J \subseteq \partial Z$ with $|J| \leq \mathfrak{l}_{m+1}$ satisfies $\mathcal{W}^{+,np}(J) \leq K/\boldsymbol{\chi}_a$, where $\boldsymbol{\chi}_a$ is some constant to be determined.

Note that if $|I_1| \leq \mathfrak{l}_{m+1}$, then we may take $\boldsymbol{\chi}_a \geq C_1$ and obtain a contradiction. Thus, we may assume $\mathfrak{l}_{m+1} < |I_1| \leq \mathfrak{l}_m$. Then the symmetric difference

$$(f^{N\mathfrak{q}_{m+1}p}(I_1) - I_1) \cup (I_1 - f^{N\mathfrak{q}_{m+1}p}(I_1))$$

consists of $2N$ number of level $m+1$ combinatorial intervals. Note by assumption, the widths $\mathcal{W}^{+,np}$ for these combinatorial intervals are bounded by K. Thus, the width $\mathcal{W}^{+,np}$ for the union of these $2N$ intervals is bounded from above by $2NK/\boldsymbol{\chi}_a$. Thus there exists a rectangle in $\widehat{X}_f$ with base $f^{N\mathfrak{q}_{m+1}p}(I_1)$ so that the family $\widetilde{\mathcal{F}}$ of its vertical arcs satisfies

$$|\mathcal{W}(\widetilde{\mathcal{F}}) - \mathcal{W}(\mathcal{F}_1)| = O(K/\boldsymbol{\chi}_a).$$

Let $\mathcal{G}$ be the pull back $\widetilde{\mathcal{F}}$ under $f^{N\mathfrak{q}_{m+1}p}$ that starts at the interval I_1. Since $f^{N\mathfrak{q}_{m+1}p}$ is univalent on I_1, we have $|\mathcal{W}(\mathcal{G}) - \mathcal{W}(\widetilde{\mathcal{F}})| = O(1)$ (see [DL22, Lemma A.10]). Thus,

$$|\mathcal{W}(\mathcal{G}) - \mathcal{W}(\mathcal{F}_1)| = O(K/\boldsymbol{\chi}_a).$$

After removing two $3NK/\boldsymbol{\chi}_a$-buffers from $\mathcal{F}_1$, we get a subfamily $\mathcal{F}_{1,new} \subseteq \mathcal{F}_1$ starting at some interval $I_{1,new}$. Note that by our assumption, the length of each of the intervals in $I_1 - I_{1,new}$ is at least $3N\mathfrak{l}_{m+1}$, and at most $O(1)$ curves in $\mathcal{F}_{1,new}$ can cross the 1-buffers of $\mathcal{G}$ starting at $I_1 - I_{1,new}$. Thus, by removing these curves if necessary, we obtain a subfamily $\mathcal{F}_{1,new} \subseteq \mathcal{F}_1$ with

$$|\mathcal{W}(\mathcal{F}_{1,new}) - \mathcal{W}(\mathcal{F}_1)| = O(K/\boldsymbol{\chi}_a)$$

that overflows $\mathcal{G}$ (see Figure 6.1).

Note that $N\mathfrak{q}_{m+1}p \geq N$, arcs in $\mathcal{G}$ do not connect periodic pseudo-Siegel disks by the definition of pulled-off constant. Thus, $\mathcal{G}$ consists of homotopically equivalent arcs connecting $\partial\widehat{Z}$ with some strictly pre-periodic pseudo-Siegel disk $\partial\widehat{U}$.

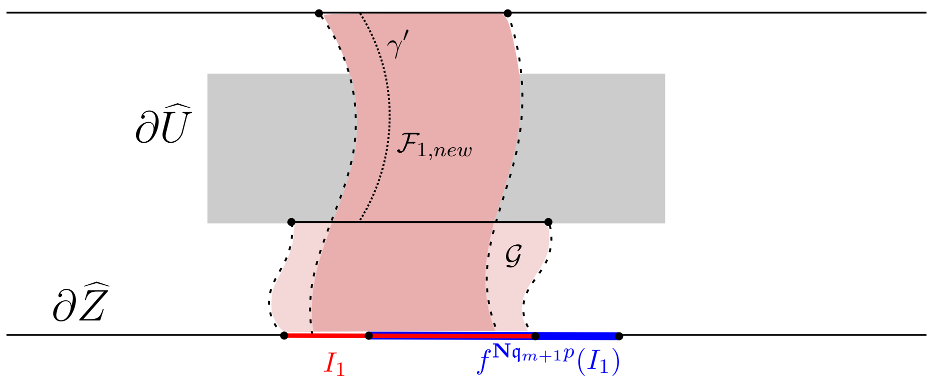

FIGURE 6.1. The configuration of the families $\mathcal{F}_{1,new}$ and $\mathcal{G}$.

Let γ' be the part of $\gamma \in \mathcal{F}_{1,new}$ after its first intersection with $\partial\widehat{U}$. Consider $\mathcal{F}' := \{\gamma' : \gamma \in \mathcal{F}_{1,new}\}$. Note that for each arc $\gamma' \in \mathcal{F}'$, its image $f^{N\mathfrak{q}_{m+1}p}(\gamma')$ contains some non-peripheral arc. Since the iteration $N\mathfrak{q}_{m+1}p$ is bounded, there exists a constant $C_2 = C_2(a)$ so that

$$\mathcal{W}(\mathcal{F}') \leq C_2 \mathcal{W}_{arc}(\widehat{X}_f).$$

By Proposition 3.3, $\mathcal{W}(\mathcal{F}_1) \geq \mathcal{W}_{arc}(\widehat{X}_f(-N\mathfrak{q}_{m+1}p))/aC_1 \geq \mathcal{W}_{arc}(\widehat{X}_f)/4aC_1$. Thus there exists a constant $C_3 = C_3(a)$ so that

$$\mathcal{W}(\mathcal{F}') \leq C_3 \mathcal{W}(\mathcal{F}_1).$$

By the series law, we have

$$\mathcal{W}(\mathcal{F}_{1,new}) \leq \mathcal{W}(\mathcal{G}) \oplus \mathcal{W}(\mathcal{F}').$$

Equivalently, we have

$$1/\mathcal{W}(\mathcal{G}) + 1/\mathcal{W}(\mathcal{F}') \leq 1/\mathcal{W}(\mathcal{F}_{1,new}).$$

Using our estimates on $\mathcal{W}(\mathcal{F}_{1,new}), \mathcal{W}(\mathcal{G}), \mathcal{W}(\mathcal{F}')$, we have

$$1/(\mathcal{W}(\mathcal{F}_1) + O(K/\boldsymbol{\chi}_a)) + 1/(C_3\mathcal{W}(\mathcal{F}_1)) \leq 1/(\mathcal{W}(\mathcal{F}_1) + O(K/\boldsymbol{\chi}_a)).$$

Note that $\mathcal{W}(\mathcal{F}_1) \in [K/C_1, K]$. Let $\mathcal{W}(\mathcal{F}_1) = cK$ for $c \in [1/C_1, 1]$. Thus, by cancelling the common term K, we obtain

$$1/(c + O(1/\boldsymbol{\chi}_a)) + 1/(C_3 c) \leq 1/(c + O(1/\boldsymbol{\chi}_a)).$$

This is a contradiction as such an inequality cannot hold if $\boldsymbol{\chi}_a \gg 1$. This concludes the proof of Theorem 6.1. □

7. BOUNDS ON ARC DEGENERATION

In this section, we shall prove that $\mathcal{W}_{arc}(\widehat{X}_f)$ is bounded in terms of the pulled-off constant. By Theorem 4.1, this would immediately imply that $\mathcal{W}_{arc}(\widehat{X}_f)$ is uniformly bounded for eventually-golden-mean maps on the boundary of a Sierpinski carpet hyperbolic component of disjoint type. More precisely, we will show

Theorem 7.1. *Let $\mathcal{H}$ be a hyperbolic component of disjoint type. Let $[f] \in \partial\mathcal{H}$ be an eventually-golden-mean map with pulled-off constant $N = N([f])$. There exists a constant $\mathbf{K}$ depending only on N and the multipliers of attracting cycles of f with the following properties.*

(1) For each Siegel disk $Z_{i,j}$ of f, there exists a pseudo-Siegel disk $\widehat{Z}_{i,j}$ which is a $\mathbf{K}$-quasiconformal closed disk.
(2) For each attracting domain $D_{i,j}$, there exists a valuable-attracting domain $\widehat{D}_{i,j}$ with $\operatorname{mod}(D_{i,j} - \widehat{D}_{i,j}) \geq 2\pi/\mathbf{K}$.
(3) The pseudo-core surface $\widehat{X}_f := \widehat{\mathbb{C}} - \bigcup \operatorname{Int}(\widehat{D}_{i,j}) - \bigcup \operatorname{Int}(\widehat{Z}_{i,j})$ has uniformly bounded arc degeneration

$$\mathcal{W}_{arc}(\widehat{X}_f) \leq \mathbf{K}.$$

Let us outline the strategy of the argument. As a preparation, we first construct pseudo-Siegel disks as in Theorem 3.4 with bounds depending on $\mathcal{W}_{arc}(\widehat{X}_f)$. Let $\widehat{X}_f$ be the pseudo-core surface. To prove Theorem 7.1, by Theorem 3.4, it suffices to show that $\mathcal{W}_{arc}(\widehat{X}_f)$ is uniformly bounded.

We will argue by contradiction. Suppose $\mathcal{W}_{arc}(\widehat{X}_f)$ is sufficiently large. Then

i) we can first localize the arc degeneration (Theorem 5.1) and obtain a small grounded interval I_1 with comparable local degeneration

$$\mathcal{W}^{+,np}(I) + \mathcal{W}^{+,per}_{\lambda}(I) \succeq \mathcal{W}_{arc}(\widehat{X}_f).$$

ii) By Property (2)(b) in Theorem 3.4, the peripheral part $\mathcal{W}^{+,per}_{\lambda,m}(I)$ is relatively small.
iii) This means $\mathcal{W}^{+,np}_{m}(I)$ is large. We will apply the calibration lemma (see Theorem 6.1), and construct an interval on a deeper level with big local degeneration, which contradicts Property (2)(a) in Theorem 3.4.

7.1. **Choosing the constants.** There are many constants in the proof. We summarize their relations and the order we choose them here.

- $\mathbf{a}$ is the constant in the localization lemma (Theorem 5.1), and we assume $\mathbf{a} > N$;
- $\boldsymbol{\chi}$ is chosen so that it satisfies the calibration lemma (with constant $a = \mathbf{a}$)(Theorem 6.1);
- $\boldsymbol{\lambda}$ is chosen so that $\boldsymbol{\lambda} \gg \boldsymbol{\Lambda}, \boldsymbol{\chi}$, where $\boldsymbol{\Lambda}$ is the constant in the localization lemma (Theorem 5.1);

Let $[f] \in \partial\mathcal{H}$ be an eventually-golden-mean map. Let $\theta_1, ..., \theta_l$ be the list of rotation numbers for Siegel disks of f.

- $\boldsymbol{\epsilon}$ is chosen so that

$$\boldsymbol{\epsilon} \ll 1/\boldsymbol{\chi}\mathbf{a}.$$

We also assume that all psuedo-Siegel disks are T-stable where

$$T \gg 4\mathbf{a}N \max\{p_i\}$$

where p_i is the period of Siegel disks (see Remark 3.5). We remark that all the constants above depend only on the pulled-off constant N.

7.2. **Uniform geometric control.**

Lemma 7.2. *There exists a constant* $\mathbf{K}$ *depending on the pulled-off constant* N *so that* $\mathcal{W}_{arc}(\widehat{X}_f) \leq \mathbf{K}$.

