WEEK 7 HOMEWORK

From Gathmann's text, do 7.7, 7.27, 7.30.

Extra problems

1. (Shafarevich, Exercise I.6.12) Let $F(x_0, x_1, x_2, x_3)$ be a homogeneous polynomial of degree 4. Prove that there exists a polynomial Φ in the coefficients of F such that $\Phi(F) = 0$ is a necessary and sufficient condition for the surface $V(F) \subseteq \mathbb{P}^3$ to contain a line.

2. (Shafarevich, Exercise I.6.13) Let $Q \subseteq \mathbb{P}^3$ be an irreducible quadric surface, and let $X \subseteq G(2,4)$ be the set of lines in $\mathbb{P}^3$ that are contained in Q. Prove that X consists of two disjoint curves (whose images under the Plücker embedding are conics in $\mathbb{P}^5$).

Bonus problems (optional)

3. (From a discussion with Vinny.) Consider the subset

$$X = \{ (x,y) \in \mathbb{A}^2 \mid y \neq 0 \text{ or } x = y = 0 \}.$$

Show that it is not possible to make X into an algebraic variety, in such a way that the inclusion $X \hookrightarrow \mathbb{A}^2$ becomes a morphism of algebraic varieties.

4. By contrast, consider the subring

$$R = K[x, y, x^2/y, x^3/y^2, \dots] \subseteq K[x, y, 1/y].$$

Show that the set of maximal ideals of R (with the Zariski topology) is isomorphic to the set X from above (with the subspace topology). Why does this not contradict the result in the previous problem?

5. In fact, the problem above is a special case of the following result:

Theorem. *Let Z be a closed subset of an algebraic variety X. If the open subvariety $X \setminus Z$ is affine, then every irreducible component of Z has codimension 1 in X.*

The proof needs a certain amount of commutative algebra.

a. Reduce the problem to the case where X is affine and irreducible and $Z \neq \emptyset$. Define $A = \mathscr{O}_X(X)$ and $B = \mathscr{O}_X(U)$; then A is a subring of B, and both rings are integral domains and noetherian.
b. Let $f \in I(Z)$ be any nonzero element in the ideal of Z. Show that for every $b \in B$, there is some $n \in \mathbb{N}$ such that $f^n b \in A$.
c. Consider the A-module $M = B/A$ and the collection of submodules

$$M[f] = \{ m \in M \mid fm = 0 \}.$$

Show that we have an exact sequence

$$0 \to M[f] \to A/fA \to B/fB \to M/fM \to 0,$$

and deduce that $M[f]$ is a finitely generated A-module. Further, show that

$$M = \bigcup_{n\in\mathbb{N}} M[f^n].$$

d. Let $Z_0 \subseteq Z$ be any irreducible component, and $P \subseteq A$ the corresponding prime ideal. Show that the maximal ideal of the local ring A_P is the only associated prime of the localized module M_P. Deduce that the submodule $M_P[f]$ is an A_P-module of finite length.

e. Denote by ℓ_n the length of the A_P-module $M_P[f^n]$. Show that $\ell_n \leq \ell_{n+1}$. Then use the short exact sequence

$$0 \to M_P[f] \to M_P[f^{n+1}] \xrightarrow{f} M_P[f^n] \to M_P/fM_P$$

to show that M_P/fM_P also has finite length.

f. Deduce that B_P/fB_P is a finitely generated A_P-module, and hence that

$$A_P/fA_P \to B_P/fB_P$$

is an integral ring homomorphism.

g. Use the Going-Up Theorem to show that B_P/fB_P must be the zero ring, and hence that $bf = s$ for some $s \in A \setminus P$ and some $b \in B$.

h. Conclude that there is an affine open subset $U \subseteq X$ such that the intersection $Z_0 \cap U$ is nonempty and satisfies

$$Z_0 \cap U = \left\{\, x \in U \;\middle|\; f(x) = 0 \,\right\}.$$

Deduce that $\operatorname{codim} Z_0 = 1$, as claimed.