# Math 589
# Problem Set 9

due Wednesday, May 6

Most problems are from Hartshorne, *Algebraic Geometry*. To typeset this problem set, I used **typst**, which is a faster alternative to LaTeX.

1. This problem shows that the higher Čech cohomology groups of a quasi-coherent sheaf on an affine variety are trivial. Let $X = \operatorname{Spec} A$ be an affine scheme, and let $\mathscr{F} = \widetilde{M}$ be a quasi-coherent sheaf. Let $\mathcal{U}$ be an arbitrary covering of $X$ by affine open subsets. Prove that the augmented Čech complex

$$0 \longrightarrow \mathscr{F}(X) \xrightarrow{\varepsilon} C^\bullet(\mathcal{U}, \mathscr{F})$$

   is exact, following the steps below.

   (a) Reduce the problem to the case where $\mathcal{U}$ is a finite cover of $X$ by basic open sets $D(f_1), \ldots, D(f_n)$. Show that the augmented Čech complex is then

$$0 \longrightarrow M \xrightarrow{\varepsilon} M \otimes B \xrightarrow{d} M \otimes B \otimes B \xrightarrow{d} M \otimes B \otimes B \otimes B \xrightarrow{d} \cdots$$

   where $B = A_{f_1} \oplus A_{f_2} \oplus \cdots \oplus A_{f_n}$ and all tensor products are over the ring $A$. Show that the augmentation map is $\varepsilon(m) = m \otimes 1$, while the differential is given by the formula

$$d(m \otimes b_0 \otimes b_1 \otimes \cdots \otimes b_k) = \sum_{i=0}^{k+1} (-1)^i m \otimes b_0 \otimes \cdots \otimes b_{i-1} \otimes b_{i+1} \otimes \cdots \otimes b_k.$$

   (b) Show that $B$ is a faithfully flat $A$-algebra (see Matsumura, p.45) and that the ring homomorphism $A \to B$ is injective.

   (c) Assume that $B$ has a *section*, meaning that there is a homomorphism of $A$-algebras $s : B \to A$. Prove that the complex is exact by constructing a null homotopy

   (d) Prove the exactness of the complex in general.

   (e) Conclude that $H^i(\mathcal{U}, \mathscr{F}) = 0$ for $i > 0$.

2. Hartshorne, III.2.3a-e
3. Harthorne, III.4.3
4. Hartshorne, III.5.2a
5. Hartshorne, III.5.10