Proof. Choose $\mathbf{K} \gg 1$ so that it is much bigger than the threshold in Theorem 5.1 (with constant $\epsilon = \boldsymbol{\epsilon}, \lambda = \boldsymbol{\lambda}$ and N), Theorem 6.1 (with constant $a = \mathbf{a}$).

Suppose by contradiction that $\mathcal{W}_{arc}(\widehat{X}_f) = K \geq \mathbf{K}$. By the localization lemma (Theorem 5.1), there exist a pseudo-Siegel disk $\widehat{Z}$ and a grounded interval I of $\widehat{Z}^m$ with

- $|I| \leq \boldsymbol{\epsilon} \ll 1/\boldsymbol{\chi}\mathbf{a}$; and
- $\mathcal{W}^{+,np}_m(I) + \mathcal{W}^{+,per}_{\boldsymbol{\lambda},m}(I) \geq 2K/\mathbf{a}$

Suppose that $\mathfrak{l}_{m+1} < |I| \leq \mathfrak{l}_m$. By Property (2)(b) of Theorem 3.4, the peripheral part

$$\mathcal{W}^{+,per}_{\boldsymbol{\lambda},m}(I) = O(\sqrt{\mathfrak{l}_m K} + 1) \ll K/\mathbf{a},$$

for all sufficiently large K.

Thus, $\mathcal{W}^{+,np}_m(I) \geq K/\mathbf{a}$. By Property (2)(a) for level m of Theorem 3.4, $K/\mathbf{a} \leq \mathcal{W}^{+,np}_m(I) + \mathcal{W}^{+,per}_{\boldsymbol{\lambda},m}(I) = O(K\mathfrak{l}_m + 1)$, so $\mathfrak{l}_m \geq \dfrac{1}{4\mathbf{a}}$. Therefore, we can apply Theorem 6.1, and obtain a grounded interval J with $|J| \leq \mathfrak{l}_{m+1}$ with

$$\mathcal{W}^{+,np}_{m+1}(J) \geq K/\boldsymbol{\chi}\mathbf{a}.$$

By Property (2)(a) for level $m+1$ of Theorem 3.4, we have

$$\mathcal{W}^{+,np}_{m+1}(J) = O(\mathfrak{l}_{m+1}K + 1).$$

Since $\mathfrak{l}_{m+1} \leq |I| \leq \boldsymbol{\epsilon} \ll 1/\boldsymbol{\chi}\mathbf{a}$, increase $\mathbf{K}$ if necessary, we have

$$\mathcal{W}^{+,np}_{m+1}(J) \ll K/\boldsymbol{\chi}\mathbf{a},$$

which is a contradiction. The lemma now follows. □

Proof of Theorem 7.1. The theorem follows by combining Theorem 3.4, Lemma 2.7 and Lemma 7.2. □

8. Dynamics on limiting trees and bounds on loop degeneration

In this section, we shall prove the following theorem giving uniform bounds of loop degeneration for eventually-golden-mean maps.

Theorem 8.1. *Let $\mathcal{H}$ be a hyperbolic component of disjoint type. Let $[f] \in \partial\mathcal{H}$ be an eventually-golden-mean map with (Siegel) pulled-off constant $N = N_{Siegel}([f])$. Let*

$$\widehat{X}_f := \widehat{\mathbb{C}} - \bigcup \mathrm{Int}(\widehat{D}_{i,j}) - \bigcup \mathrm{Int}(\widehat{Z}_{i,j})$$

be the Riemann surface as in Theorem 7.1. There exists a constant $\mathbf{K}_{loop}$ depending only on N and the multipliers of the attracting cycle of $[f]$ so that

$$\mathcal{W}_{loop}(\widehat{X}_f) \leq \mathbf{K}_{loop}.$$

8.1. **Limiting maps on trees.** Recall that the marked hyperbolic component $\mathcal{H}$ are parameterized by the $2d-2$ multipliers $\rho_1, ..., \rho_{2d-2}$:

$$\mathcal{H} \cong \mathbb{D}_1 \times ... \times \mathbb{D}_{2d-2}.$$

Fixing $a_1, ..., a_k \in \mathbb{D}$ and a constant N, and consider the slice

$$\begin{aligned}\mathfrak{A} := \{[f] \in \partial\mathcal{H} : &[f] \text{ is an eventually-golden-mean map, with} \\ &\rho_i = a_i, i = 1, ..., k, |\rho_i| = 1, i = k+1, ..., 2d-2, N_{Siegel}([f]) \leq N\}.\end{aligned}$$

To prove Theorem 8.1, it suffices to show that there exists a constant $\mathbf{K}_{loop}$ with $\mathcal{W}_{loop}(\widehat{X}_f) \leq \mathbf{K}_{loop}$ for any map $[f] \in \mathfrak{A}$.

The proof is by contradiction. We first show that, after passing to a subsequence, any sequence $[f_n] \in \mathfrak{A}$ converges to a limiting map on a tree of Riemann spheres. If there is no such constant $\mathbf{K}_{loop}$, then the limiting tree is non-trivial. The dynamics on the tree is recorded by a Markov matrix M and a degree matrix D. We show that there exists a non-negative vector $\vec{v} \neq \vec{0}$ with $M\vec{v} = D\vec{v}$. As matrices with non-negative entries, we show that $D^{-1}M$ is no bigger than the Thurston matrix for f_n for sufficiently large n. So the spectral radius of the Thurston's matrix is greater or equal to 1, giving a contradiction.

Following the notations in [Luo22b], we define

Definition 8.2. A *tree of Riemann spheres* $(\mathscr{T}, \widehat{\mathbb{C}}^{\mathscr{V}})$ consists of a finite tree $\mathscr{T}$ with vertex set $\mathscr{V}$, a disjoint union of Riemann spheres $\widehat{\mathbb{C}}^{\mathscr{V}} := \bigcup_{a \in \mathscr{V}} \widehat{\mathbb{C}}_a$, together with markings $\xi_a : T_a\mathscr{T} \hookrightarrow \widehat{\mathbb{C}}_a$ for $a \in \mathscr{V}$.

The image $\Xi_a := \xi_a(T_a\mathscr{T})$ is called the *singular set* at a, and $\Xi = \bigcup_{a \in \mathscr{V}} \Xi_a$.

A *rational map* (F, R) on $(\mathscr{T}, \widehat{\mathbb{C}}^{\mathscr{V}})$ is a map

$$F : (\mathscr{T}, \mathscr{V}) \longrightarrow (\mathscr{T}, \mathscr{V}) \text{ that is injective on edges,}$$

and a union of maps $R := \bigcup_{a \in \mathscr{V}} R_a$ so that

- $R_a : \widehat{\mathbb{C}}_a \longrightarrow \widehat{\mathbb{C}}_{F(a)}$ is a rational map;
- $R_a \circ \xi_a = \xi_{F(a)} \circ DF_a$.

It is said to have degree d if R has $2d-2$ critical points in $\widehat{\mathbb{C}}^{\mathscr{V}} - \Xi$.

A sequence f_n of degree d rational maps is said to *converge* to (F, R) on $(\mathscr{T}, \widehat{\mathbb{C}}^{\mathscr{V}})$ if there exist rescalings $A_{a,n} \in \mathrm{PSL}_2(\mathbb{C})$ for $a \in \mathscr{V}$ such that

- $A^{-1}_{F(a),n} \circ f_n \circ A_{a,n}(z) \to R_a(z)$ compactly on $\widehat{\mathbb{C}}_a - \Xi_a$;
- $A^{-1}_{b,n} \circ A_{a,n}(z)$ converges to the constant map $\xi_a(v)$, where $v \in T_a\mathscr{T}$ is the tangent vector in the direction of b.

Theorem 8.3. *Let $[f_n] \in \mathfrak{A}$. Then after passing to a subsequence $[f_n]$ converges to a degree d rational map (F, R) on $(\mathscr{T}, \widehat{\mathbb{C}}^{\mathscr{V}})$.*

Moreover, $[f_n]$ converges in $\mathcal{M}_{d,fm}$ if and only if $\mathscr{T}$ is trivial, i.e., $\mathscr{T}$ consists of a single vertex.

We remark that a similar result is proved for *quasi-post critically finite* degenerations of arbitrary rational maps in [Luo22b]. For quasi-post critically finite degeneration, the orbit of the critical points is controlled uniformly throughout the sequence. In our setting, the critical orbits are controlled in those valuable-attracting domains, as the multipliers stay constant, and Theorem 7.1 gives a uniform bound for pseudo-Siegel disks. More precisely, we use these uniform bounds crucially in two places.

(1) We use the fact that valuable-attracting domains and pseudo-Siegel disks are uniform quasi disks to show that their rescaling limits converge (Definition 8.5 and Lemma 8.6).
(2) We use the uniform bound on arc degeneration to control the holes for the rescaling limit (Proposition 8.16), which allow us to construct Thurston obstructions for large n (Proposition 8.19).

With these modifications, the proof is similar to the quasi-post critically finite case as in [Luo22b].

We also remark that the same proof of Theorem 8.3 also gives the following

Theorem 8.4. *Let $[f_n] \in \mathcal{H}$ be a seuqnece of eventually-golden-mean maps with uniformly bounded $\mathcal{W}_{arc}(\widehat{X}_f)$. Then after passing to a subsequence $[f_n]$ converges to a degree d rational map (F, R) on $(\mathscr{T}, \widehat{\mathbb{C}}^{\mathscr{V}})$.*

Moreover, $[f_n]$ converges in $\mathcal{M}_{d,fm}$ if and only if $\mathscr{T}$ is trivial, i.e., $\mathscr{T}$ consists of a single vertex.

Since maps in the slice $\mathfrak{A}$ are marked, we use $\mathcal{V}$ to denote the collection of valuable-attracting domains and open pseudo-Siegel disks.

More precisely, this means that if $U \in \mathcal{V}$ and $[f] \in \mathfrak{A}$, then $U(f)$ is either a valuable-attracting component or the interior of a pseudo-Siegel disk for f. Note that we have an induced dynamics

$$f_* : \mathcal{V} \longrightarrow \mathcal{V}.$$

Construction of the rescaling. Let us fix a sequence $[f_n] \in \mathfrak{A}$. We define the rescaling for $U \in \mathcal{V}$ as follows.

Definition 8.5. Let $U \in \mathcal{V}$. Let $\alpha_n \in U(f_n)$ be the corresponding non-repelling periodic point. A sequence $A_{U,n} \in \mathrm{PSL}_2(\mathbb{C})$ is called a *rescaling* for U if

- $A_{U,n}(0) = \alpha_n \in U(f_n)$;
- $A_{U,n}(1) \in \partial U(f_n)$;
- $A_{U,n}(\infty) \in \widehat{\mathbb{C}} - \overline{U(f_n)}$.

Lemma 8.6. *After passing to a subsequence, $A_{U,n}^{-1}(U(f_n))$ converges in Hausdorff topology to some quasiconformal disk.*

Proof. Since $U(f_n)$ are uniformly quasiconformal disks by Theorem 7.1, there is a sequence $\Psi_n : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ of uniformly quasiconformal maps so that $\Psi_n(\mathbb{D}) = U(f_n)$, normalized so that $\Psi_n(0) = A_{U,n}(0), \Psi_n(1) = A_{U,n}(1)$ and $\Psi_n(\infty) = A_{U,n}(\infty)$. Then $A_{U,n}^{-1} \circ \Psi_n$ is uniformly quasiconformal and fixes $0, 1, \infty$. Therefore, after passing to a subsequence, $A_{U,n}^{-1} \circ \Psi_n$ converges to a quasiconformal map Ψ. Thus, $A_{U,n}^{-1}(U(f_n))$ converges to the quasiconformal disk $\Phi(\mathbb{D})$. □

The following lemma follows from the same argument as in [Luo22b, Lemma 4.3].

Lemma 8.7. *If $A_{U,n}, B_{U,n}$ are two rescalings for $U \in \mathcal{V}$, then the sequence $B_{U,n}^{-1} \circ A_{U,n}$ is bounded.*

Equivalently, if we identify the hyperbolic 3-space $\mathbb{H}^3$ as the unit ball and $\mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{Isom}(\mathbb{H}^3)$, then

$$d_{\mathbb{H}^3}(A_{U,n}(\mathbf{0}), B_{U,n}(\mathbf{0})) \text{ is bounded,}$$

where $\mathbf{0} \in \mathbb{H}^3$ is the center of the unit ball.

Let us now fix rescaling $A_{U,n}$ for $U \in \mathcal{V}$.

Lemma 8.8. *Let $U \in \mathcal{V}$. Let $A_{U,n}, A_{f_*(U),n}$ be rescalings for U and $f(U)$. Then after passing to a subsequence,*

$$A_{f_*(U),n}^{-1} \circ f_n \circ A_{U,n}$$

converges (away from finitely many points) to a non-constant map.

Proof. Note that $A_{f_*(U),n}^{-1} \circ f_n \circ A_{U,n}$ is a sequence of rational maps, so after passing to a subsequent, it converges (away from finitely many points) to a rational map with degree $\leq d$. By Lemma 8.6, after passing to a subsequence, $A_{U,n}^{-1}(U(f_n))$ and $A_{f_*(U),n}^{-1}(f_*U(f_n))$ converge to U_∞ and $(f_*U)_\infty$. By [McM94, Theorem 5.6], $A_{f_*(U),n}^{-1} \circ f_n \circ A_{U,n}$ cannot converge to a constant map on U_∞, and the lemma follows. □

After passing to a subsequence, we may assume for different $U, V \in \mathcal{V}$, $A_{U,n}^{-1} \circ A_{V,n}$ converges to either

- a Möbius transformation; or

- a constant map.

This defines an equivalence relation on $\mathcal{V}$: $U \sim V$ if and only if $A_U^{-1} \circ A_V$ converges to a Möbius transformation. It follows from the same proof of [Luo22b, Lemma 4.7] that if $U \sim V$, then $f_*(U) \sim f_*(V)$.

Let $\Pi := \mathcal{V}/\sim$ be the set of equivalence classes. By the previous remark, we have an induced map

$$F : \Pi \longrightarrow \Pi.$$

Definition 8.9. For each equivalence class $a \in \Pi$, we choose a representative $U \in a$, and define the *rescaling* at a by

$$A_{a,n} := A_{U,n} \in \mathrm{PSL}_2(\mathbb{C}).$$

Construction of the tree of Riemann spheres $(\mathscr{T}, \widehat{\mathbb{C}}^{\mathscr{V}})$**.** Recall that we identify the hyperbolic 3-space $\mathbb{H}^3$ as the unit ball in $\mathbb{R}^3$ and $\widehat{\mathbb{C}}$ as the conformal boundary of $\mathbb{H}^3$. Denote $\mathbf{0} \in \mathbb{H}^3$ as the center of the unit ball. We denote $x_{a,n} = A_{a,n}(\mathbf{0}) \in \mathbb{H}^3$ and $\Pi_n = \{x_{a,n} \in \mathbb{H}^3 : a \in \Pi\}$. Note that by our construction, $A_{a,n}^{-1} \circ A_{b,n} \to \infty$ in $\mathrm{PSL}_2(\mathbb{C})$, so

$$d_{\mathbb{H}^3}(x_{a,n}, x_{b,n}) \to \infty \text{ if } a \neq b.$$

Thus, the hyperbolic polyhedra $\operatorname{Cvx}\operatorname{Hull}(\Pi_n)$ is degenerating. One can construct a sequence of trees $\mathscr{T}_n$ as the spine for $\operatorname{Cvx}\operatorname{Hull}(\Pi_n)$ capturing the degenerations of the polyhedra. We summarize some properties for $\mathscr{T}_n$ and refer the readers to [Luo22b, §3 and §4] for more details.

- The vertex set $\mathscr{V}_n$ for $\mathscr{T}_n$ is a finite set consisting of Π_n and branched points of $\mathscr{T}_n$;
- Each edge of $\mathscr{T}_n$ is a hyperbolic geodesic segment whose length goes to ∞ as $n \to \infty$;
- There exists a uniform lower bound on the angle between two adjacent edges of $\mathscr{T}_n$;
- The finite tree $\mathscr{T}_n \subseteq \operatorname{Cvx}\operatorname{Hull}(\Pi_n)$ and any point $x \in \operatorname{Cvx}\operatorname{Hull}(\Pi_n)$ is within uniform bounded distance from $\mathscr{T}_n$.

Since there are a bounded number of endpoints for $\mathscr{T}_n$, after passing to a subsequence, we assume that $\mathscr{T}_n$ are isomorphic as finite trees. Denote this isomorphic class of finite trees as $(\mathscr{T}, \mathscr{V})$, and we have a marking for each n

$$\Psi_n : (\mathscr{T}, \mathscr{V}) \longrightarrow (\mathscr{T}_n, \mathscr{V}_n).$$

We remark that $\Pi \subseteq \mathscr{V}$, and any point $a \in \mathscr{V} - \Pi$ is a branch point. We extend the definition of rescalings for $\mathscr{V} - \Pi$. Let $a \in \mathscr{V} - \Pi$, a sequence $A_{a,n} \in \mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{Isom}(\mathbb{H}^3)$ is defined to be a *resacling* at a if

$$A_{a,n}(\mathbf{0}) = \Psi_n(a).$$

Note that different choices of rescalings at a are differed by pre-composing with a rotation that fixes $\mathbf{0}$, which form a compact group.

Denote $\overline{\mathbb{B}} = \mathbb{H}^3 \cup \widehat{\mathbb{C}}$, and $\overline{\mathbb{B}}_a = \mathbb{H}^3_a \cup \widehat{\mathbb{C}}_a$ for $a \in \mathscr{V}$. We define a sequence $z_n \in \overline{\mathbb{B}}$ converges to $z \in \overline{\mathbb{B}}_a$ *in a-coordinate* or *with respect to the rescaling at* a, denoted by $z_n \to_a z$ or $z = \lim_a z_n$ if

$$\lim_{n\to\infty} A_{a,n}^{-1}(z_n) = z.$$

By construction, $A_{a,n}^{-1} \circ A_{b,n}$ converges to a constant map x_b for $a \neq b \in \mathscr{V}$. Thus, we can associate the point $x_b \in \widehat{\mathbb{C}}_a$ to b. It is interpreted that the Riemann sphere $\widehat{\mathbb{C}}_b$ converges to x_b in the rescaling coordinate $\widehat{\mathbb{C}}_a$. We denote

$$\Xi_a := \bigcup_{b\neq a} x_b \subseteq \widehat{\mathbb{C}}_a$$

as the *singular set* at a and $\Xi := \bigcup_{a\in\mathscr{V}} \Xi_a \subseteq \widehat{\mathbb{C}}^{\mathscr{V}}$ as the *singular set.*

It follows from the construction that $\Psi_n(b) \to_a x_b \in \widehat{\mathbb{C}}_a$ if $a \neq b \in \mathscr{V}$. Since the angle $\angle\Psi_n(a)\Psi_n(b)\Psi_n(c)$ is uniformly bounded below from 0 for any distinct triple $a, b, c \in \mathscr{V}$, the singular set Ξ_a is in correspondence with the tangent space $T_a\mathscr{T}$ at a. We denote this correspondence by

$$\xi_a : T_a\mathscr{T} \longrightarrow \Xi_a.$$

Construction of the rescaling rational maps. The following lemma allows us to construct rescaling rational maps.

Lemma 8.10. *Let $a \in \mathscr{V}$, after passing to a subsequence, there exists a unique $b \in \mathscr{V}$ so that*

$$A_{b,n}^{-1} \circ f_n \circ A_{a,n}$$

converges to a rational map $R_a = R_{a\to b}$ of degree at least 1.

Moreover, the holes of R_a are contained in Ξ_a.

Proof. If $a \in \Pi \subseteq \mathscr{V}$, i.e., if a is represented by some $U \in \mathcal{V}$, then Lemma 8.10 follows immediately from Lemma 8.8.

Otherwise, the proof is the same as [Luo22b, Lemma 4.12]. □

The above lemma allows us to define $F : \mathscr{V} \longrightarrow \mathscr{V}$ by setting $F(a)$ as the unique vertex b in Lemma 8.10 extending the map $F : \Pi \subseteq \mathscr{V} \longrightarrow \Pi \subseteq \mathscr{V}$.

We define the map $F : \mathscr{T} \longrightarrow \mathscr{T}$ by extending continuously on any edge $[a, b]$ to the arc $[F(a), F(b)]$.

Modulus estimate for dynamics on edges. Let $E = [a, b]$ be an edge of $\mathscr{T}$. In the following, we shall associate it with annuli $\mathcal{A}_{E,n}$ and define the local degree of E.

Let $x_b \in \Xi_a \subseteq \widehat{\mathbb{C}}_a$ and $x_a \in \Xi_b \subseteq \widehat{\mathbb{C}}_b$ be the points associated to b and a respectively. Choose a small closed curve $C_a \subseteq \widehat{\mathbb{C}}_a$ around x_b. We assume C_a bounds no holes nor critical points of R_a other than possibly x_b and $R_a : C_a \longrightarrow R_a(C_a)$ is a covering map of degree $\deg_{\xi_a(v_b)}(R_a)$. Similarly, we define $C_b \subseteq \widehat{\mathbb{C}}_b$ around x_a.

Since $A_{F(a),n}^{-1} \circ f_n \circ A_{a,n}$ converges uniformly to R_a in a neighborhood of C_a, we can find a sequence of closed curves $C_{a,n}$ such that

- $f_n : C_{a,n} \longrightarrow f_n(C_{a,n})$ is a covering of degree $\deg_{\xi_a(v_b)}(R_a)$;
- $A_{a,n}^{-1}(C_{a,n})$ converges in Hausdorff topology to C_a.

Similarly, let $C_{b,n}$ be the corresponding closed curves for C_b. Note that $C_{a,n}$ and $C_{b,n}$ are disjoint for sufficiently large n and bounds an annulus $\mathcal{A}_{E,n} \subseteq \widehat{\mathbb{C}}$. We call $\mathcal{A}_{E,n}$ a sequence of annuli associated to E.

We claim that $\mathcal{A}_{E,n}$ contains no critical points of f_n for sufficiently large n. Indeed, otherwise, after passing to a subsequence, let $c_n \in \mathcal{A}_{E,n}$, and let $c \in \Pi$ corresponds to the sequence (c_n). Consider the projection

$$\mathrm{proj}_{[\Psi_n(a),\Psi_n(b)]}(\Psi_n(c))$$

of $\Psi_n(c) \in \mathbb{H}^3$ onto the geodesics $[\Psi_n(a), \Psi_n(b)]$. Since we assume C_a bounds no holes nor critical points other than possibly $\xi_a(v_b)$ and similarly for C_b,

$$d_{\mathbb{H}^3}(\mathrm{proj}_{[\Psi_n(a),\Psi_n(b)]}(\Psi_n(c)), \partial[\Psi_n(a), \Psi_n(b)]) \to \infty.$$

This contradicts that $[a, b]$ is an edge of $\mathscr{T}$.

Therefore, $f_n : \mathcal{A}_{E,n} \longrightarrow f_n(\mathcal{A}_{E,n})$ is a covering map. Thus, $\deg_{x_b}(R_a) = \deg_{x_a}(R_b)$, and we define the local degree at E

$$\delta(E) := \deg_{x_b}(R_a) = \deg_{x_a}(R_b),$$

as the degree of this covering map.

The proof of following modulus estimate can be found in [Luo22b, Proposition 4.15].

Proposition 8.11. *Let $\mathcal{A}_{E,n}$ associated to an edge $E = [a, b]$ of $\mathscr{T}$. There exists a constant K such that the modulus*

$$|\operatorname{mod}(\mathcal{A}_{E,n}) - \frac{d_{\mathbb{H}^3}(\Psi_n(a), \Psi_n(b))}{2\pi}| \leq K,$$

and

$$|\operatorname{mod}(f_n(\mathcal{A}_{E,n})) - \frac{d_{\mathbb{H}^3}(\Psi_n(F(a)), \Psi_n(F(b)))}{2\pi}| \leq K.$$

The above modulus estimate gives the following corollaries.

Corollary 8.12. *Let $E = [a, b]$ be an edge of $\mathscr{T}$. Then*

$$d_{\mathbb{H}^3}(\Psi_n(F(a)), \Psi_n(F(b))) = \delta(E) d_{\mathbb{H}^3}(\Psi_n(a), \Psi_n(b)) + O(1).$$

Corollary 8.12 implies that the map F is injective on edges. Thus, we can define the tangent map $DF_a : T_a\mathscr{T} \longrightarrow T_{F(a)}\mathscr{T}$. Since $A_{b,n}^{-1} \circ f_n \circ A_{a,n}$ converges to a rational maps away from the singular set, we also have the following compatibility property.

Corollary 8.13. *Let $a \in \mathscr{T}$. Then*

$$R_a \circ \xi_a = \xi_{F(a)} \circ DF_a.$$

Proof of Theorem 8.3. By our construction, f_n converges to (R,F) and (R,F) is a degree d rational map on $(\mathscr{T},\widehat{\mathbb{C}}^{\mathscr{V}})$.

For the moreover part, we note that if $[f_n]$ converges in $\mathcal{M}_{d,\mathrm{fm}}$, then all rescaling limits $A_{U,n}$ are equivalent for $U \in \mathcal{V}$, so $\mathscr{T}$ consists of a single vertex. On the other hand, if all rescaling limits $A_{U,n}$ are equivalent, then $A_{U,n}^{-1} \circ f_n \circ A_{U,n}$ converges to a degree d rational map, so $[f_n]$ converges in $\mathcal{M}_{d,\mathrm{fm}}$. □

Matrix encoding. Index the set of edges of $\mathscr{T}$ by $\{E_1, ..., E_k\}$. We define the following two matrices to encode the dynamics $F : \mathscr{T} \longrightarrow \mathscr{T}$.

- (Markov matrix): $M_{i,j} = \begin{cases} 1 & \text{if } E_i \subseteq F(E_j) \\ 0 & \text{otherwise} \end{cases}$
- (Degree matrix): $D_{i,j} = \begin{cases} \delta(E_i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

Proposition 8.14. *Let M and D be the Markov matrix and the degree matrix respectively. If $\mathscr{T}$ is not trivial, then there exists a non-negative vector $\vec{v} \neq \vec{0}$ so that*

$$M\vec{v} = D\vec{v}.$$

Proof. Let $\vec{v}_n = \begin{bmatrix} l(\Psi_n(E_1)) \\ \vdots \\ l(\Psi_n(E_k)) \end{bmatrix}$, where $l(\Psi_n(E_i))$ is the hyperbolic length of the edge $\Psi_n(E_i)$. Let $\rho_n = \max_{i=1,\dots,k} l_{\mathbb{H}^3}(\Psi_n(E_i)) \to \infty$ After passing to a subsequence, we assume the limit $\vec{v} = \lim_{n\to\infty} \vec{v}_n/\rho_n$ exists. Then $\vec{v}$ is non-negative and $\vec{v} \neq \vec{0}$.

If suffices to check $M\vec{v} = D\vec{v}$. If $a, b \in \mathscr{V}$ are connected by a sequence of edges $E_{i_1} \cup E_{i_2} \cup ... E_{i_m}$, since the angles between different incident edges at a vertex of $\mathscr{T}_n$ are uniformly bounded below from 0, we have

$$d_{\mathbb{H}^3}(\Psi_n(a), \Psi_n(b)) = \sum_{j=1}^{m} l_{\mathbb{H}^3}(\Psi_n(E_{i_j})) + O(1).$$

Thus, by Corollary 8.12, if $F(E_i) = E_{i_1} \cup E_{i_2} \cup ... E_{i_m}$, then

$$\delta(E_i) l_{\mathbb{H}^3}(\Psi_n(E_i)) = \sum_{j=1}^{m} l_{\mathbb{H}^3}(\Psi_n(E_{i_j})) + O(1).$$

Dividing both sides by ρ_n and taking limits, we conclude the result. □

8.2. **Thurston's obstruction.** Let $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be a rational map with post-critical set P_f. Let $P_f \subseteq U$ be a forward invariant set, i.e., $f(U) \subseteq U$. A simple closed curve σ on $\widehat{\mathbb{C}} - U$ is *essential* if it does not bound a disk in $\widehat{C} - U$, and a curve is *peripheral* if it encloses a single point of U. Two simple curves are *parallel* if they are homotopic in $\mathbb{C} - U$.

A *curve system* $\Sigma = \{\sigma_i\}$ in $\widehat{\mathbb{C}} - U$ is a finite nonempty collection of disjoint simple closed curves, each essential and non-peripheral, and no two parallel. A curve system determines a *transition matrix* $A(\Sigma) : \mathbb{R}^\Sigma \longrightarrow \mathbb{R}^\Sigma$ by the formula

$$A_{\sigma\tau} = \sum_\alpha \frac{1}{\deg(f : \alpha \to \tau)}$$

where the sum is taken over components α of $f^{-1}(\tau)$ isotopic to σ.

Let $\lambda(\Sigma) \geq 0$ denote the spectral radius of $M(\Sigma)$. Since $A(\Sigma) \geq 0$, the Perron-Frobenius theorem guarantees that $\lambda(\Sigma)$ is an eigenvalue for $A(\Sigma)$ with a non-negative eigenvector.

The same proof of [McM94, Theorem B.4] gives

Proposition 8.15. *Let $[f] \in \mathfrak{A}$ be an eventually-golden-mean Siegel map. Let $\mathcal{U}_f$ be the union of Siegel disks and valuable-attracting domains. Let Σ be a curve system in $\widehat{\mathbb{C}} - \mathcal{U}_f$, then $\lambda(\Sigma) < 1$.*

Curve system for edges of $\mathscr{T}$. Let $\mathcal{U}_n$ be the union of Siegel disks and valuable-attracting domains for f_n, and let $\widehat{\mathcal{U}}_n$ be the union of open pseudo-Siegel disks and valuable-attracting domains for f_n. Note that $\mathcal{U}_n$ is forward invariant, and $\mathcal{U}_n \subseteq \widehat{\mathcal{U}}_n$. Let $\mathcal{A}_{E,n}$ be the annulus associated to an edge E.

After passing to a subsequence, we may assume that for any $a \in \mathscr{V}$, and any pseudo-Siegel disk or a valuable-attracting domain $U(f_n)$, the limit $A_{a,n}^{-1}(U(f_n))$ exists. Since $U(f_n)$ are uniformly quasiconformal disks, the limit is either a point or a quasiconformal disk. We say $U(f_n)$ is trivial for a if the limit is a point, and non-trivial otherwise.

The following lemma is the crucial step that we use the geometric control of valuable-attracting domains and pseudo-Siegel disks.

Proposition 8.16. *Let $a \in \mathscr{V}$. The singular set Ξ_a is disjoint from the closure of any non-trivial limits of pseudo-Siegel disks and valuable-attracting domains in $\widehat{\mathbb{C}}_a$.*

Proof. Let $U = \lim A_{a,n}^{-1}(U(f_n)) \subseteq \widehat{\mathbb{C}}_a$ be a non-trivial limit. It is easy to see that the singular set is disjoint from U. Now suppose $x \in \Xi_a \cap \partial U$. Since x is a singular point, there exists a sequence of pseudo-Siegel disk or a valuable-attracting domain $W(f_n)$ with $\lim A_{a,n}^{-1}(W(f_n)) = x$. Without loss of generality, we assume a is fixed. Since there is a critical point on the boundary of U, R_a has degree at least 2. Therefore, $\widehat{\mathbb{C}}_a$ contains at least two non-trivial limit of pseudo-Siegel disks or a valuable-attracting domains. Consider a small arc $\gamma \subseteq \widehat{\mathbb{C}}_a - U$ with $\partial\gamma \subseteq \partial U$ that encloses x. Then the corresponding arc with end points in $\partial U(f_n)$ for f_n is non-peripheral and its extremal width goes to infinity as $n \to \infty$. This is a contradiction to Theorem 7.1. $\square$

As a corollary, we have

Corollary 8.17. *For sufficiently large n, the core curve $\sigma_{E,n}$ of $\mathcal{A}_{E,n}$ is a curve in $\widehat{\mathbb{C}} - \widehat{\mathcal{U}}_n \subseteq \widehat{\mathbb{C}} - \mathcal{U}_n$.*

Let

$$\Sigma_n = \{\sigma_{E,n} : E \text{ is an edge of } \mathscr{T}\}.$$

Then Σ_n is a curve system in $\widehat{\mathbb{C}} - \mathcal{U}_n$ for all sufficiently large n.

Lemma 8.18. *If $E \subseteq F(E')$, then for sufficiently large n, $\sigma_{E,n}$ has a lift of degree $\delta(E')$ homotopic to $\sigma_{E',n}$ in $\widehat{\mathbb{C}} - \mathcal{U}_n$.*

Proof. Let $\mathcal{A}_{E,n}$ and $\mathcal{A}_{E',n}$ be annuli associated to E and E' respectively. Modify the boundaries of $\mathcal{A}_{E,n}$ if necessary, we may assume $\mathcal{A}_{E,n} \subseteq f_n(\mathcal{A}_{E',n})$, where the inclusion induces an isomorphism on the fundamental group. So for sufficiently large n, there exists an essential simple closed curve $\gamma_n \subseteq \mathcal{A}_{E,n}$ that has a degree $\delta(E)$ lift $\gamma_n' \subseteq \mathcal{A}_{E',n}$. By Proposition 8.16, these γ_n and γ_n' are homotopic to core curves $\sigma_{E,n}$ and $\sigma_{E',n}$ in $\widehat{\mathbb{C}} - \mathcal{U}_n$, and the lemma follows. □

Combining Proposition 8.14 and Lemma 8.18, we have

Proposition 8.19. *Let Σ_n be the curve system associated with the edges of $\mathscr{T}$ in $\widehat{\mathbb{C}} - \mathcal{U}_n$. If $\mathscr{T}$ is not trivial, i.e. it contains more than one vertex, then for sufficiently large n, the spectral radius $\lambda(\Sigma_n) \geq 1$.*

Proof of Theorem 8.1. Suppose for contradiction that Theorem 8.1 does not hold. Then there exists a sequence $[f_n] \in \mathfrak{A}$ with $\mathcal{W}_{loop}(X_{f_n}) \to \infty$. After passing to a subsequence, $[f_n]$ converges to a degree d rational map (F, R) on $(\mathscr{T}, \widehat{\mathbb{C}}^{\mathscr{V}})$ by Theorem 8.3. Since $[f_n] \to \infty$ in $\mathcal{M}_{d,\mathrm{fm}}$, the tree $\mathscr{T}$ is not trivial. By Proposition 8.19, the curve system Σ_n has spectral radius $\lambda(\Sigma_n) \geq 1$ for all sufficiently large n. This is a contradiction to Proposition 8.15. □

The same proof also gives Theorem 1.4 and Theorem A.

Proof of Theorem 1.4. Since the arc degeneration is uniformly bounded, f_n converges to a degree d rational map (F, R) on $(\mathscr{T}, \widehat{\mathbb{C}}^{\mathscr{V}})$ by Theorem 8.4. Suppose for contradiction that $[f_n]$ diverges, then the tree $\mathscr{T}$ is non-trivial. Therefore, $\mathcal{W}_{loop}(X_{f_n}) \to \infty$ which is a contradiction. Since the degeneration is uniformly bounded, the psuedo-Siegel disks and valuable-attracting domains do not collide. So $[f]$ has $2d-2$ non-repelling cycles. □

Proof of Theorem A. It suffices to show that the marked hyperbolic component $\mathcal{H}$ is bounded. Note that $\mathcal{H}$ is identified with $\mathbb{D}_1 \times ... \times \mathbb{D}_{2d-2}$. It suffices to realize the multiplier $(\lambda_1, ..., \lambda_{2d-2}) \in \partial(\mathbb{D}_1 \times ... \times \mathbb{D}_{2d-2})$ by a map $[f] \in \partial\mathcal{H}$. Let $[f_n] \in \partial\mathcal{H}$ be a sequence of eventually-golden-mean maps with the corresponding multipliers $(\lambda_{1,n}, ..., \lambda_{2d-2,n})$ converging to $(\lambda_1, ..., \lambda_{2d-2})$ strongly (see Definition 2.5).

By Theorem 4.1, the pulled-off constant is uniformly bounded in this case. By Theorem 8.1, $[f_n]$ has uniformly bounded degeneration, so $[f_n] \to [f] \in \mathcal{M}_d$ by Theorem 1.4. By construction, the corresponding multiplier profile of f is $(\lambda_1, ..., \lambda_{2d-2})$, and the theorem follows. □

Appendix A. Degenerations of Riemann surfaces

In this section, we introduce some terminologies to study degenerations of Riemann surfaces using extremal length and extremal width. There is a wealth of sources containing background material on this topic (see [Ahl73], [Kah06] or [KL09, Appendix 4]). We will briefly summarize the necessary minimum.

A.1. **Arcs and simple closed curves.** Let X be a compact Riemann surface with boundary. An *arc* γ of X is a continuous map

$$h : [0,1] \longrightarrow X$$

with $h(0), h(1) \in \partial X$. We shall not differentiate the continuous map with its image γ in X.

We say two arcs γ_0, γ_1 are homotopic, denoted by $\gamma_0 \sim \gamma_1$, if there exists a continuous path in the space of all arcs that connects γ_0 and γ_1. This means that there exists a continuous map

$$H : [0,1] \times [0,1] \longrightarrow X$$

with $H(t,0) = \gamma_0(t)$, $H(t,1) = \gamma_1(t)$, $H(0,s), H(1,s) \in \partial X$.

We remark that this is different from homotopy relative to ∂X, as we allow the homotopy to slide points on the boundary ∂X.

An arc γ is said to be *peripheral* if it is a homotopic to an arc that is contained in a boundary component of X. Note that each component of ∂X is a circle and an arc is peripheral if and only if it is homotopic to a point.

Similarly, a *closed curve* α of the Riemann surface X is a continuous map

$$h : \mathbb{S}^1 \longrightarrow X.$$

We do not differentiate the continuous map with its image α in X. Two closed curves are *homotopic* if the two continuous maps are homotopic. We denote this by $\alpha_0 \sim \alpha_1$. It is said to be *simple* if h is an embedding.

For simplicity, we refer to both arc and closed curve as curves.

A.2. **Extremal length and extremal width.** Let $\mathcal{F}$ be a family of curves on X. Given a (measurable) conformal metric $\rho = \rho(z)|dz|$ on X, let

$$L(\mathcal{F}, \rho) := \inf_{\gamma \in \mathcal{F}} L(\gamma, \rho),$$

where $L(\gamma, \rho)$ stands for the ρ-length of γ. The extremal length of $\mathcal{F}$ is

$$\mathcal{L}_X(\mathcal{F}) := \sup_{\rho} \frac{L(\mathcal{F}, \rho)^2}{A(X, \rho)},$$

where $A(U, \rho)$ is the area of X with respect to the measure ρ^2, and the supremum is taken over all ρ subject to the condition $0 < A(X, \rho) < \infty$. The extremal width of $\mathcal{F}$ is defined as the inverse of the extremal length:

$$\mathcal{W}_X(\mathcal{F}) = \frac{1}{\mathcal{L}_X(\mathcal{F})}.$$

A.2.1. *Series law and parallel law.* One of the key properties of the extremal width is that it behaves like resistance in an electric circuit.

We say a family of curves $\mathcal{F}$ *overflows* another family of curves $\mathcal{G}$ if every curve $\gamma \in \mathcal{F}$ contains a subcurve $\gamma' \in \mathcal{G}$. By definition, if $\mathcal{F}$ overflows $\mathcal{G}$, then

$$\mathcal{W}_X(\mathcal{F}) \leq \mathcal{W}_X(\mathcal{G}).$$

We say $\mathcal{F}$ *disjointly overflows* two families $\mathcal{G}_1, \mathcal{G}_2$ if any curve $\gamma \in \mathcal{F}$ contains the disjoint union $\gamma_1 \sqcup \gamma_2$ of two curves $\gamma_i \in \mathcal{G}_i$ (see Figure A.1). If $\mathcal{F}$ disjointly overflows $\mathcal{G}_1, \mathcal{G}_2$, then the Grötzsch inequality states that

$$\mathcal{W}_X(\mathcal{F}) \leq \mathcal{W}_X(\mathcal{G}_1) \bigoplus \mathcal{W}_X(\mathcal{G}_2), \tag{A.1}$$

where $x \bigoplus y = \dfrac{1}{\frac{1}{x} + \frac{1}{y}}$ is the harmonic sum. We shall refer to Equation A.1 the series law.

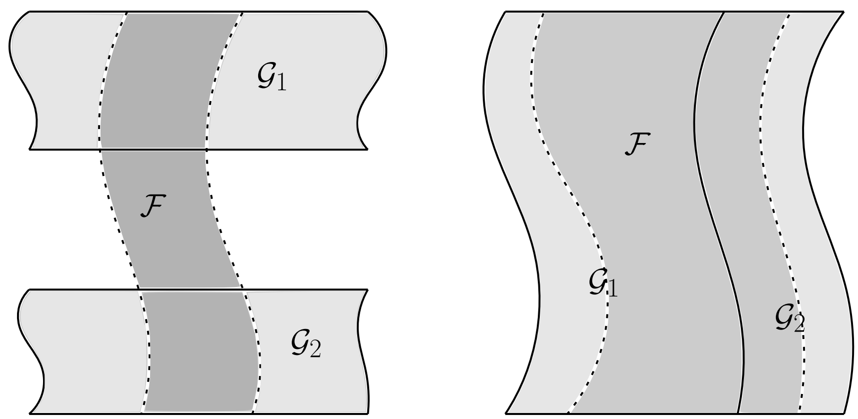

FIGURE A.1. An illustration of the series law on the left and parallel law on the right.

On the other hand, if $\mathcal{F} \subseteq \mathcal{G}_1 \cup \mathcal{G}_2$, i.e., every curve in $\mathcal{F}$ is either a curve in $\mathcal{G}_1$ or a curve in $\mathcal{G}_2$ (see Figure A.1), then

$$\mathcal{W}_X(\mathcal{F}) \leq \mathcal{W}_X(\mathcal{G}_1) + \mathcal{W}_X(\mathcal{G}_2). \tag{A.2}$$

We shall refer to Equation A.2 the parallel law.

A.2.2. *Extremal width between two sets.* Let $I, J \subseteq X$ be subsets of X. By an arc connecting I and J in X, we mean an arc parameterized by a continuous map $\gamma : [0,1] \longrightarrow X$ with $\gamma(0) \in I$, $\gamma(1) \in J$ and $\gamma((0,1)) \subseteq X - (I \cup J)$.

We use $\mathcal{W}_X(I, J)$ to denote the conformal widths of the family of arcs connecting I and J in X. When the underlying Riemann surface $X = \widehat{\mathbb{C}}$, we will sometimes omit the subindex, and simply write

$$\mathcal{W}(I,J) := \mathcal{W}_{\widehat{\mathbb{C}}}(I,J).$$

A.3. **Euclidean rectangles and (topological) rectangles.** A *Euclidean rectangle* is a rectangle $E_x := [0,x]\times[0,1] \subset \mathbb{C}$, where:

- $(0,0),(x,0),(x,1),(0,1)$ are four vertices of E_x,
- $\partial^h E_x = [0,x]\times\{0,1\}$ is the horizontal boundary of E_x,
- $\partial^{h,0}E_x := [0,x]\times\{0\}$ is the *base* of E_x,
- $\partial^{h,1}E_x := [0,x]\times\{1\}$ is the *roof* of E_x,
- $\partial^v E_x = \{0,x\}\times[0,1]$ is the *vertical* boundary of E_x,
- $\partial^{v,\ell}E_x := \{0\}\times[0,1]$, $\partial^{v,\rho}E_x := \{x\}\times[0,1]$ is the *left* and *right vertical* boundaries of E_x;
- $\mathcal{F}(E_x) := \{\{t\}\times[0,1] \mid t\in[0,x]\}$ is the *vertical foliation* of E_x,
- $\mathcal{F}^{\text{full}}(E_x) := \{\gamma : [0,1]\to E_x \mid \gamma(0)\in\partial^{h,0}E_x,\ \gamma(1)\in\partial^{h,1}E_x\}$ is the *full family of curves* in E_x;
- $\mathcal{W}(E_x) = \mathcal{W}(\mathcal{F}(E_x)) = \mathcal{W}(\mathcal{F}^{\text{full}}(E_x)) = x$ is the *width* of E_x,
- $\operatorname{mod}(E_x) = 1/\mathcal{W}(E_x) = 1/x$ the extremal length of E_x.

By a *(topological) rectangle* in a Riemann surface we mean a closed Jordan disk $\mathcal{R}$ together with a conformal map $g : \mathcal{R} \longrightarrow E_x$. We call the preimage $\partial^{h,0}\mathcal{R}$ of $[0,x]\times\{0\}$ the *base*, and the preimage $\partial^{h,1}\mathcal{R}$ of $[0,x]\times\{1\}$ the *roof*. We denote the horizontal boundaries by

$$\partial^h\mathcal{R} := \partial^{h,0}\mathcal{R}\cup\partial^{h,1}\mathcal{R}.$$

Similarly, we denote the vertical boundaries by

$$\partial^v\mathcal{R} := \partial^{v,0}\mathcal{R}\cup\partial^{v,1}\mathcal{R}.$$

The width of a rectangle R is

$$\mathcal{W}(\mathcal{R}) := \mathcal{W}_{\mathcal{R}}(\partial^{h,0}\mathcal{R},\partial^{h,1}\mathcal{R}) = x.$$

A K-*buffer* of a rectangle $\mathcal{R}$ is the image $g([0,K]\times[0,1]\cup[x-K,x]\times[0,1])$.

The collection of vertical arcs

$$\mathcal{F}_{v,\mathcal{R}} := \{g(\{t\}\times[0,1]) : t\in[0,x]\}$$

is called the *vertical foliation* of the rectangle $\mathcal{R}$. Similarly, the *horizontal foliation* of $\mathcal{R}$ is the collection

$$\mathcal{F}_{h,\mathcal{R}} := \{g([0,x]\times\{t\}) : t\in[0,1]\}.$$

Abusing the notations, when we say remove K-buffers for the vertical foliation, we mean the foliation

$$\mathcal{F} := \{g(\{t\}\times[0,1]) : t\in[K,x-K]\}.$$

A *genuine subrectangle* of E_x is any rectangle of the form $E' = [x_1,x_2]\times[0,1]$, where $0\le x_1 < x_2 \le x$; it is identified with the standard Euclidean rectangle $[0,x_2-x_1]\times[0,1]$ via $z\mapsto z-x_1$. A genuine subrectangle of a topological rectangle is defined accordingly.

A.4. **Arc and loop degenerations.** Let γ be a non-peripheral arc of X, and $\mathcal{F}(\gamma)$ be the family of arcs homotopic to γ. We define the degeneration for γ as the extremal width

$$\mathcal{W}(\gamma) := \mathcal{W}(\mathcal{F}(\gamma)).$$

Since majority of wide rectangles do not intersect (see, for example, [Kah06, §3]), there are only finitely many homotopy classes of non-peripheral arcs γ with $\mathcal{W}(\gamma) \geq 2$. In fact, this number is bounded by the topological complexity of X. We define the arc degeneration for X as

$$\mathcal{W}_{arc}(X) = \sum_{\gamma : \mathcal{W}(\gamma) \geq 2} \mathcal{W}(\gamma).$$

Similarly, if Z is a component of ∂X, we defined

$$\mathcal{W}^{loc}_{arc}(Z) = \sum_{\gamma \in \Gamma_1 : \mathcal{W}(\gamma) \geq 2} \mathcal{W}(\gamma) + 2 \sum_{\gamma \in \Gamma_2 : \mathcal{W}(\gamma) \geq 2} \mathcal{W}(\gamma)$$

where Γ_1 (or Γ_2) contains homotopy classes of non-peripheral arcs with exactly one endpoint (or two endpoints) on Z. Note that by definition,

$$2\mathcal{W}_{arc}(X) = \sum_{Z} \mathcal{W}^{loc}_{arc}(Z),$$

where the sum is over all boundary components of X.

Similarly, let α be a homotopically non-trivial simple closed curve, and let $\mathcal{G}$ be the family of simple closed curves isotopic to α. We remark here that the curve α is allowed to be homotopic to a boundary component of X. We define the *degeneration* for α of X as the extremal width

$$\mathcal{W}(\alpha) := \mathcal{W}(\mathcal{G}).$$

We define the *loop degeneration* for X as

$$\mathcal{W}_{loop}(X) = \sum_{\alpha : \mathcal{W}(\alpha) \geq 2} \mathcal{W}(\alpha).$$

A.5. **The Thin-Thick Decomposition.** Here, we summarize a few variations of the fundamental fact that wide families of curves are supported within finitely many pairwise disjoint wide rectangles. We refer the readers to [Lyu, §7.6] for more details.

Let $I, J \subset \partial X$ be two intervals. Let $\mathcal{F}_X(I, J)$ be the family of arcs connecting I and J in X. Then the *Thin-Thick Decomposition* of X rel the pair I, J says that, up to $O_{\chi(X)}(1)$, we can replace the family $\mathcal{F}_X(I, J)$ by a finitely many disjoint rectangles. More precisely, there exist finitely many pairwise disjoint non-homotopic rectangles $\mathcal{R}_1, ..., \mathcal{R}_s$ connecting I and J, i.e,

$$\partial^{h,0}\mathcal{R}_i \subset I \qquad \text{and} \qquad \partial^{h,1}\mathcal{R}_i \subset J,$$

such that

$$\sum_{i=1}^{s} \mathcal{W}(\mathcal{R}_i) = \mathcal{W}_X(I, J) - O_{\chi(X)}(1), \tag{A.3}$$

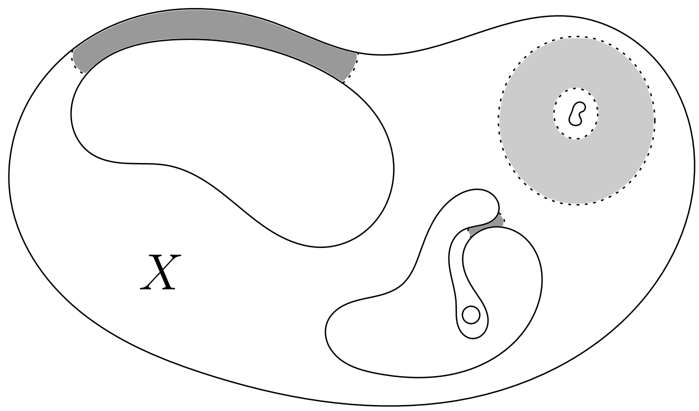

FIGURE A.2. An illustration of arc and loop degenerations. Here X is a genus 0 Riemann surface with 5 boundary components. Arc and loop degenerations are indicated by dark and light grey respectively.

where $\chi(X)$ is the Euler characteristic of X. We remark that since the rectangle are disjoint and non-homotopic, s is bounded by the topological complexity of the surface X.

The *Thin-Thick Decomposition* of X says that there are finitely many pairwise disjoint non-homotopic rectangles and annuli in X

$$\mathcal{T}_X = (\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_t, \mathcal{A}_1, \mathcal{A}_2, \dots \mathcal{A}_s), \qquad \partial^h R \subset \partial X$$

such that

- families
$$\mathcal{F}(\gamma), \quad \mathcal{F}_{arc}(X), \quad \mathcal{F}^{loc}_{arc}(Z)$$
for the corresponding
$$\mathcal{W}(\gamma), \quad \mathcal{W}_{arc}(X), \quad \mathcal{W}^{loc}_{arc}(Z)$$
introduced in §A.4 are supported, up to $O_{\chi(X)}(1)$, within finitely rectangles from $\mathcal{T}_X$,
- the family $\mathcal{F}_{loop}(X)$ (for $\mathcal{W}_{loop}(X)$) is formed by the annuli from $\mathcal{T}_X$.

Given a component Z of ∂X, the covering annuli $\mathbb{A}(X, Z)$ of X with respect to Z is obtained by opening up all loops except Z; see [DL23, §3.3.3] for a more detailed description. Then the family $\mathcal{F}\big(\mathbb{A}(X, Z)\big)$ of curves in $\mathbb{A}(X, Z)$ connecting its boundary components is, up to $O_{\chi(X)}(1)$, supported in the univalent lifts $\mathcal{R}^\tau_i \subset \mathbb{A}(X, Z), \tau \in \{0, 1\}$ of the rectangles $\mathcal{R}_i$ from $\mathcal{T}_X$ with

- $\tau = 0$ if $\partial^{h,0}\mathcal{R}_i \subset Z$,
- $\tau = 1$ if $\partial^{h,1}\mathcal{R}_i \subset Z$.

In particular, this means that

$$\begin{aligned}\mathcal{W}\left(\mathbb{A}(X,Z)\right) &= \sum_{i,\tau} \mathcal{W}(\mathcal{R}_i^\tau) + O_{\chi(X)}(1) \\ &= \sum_{\mathcal{R}_i \in \mathcal{T}_X,\, \partial^{h,0}\mathcal{R}_i \subset Z} \mathcal{W}(\mathcal{R}_i) + \sum_{\mathcal{R}_i \in \mathcal{T}_X,\, \partial^{h,1}\mathcal{R}_i \subset Z} \mathcal{W}(\mathcal{R}_i) + O_{\chi(X)}(1) \\ &= \mathcal{W}^{loc}_{arc}(Z) + O_{\chi(X)}(1).\end{aligned} \tag{A.4}$$

Appendix B. Siegel $\psi^\bullet$-ql maps and psuedo-Siegel disks

In this appendix, we summarize pseudo-Siegel bounds from [DL22] adopted to $\psi^\bullet$-ql maps.

We recall that ψ-quadratic-like maps were introduced in [Kah06]. They generalize the notion of quadratic-like maps with the goal of explicitly relating the geometry of various renormalizations of quadratic polynomials. It is essential for the theory that the post-critical set of the map is ι-proper (see §B.1). In [Kah06], it is assumed that the filled-in Julia set itself is ι-proper in the definition of ψ-ql maps. In this appendix, we will consider $\psi^\bullet$-ql Siegel maps with the requirement that the closed Siegel disk (and its iterated preimages) is ι-proper. We refer to [DL23] for a related notion of $\psi^\bullet$-ql "bush" maps. For technical reasons, we require in Item (II) that ι is a covering onto its image in the complement of the Siegel disk.

B.1. **$\psi^\bullet$-ql Siegel maps.** A map $\iota\colon A \to B$ between open Riemann surfaces is called an *immersion* if every $x \in X$ has a neighborhood U_x such that $\iota\colon U_x \to \iota(U_x)$ is a conformal isomorphism. Immersions arising in applications are compositions of covering maps and embeddings in various orders. A compact subset $S \Subset B$ is called *ι-proper* if $\iota \mid \iota^{-1}(S) \to S$ is a homeomorphism.

A *pseudo$^\bullet$-quadratic-like Siegel map* ("$\psi^\bullet$-ql Siegel map") is a pair of holomorphic maps

$$F = (f, \iota)\colon \quad (U, \overline{Z}_U) \rightrightarrows (V, \overline{Z}), \qquad \text{so} \quad \overline{Z}_U \subseteq f^{-1}(\overline{Z}) \cap \iota^{-1}(\overline{Z}) \tag{B.1}$$

between two conformal disks U, V with the following properties:

(I) $f\colon U \to V$ is a double branched covering with a unique critical point c_0;

(II) $\iota\colon U \to V$ is an immersion such that

$$\iota\colon U \setminus f^{-1}(\overline{Z}_U) \longrightarrow \iota\left(U \setminus f^{-1}(\overline{Z}_U)\right)$$

is a covering map;

(III) $\overline{Z}_U = \iota^{-1}(\overline{Z})$ is ι-proper; in particular, $\iota\colon \overline{Z}_U \xrightarrow{\simeq} \overline{Z}$;

define inductively $K_0 \coloneqq \overline{Z}$, $K_{1,U} \coloneqq f^{-1}(\overline{Z})$, $K_1 \coloneqq \iota(K_{1,U})$, and, for $n \ge 1$

$$K_{n,U} \coloneqq f^{-1} \circ \left(\iota \circ f^{-1}\right)^{n-1}(\overline{Z}) \qquad \text{and} \qquad K_n \coloneqq \iota(K_{n,U}) = \left(\iota \circ f^{-1}\right)^{n}(\overline{Z});$$

(IV) for all $n \ge 0$, the restriction $\iota\colon\ K_{n,U} \xrightarrow{\simeq} K_n$ is a homeomorphism;

(V) there exist neighborhoods $X_U \supset \overline{Z}_U$ and $X \supset \overline{Z}$ with the following property: $\iota : X_U \to X$ is a conformal isomorphism such that

$$f_X \coloneqq f \circ \left(\iota \mid X_U\right)^{-1} : X \to f(X_U) =: Y \tag{B.2}$$

is a *Siegel map*: $\overline{Z} \Subset X \cap Y$ is the closed qc Siegel disk around the fixed point $\alpha \in Z = \operatorname{int} \overline{Z}$ with bounded-type rotation number.

Since ι is a conformal isomorphism in a neighborhood of $\overline{Z}$, we will below identify

$$\overline{Z} \simeq \overline{Z}_U \equiv \overline{Z}_F \qquad \text{and write} \quad F\colon (U, \overline{Z}) \rightrightarrows (V, \overline{Z}) \qquad \text{or} \quad F\colon U \rightrightarrows V.$$

Similarly, we identify $K_n \simeq K_{n,U}$.

The *width* of F is

$$\mathcal{W}^\bullet(F) \coloneqq \mathcal{W}(V \setminus \overline{Z}).$$

If $\mathcal{W}^\bullet(F) \le K$, then $X_U \simeq X$ in Item (V) can be selected so that

$$\operatorname{mod}\,(X \setminus \overline{Z}) \ge \varepsilon(K). \tag{B.3}$$

Thereofre, $\psi^\bullet$-ql Siegel maps f with $\mathcal{W}^\bullet(f) \le K$ form a compact set.

Example B.1. *Consider a quadratic rational map $g \in \partial_{\mathrm{egm}}\mathcal{H}_{z^2}$, where H_{z^2} is the hyperbolic component of $z \mapsto z^2$, see §1.4. Assume that g has closed Siegel qc-disks $\overline{Z}_0, \overline{Z}_\infty$ at 0 and ∞. We naturally obtain two $\psi^\bullet$-ql maps:*

$$G_0 = (g, \hookrightarrow)\ :\ (U_0, \overline{Z}_0) \rightrightarrows (V_0, \overline{Z}_0), \qquad V_0 = \widehat{\mathbb{C}} \setminus \overline{Z}_\infty,\ U_0 = g^{-1}(V_0),$$

$$G_\infty = (g, \hookrightarrow)\ :\ (U_\infty, \overline{Z}_\infty) \rightrightarrows (V_\infty, \overline{Z}_\infty), \qquad V_\infty = \widehat{\mathbb{C}} \setminus \overline{Z}_0,\ U_\infty = g^{-1}(V_\infty),$$

where the immersion $\iota =$ "$\hookrightarrow$" is an embedding. We have:

$$\mathcal{W}^\bullet(G_0) = \mathcal{W}^\bullet(G_\infty) = \mathcal{W}(\widehat{\mathbb{C}} \setminus [\overline{Z}_0 \cup \overline{Z}_\infty]).$$

B.2. **$\psi^\bullet$-ql renormalization.** Consider a disjoint type hyperbolic components $\mathcal{H}$ and an eventually-golden mean map $[f] \in \partial_{\mathrm{egm}}\mathcal{H}$; see Definitions 2.1 and 2.4. The construction below is an adaptation of ψ-ql renormalization from [Kah06]; see also [DL23, §3].

Consider a periodic Siegel disk $Z_s = Z_{i,j}$ of f with period $p \ge 1$. We will now define a $\psi^\bullet$-ql map associated with Z. Write

$$X \coloneqq \widehat{\mathbb{C}} - \bigcup_{i,j} D_{i,j} - \bigcup_{i,j} Z_{i,j} \qquad \text{and} \qquad X' \coloneqq f^{-p}(X).$$

Since $X' \subset X$, we obtain a correspondence:

$$(f^p,\ \hookrightarrow)\colon\ X' \rightrightarrows X, \tag{B.4}$$

where $\hookrightarrow$ is a natural embedding. Consider the covering $\widetilde{X} \to X$ opening up all loops except ∂Z; in particular, $\widetilde{X}$ is an annulus. Similarly, the covering $\widetilde{X}' \to X'$ opens up all loops except (slightly thickened) $\partial(Z \cup Z')$, where Z' is the unique preperiodic f^p-lift of Z attached to Z.

Then (B.4) induces a correspondence

$$F = (f^p, \iota)\colon \widetilde{X}' \rightrightarrows \widetilde{X},$$

where f^p is a $2:1$ covering map and ι is an immersion obtained by lifting "$\hookrightarrow$". (In fact, ι is a covering onto its image). Gluing $\widetilde{X}$ with $\overline{Z}$ and gluing $\widetilde{X}'$ with $\overline{Z \cup Z'}$, we obtain a $\psi^\bullet$-ql map

$$F = (f^p, \iota)\colon U \rightrightarrows V \tag{B.5}$$

The Thin-Thick Decomposition in §A.5, or more precisely, the Equation A.4 implies that

$$\mathcal{W}^\bullet(F) = \mathcal{W}^{loc}_{arc}(Z) + O(1),$$

where $\mathcal{W}^{loc}_{arc}$ is defined in §A.4. Moreover, the rectangles in the Thin-Thick Decomposition of X adjacent to Z lift univalently into the dynamical plane of F; their lifts are disjoint rectangles connecting ∂V and ∂Z with total width being $\mathcal{W}(F) - O(1)$.

B.3. **A priori-bounds for $\psi^\bullet$-ql Siegel maps.** The definition of pseudo-Sigel disks for $\psi^\bullet$-ql Siegel maps is the same as Definition 3.1 (for maps in $\partial_{\text{egm}}\mathcal{H}$) with no peripheral requirements as in §3.2.1 – every set in $\operatorname{int} V$ is peripheral rel $\overline{Z}$. In other words, Property (P) in §3.2.2 takes form:

(P$^\bullet$) The territory $\mathcal{X}(\widehat{Z}^m)$ is a topological disk in V.

Let $D \supset Z$ be a peripheral disk. We say a curve γ in V is *vertical* or *non-peripheral* (rel D) if γ connects ∂D and ∂V in V, and we say it is *peripheral* (rel D) if $\partial\gamma \subseteq \partial D$.

Let $\lambda \geq 1$ and let I be an interval on $\widehat{Z}^m$. The families $\mathcal{F}^{+,ver}_{\widehat{Z}^m}(I), \mathcal{F}^{+,per}_{\lambda,\widehat{Z}^m}(I)$ and their corresponding widths $\mathcal{W}^{+,ver}_{\widehat{Z}^m}(I), \mathcal{W}^{+,per}_{\lambda,\widehat{Z}^m}(I)$ are defined accordingly as in §3.6. We remark that here $+$ in the superscript means that the curves in the family have interior contained in in $V - \widehat{Z}^m$.

Let $K_F \coloneqq \mathcal{W}^\bullet(F)$ be the width of F. We define the *special transition level* $\mathbf{m}_F$ for F as follows.

- If $K_F \leq 1$, we set $\mathbf{m}_F \coloneqq -2$;
- Otherwise, we set $\mathbf{m}_F$ to be the level satisfying

$$\frac{1}{\mathfrak{l}_{\mathbf{m}_F}} < K_F \leq \frac{1}{\mathfrak{l}_{\mathbf{m}_F+1}} \qquad \text{or, equivalently,} \qquad \begin{matrix} \mathfrak{l}_{\mathbf{m}_F} K_F > 1, \text{ and} \\ \mathfrak{l}_{\mathbf{m}_F+1} K_F \leq 1. \end{matrix}$$

We recall that $\mathfrak{l}_{-1} = 1$ and $\mathfrak{l}_0 = \operatorname{dist}(x, f(x))$.

The following theorem is proved in [DL22, Theorem C.3].

Theorem B.2. *Consider an eventually-golden-mean $\psi^\bullet$-ql map F (see §B.1) of width $K_F \coloneqq \mathcal{W}^\bullet(F)$ and the transition level $\mathbf{m}_F$. Then there is an increasing sequence of pseudo-Siegel disks $\widehat{Z}^m$, $m \geq -1$ such that for every grounded interval $J \subset \partial Z$ with $\mathfrak{l}_{m+1} < |J| \leq \mathfrak{l}_m$ the following holds:*

(A) if $m > \mathbf{m}_F$, then

$$\mathcal{W}^{+,ver}_{\widehat{Z}^m}(J^m) = O(1) \qquad \text{and} \qquad \mathcal{W}^{+,per}_{10,\widehat{Z}^m}(J^m) \asymp 1,$$

(B) if $m < \mathbf{m}_F$, then

$$\mathcal{W}^{+,ver}_{\widehat{Z}^m}(J^m) \asymp |J|K_F \qquad \text{and} \qquad \mathcal{W}^{+,per}_{10,\widehat{Z}^m}(J^m) = O(1),$$

(C) if $m = \mathbf{m}_F$, then

$$\mathcal{W}^{+,ver}_{\widehat{Z}^m}(J^m) = O(\mathfrak{l}_m K_F) \qquad \text{and} \qquad \mathcal{W}^{+,per}_{10,\widehat{Z}^m}(J^m) = O(\sqrt{\mathfrak{l}_m K_F}).$$

Moreover, $\widehat{Z}^{-1}$ is $M(K_F)$-qc disk; i.e. the dilatation of $\widehat{Z}^{-1}$ is bounded in terms of K_F.

We remark that in Cases (B) and (C), we have $|J|K_F,\ \mathfrak{l}_m K_F \geq 1$.

We also remark that in all three cases, we have the following bounds

$$\begin{aligned} \mathcal{W}^{+,ver}_{\widehat{Z}^m}(J^m) &= O(\mathfrak{l}_m K_F + 1) \\ \mathcal{W}^{+,per}_{10,\widehat{Z}^m}(J^m) &= O(\sqrt{\mathfrak{l}_m K_F} + 1). \end{aligned} \tag{B.6}$$

Remark B.3. In short, $\psi^\bullet$-formalism stated in Theorem B.2 takes care of all scales except the special transition scale $m = \mathbf{m}_F$. Case (A) says that on deep scales, the geometry of F is uniformly bounded, and the estimates are equivalent to that of quadratic polynomials. Case (B) says that on shallow scales, vertical degeneration dominates peripheral and is uniformly distributed at all intervals.

Theorem B.2 does not provide a satisfactory description of $\mathcal{W}^{+,ver}$ and $\mathcal{W}^{+,per}_{10}$ in Case (C). In our paper, such information comes from the global analysis of pseudo-Core surface degenerations stated in Theorem 5.1 and Theorem 6.1; see Remark 1.10.

For an explicit construction of "geodesic" pseudo-Siegel disks satisfying Theorem B.2, see §B.5.

B.4. **Localization of submergence.** Let us say that a rectangle $\mathcal{R}$ *submerges* into a pseudo-bubble $\widehat{Z}_i$ if

- $\partial^h \mathcal{R}$ is disjoint from $\mathcal{X}(\widehat{Z}_i)$; and
- every curve $\gamma \in \mathcal{F}(\mathcal{R})$ intersects $\widehat{Z}_i$.

Lemma B.4. *Assume that a rectangle $\mathcal{R}$ with $\mathcal{W}(\mathcal{R}) = K$ submerges into a pseudo-bubble $\widehat{Z}_i$. Then for every $\lambda > 2$, there is*

- *a grounded interval $J \subset \partial \widehat{Z}_i$ with $|J| < \dfrac{1}{\lambda^2}$, and*
- *sublamination $\widetilde{\mathcal{Q}} \subset \mathcal{F}(\mathcal{R})$ that overflows a lamination $\mathcal{Q}$ outside of $\widehat{Z}_i$ with $\mathcal{W}(\mathcal{Q}) \succeq K - O(\ln \lambda)$*

such that

- *either $\mathcal{Q}$ is a lamination from J to $\partial^{h,1}\mathcal{R}$;*
- *or $\mathcal{Q} \subset \mathcal{F}^+(J, \partial \widehat{Z}_i \setminus [\lambda J]^c)$; i.e., $\mathcal{Q}$ is a lamination outside of $\widehat{Z}_i$ from J to $\partial \widehat{Z}_i \setminus (\lambda J)$.*

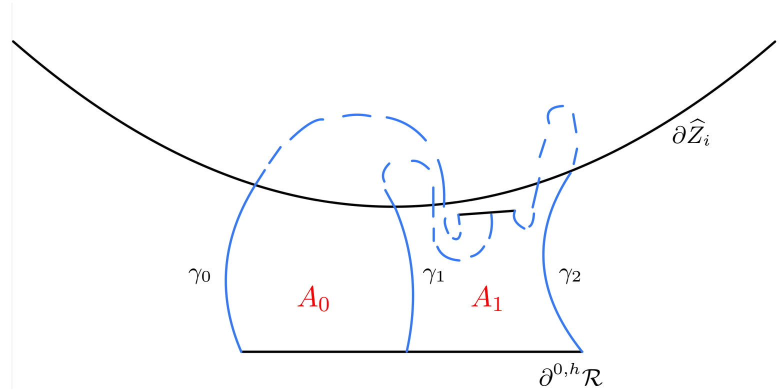

FIGURE B.1. An illustration of a rectangle submerging into a pseudo-bubble.

Proof. Let $\widetilde{\gamma_0}, \widetilde{\gamma_1}, \widetilde{\gamma_2}$ be the leftmost, middle and rightmost vertical arcs of the rectangle $\mathcal{R}$. We orient these arcs so that they connects the lower boundary $\partial^{h,0}\mathcal{R}$ to $\partial^{h,1}\mathcal{R}$. Since $\mathcal{R}$ submerges into $\widehat{Z}_i$, $\widetilde{\gamma_j}$ intersects $\widehat{Z}_i$. Let a_j be the first time $\widetilde{\gamma_j}$ enters $\widehat{Z}_i$, and let $\gamma_j \subseteq \widetilde{\gamma_j}$ be the sub arc connecting $\partial^{h,0}\mathcal{R}$ and a_j. Let A_0, A_1 be the region bounded by $\gamma_0, \gamma_1, \partial^{h,0}\mathcal{R}, \partial\widehat{Z}_i$ and $\gamma_1, \gamma_2, \partial^{h,0}\mathcal{R}, \partial\widehat{Z}_i$ as illustrated in Figure B.1). Since $\partial^{h,1}\mathcal{R}$ is disjoint from $\mathcal{X}(\widehat{Z}_i)$, at least one of the regions A_0, A_1 is disjoint from $\partial^{h,1}\mathcal{R}$. Without loss of generality, we may assume A_0 is disjoint from $\partial^{h,1}\mathcal{R}$. Consider the left rectangle $\mathcal{R}' \subseteq \mathcal{R}$ bounded by $\widetilde{\gamma_0}, \widetilde{\gamma_1}, \partial^{h,0}\mathcal{R}, \partial^{h,1}\mathcal{R}$, and let I be the interval on $\partial\widehat{Z}_i$ bounded by a_0, a_1. Then for every vertical γ arc connecting $\partial^{h,0}\mathcal{R}'$ to $\partial^{h,1}\mathcal{R}'$,

- the first intersection of γ with D is in I; and
- the last intersection of γ with D is in $I^c = \partial D \setminus I$.

With this reduction, we can directly apply [DL22, Lemma 6.9], and the lemma follows. □

B.4.1. *From $\mathcal{W}^{np}$ to $\mathcal{W}^{+,np} \sqcup \mathcal{W}^{+,p}_{\lambda}$.* We need the following submergence results in our main application to convert $\mathcal{W}^{np}$-degeneration into $\mathcal{W}^{+,np}$ and $\mathcal{W}^{+,p}_{\lambda}$-degenerations.

Lemma B.5. *Let $\widehat{Z}$ be a pseudo-Siegel disk. Let $\mathcal{R}$ be a rectangle with $\mathcal{W}(\mathcal{R}) = K$ such that the $I := \partial^{h,0}\mathcal{R}$ is a grounded interval on $\widehat{Z}$, and $\partial^{h,1}\mathcal{R}$ is disjoint from $\mathcal{X}(\widehat{Z})$. Then for every $\lambda > 2$, there is either*

- *a genuine subrectangle $\mathcal{R}_1$ of $\mathcal{R}$ with $\mathcal{W}(\mathcal{R}_1) \succeq K$ such that $\mathcal{R}_1$ is outside of* $\operatorname{int}\widehat{Z}$*; or*
- *a grounded interval $J \subset \partial\widehat{Z}$ such that $\mathcal{W}^{+,per}_{\lambda}(J) \succeq K - O(\ln\lambda)$; in particular, $|J| < \dfrac{1}{\lambda}$ if $K \gg \ln\lambda$.*

Proof. Assume that there are no genuine subrectangle $\mathcal{R}_1$ of $\mathcal{R}$ with $\mathcal{W}(\mathcal{R}_1) \succeq \mathcal{W}(\mathcal{R})$. Then a substantial part of $\mathcal{F}(\mathcal{R})$ submerges into $\operatorname{int} \widehat{Z}^m$ and we have two cases:

- either a substantial part of $\mathcal{F}(\mathcal{R})$ first submerges into $\operatorname{int} \widehat{Z}^m$ in $(\lambda^3 I)^c$;
- or a substantial part of $\mathcal{F}(\mathcal{R})$ first submerges into $\operatorname{int} \widehat{Z}^m$ in $\lambda^3 I$.

In the first case, we take $J := I$. The second case follows from [DL22, Corrolary 6.2] applied to either the pair $I \cup L_-,\ (\lambda^3 I)^c$ or to the pair $I \cup L_+,\ (\lambda^3 I)^c$, where L_-, L_+ are two intervals in $(\lambda^3 I) \setminus I$. □

B.5. **Geodesic pseudo-Siegel disks.** In this subsection, we summarize an explicit construction of *geodesic* pseudo-Siegel disks $\widehat{Z}$ satisfying Theorem B.2 for an eventually-golden-mean $\psi^\bullet$-ql map F. We refer the readers to [DL22, §C] for more discussions.

Choose an absolute but big constant $\mathbf{M} \gg 1$, and let $\mathbf{m}_F$ be the transition level defined in §B.3. We set

$$\mathbf{M}_m := \begin{cases} \mathbf{M}, \quad \text{if } m > \mathbf{m}_F, \\ \mathbf{M} + e^{\mathfrak{l}_m K_F}, \quad \text{if } m = \mathbf{m}_F, \\ \infty, \quad \text{if } m < \mathbf{m}_F. \end{cases} \tag{B.7}$$

We say that a level m is *near-parabolic* if $\mathfrak{l}_m > \mathbf{M}_m \mathfrak{l}_{m+1}$; otherwise m is *non-parabolic*. Since F is eventually-golden-mean, all sufficiently deep levels $m \gg_F 1$ are non-parabolic. In short, $\mathbf{M}_m$ will be a combinatorial threshold for regularization: if $\dfrac{\mathfrak{l}_m}{\mathfrak{l}_{m+1}} \geq \mathbf{M}_m$, then $\widehat{Z}^{m+1}$ is regularized into $\widehat{Z}^m$ at depth $e^{\sqrt{\ln \mathbf{M}_m}}$, see §B.5.1; otherwise $\widehat{Z}^m := \widehat{Z}^{m+1}$. We remark that by our definition of $\mathbf{M}_m$, if $m < \mathbf{m}_F$, then we always set $\widehat{Z}^m := \widehat{Z}^{m+1}$.

B.5.1. *Construction of parabolic fjord $\mathfrak{F}_I$ and S_I^{inn}.* Consider a parabolic level m. Let $I = [a, b] \in \mathfrak{D}_m$ be an interval in the mth diffeo-tiling of ∂Z. Choose $a', b' \in I$ with $a < a' < b' < b$ such that

$$\operatorname{dist}(a, a') = \operatorname{dist}(b', b) = \left\lfloor e^{\sqrt{\ln \mathbf{M}_m}} \right\rfloor \ \mathfrak{l}_{m+1}$$

and set β_I to be the hyperbolic geodesic of $V \setminus \overline{Z}$ connecting a', b'. This defines the parabolic fjord $\mathfrak{F}_I$, see Figure 3.2. We remark that since the level m is near-parabolic, we have

$$|I| = \operatorname{dist}(a, b) \asymp \mathfrak{l}_m \geq \mathbf{M}_m \mathfrak{l}_{m+1} \gg \left\lfloor e^{\sqrt{\ln \mathbf{M}_m}} \right\rfloor \ \mathfrak{l}_{m+1}.$$

To construct S_I^{inn}, we choose a sufficiently big $\mathbf{v} \gg 1$ with the understanding that $e^{\sqrt{\ln \mathbf{M}_m}} \gg \mathbf{v}$. Choose $a'' \in [a, a']$ and $b'' \in [b', b]$ such that

$$\operatorname{dist}(a, a'') = \operatorname{dist}(b'', b) = \left\lfloor \frac{e^{\sqrt{\ln \mathbf{M}_m}}}{\mathbf{v}} \right\rfloor \mathfrak{l}_{m+1} \gg \mathfrak{l}_{m+1}, \tag{B.8}$$

this defines S_I^{inn}, see Figure 3.2.

B.5.2. *Construction of A_I and $\overset{\star}{\mathcal{X}}_I$.* Let $\mathfrak{G}_I$ be the rectangle from $[a, a'']$ to $[b'', b]$ bounded by hyperbolic geodesics in $V \setminus \overline{Z}$; i.e.,

$$\partial^{h,0}\mathfrak{G}_I = [a, a''], \qquad \partial^{h,1}\mathfrak{G}_I = [b'', b]$$

and $\partial^v \mathfrak{G}_I$ is the pair of hyperbolic geodesics. The condition that $e^{\sqrt{\ln \mathbf{M}_m}} \gg \mathbf{v}$ implies that (see [DL22, §4])

$$\mathcal{W}(\mathfrak{G}_I) \asymp \ln\left(e^{\sqrt{\ln \mathbf{M}_m}}/\mathbf{v}\right) \gg 1.$$

We can now select a required rectangle $\mathcal{X}_I$ conformally deep in $\mathfrak{G}_I$ (i.e., conformally close to β_I) so that its width is of the size $\Delta \ll \sqrt{\ln \mathbf{M}_m}$. We can also select A_I separating $\mathcal{X}_I$ from S_I^{inn}. In particular, we can assume that the interval

$$[x_a, x_b] \coloneqq \partial \overset{\star}{\mathcal{X}}_I \cap I \qquad \text{with} \quad x_a \in [a, a''], \quad x_b \in [b'', b]$$

is defined similar to (B.8):

$$\operatorname{dist}(a, x_a) = \operatorname{dist}(x_b, b) = \left\lfloor \frac{e^{\sqrt{\ln \mathbf{M}_m}}}{\mathbf{w}} \right\rfloor \mathfrak{l}_{m+1},$$

where $\mathbf{w} \gg \mathbf{v} \gg 1$ with the understanding that we still have $e^{\sqrt{\ln \mathbf{M}_m}} \gg \mathbf{w}$.

B.5.3. *Stability of $\widehat{Z}^m$.* Since $e^{\sqrt{\ln \mathbf{M}_m}} \gg \mathbf{w}$, we see that the construction guarantees that $\operatorname{dist}(\partial^h \mathcal{X}_I, \partial I) = \left\lfloor \frac{e^{\sqrt{\ln \mathbf{M}_m}}}{\mathbf{w}} \right\rfloor \mathfrak{l}_{m+1} \gg \mathfrak{l}_{m+1}$. Thus, by the discussion in § 3.3, we see that $\widehat{Z}^m$ can be assumed to be T-stable for arbitrarily large T (see Remark 3.5).

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