MATH 546, HOMEWORK 1

1. Zero locus. Let f, g be holomorphic functions in some neighborhood of $0 \in \mathbb{C}^n$. Suppose that f is irreducible in $\mathscr{O}_n$, and that g vanishes on $Z(f) = f^{-1}(0)$, in the sense that $g(z) = 0$ for every $z \in Z(f)$. Prove that f divides g in $\mathscr{O}_n$.

2. The Riemann extension theorem. Let $D \subseteq \mathbb{C}^n$ be an open subset and $f \in \mathscr{O}(D)$ a holomorphic function. If $g\colon D \setminus Z(f) \to \mathbb{C}$ is holomorphic and bounded, prove that it extends uniquely to a holomorphic function on all of D.

3. Hensel's lemma. Let $h(z,t) \in \mathscr{O}_n[t]$ be a monic polynomial of degree d, with coefficients in the ring $\mathscr{O}_n$. Then $h(0,t)$ is a monic polynomial in t, and can therefore be factored by the fundamental theorem of algebra as

$$h(0,t) = (t - c_1)^{d_1} \cdots (t - c_r)^{d_r}$$

where $d_1 + \cdots + d_r = d$ and the $c_j \in \mathbb{C}$ are distinct. Prove that there are uniquely determined monic polynomials $p_1, \ldots, p_r \in \mathscr{O}_n[t]$, such that $h = p_1 \cdots p_r$ and $p_j(0,t) = (t - c_j)^{d_j}$ for every $j = 1, \ldots, r$.

Due on Tuesday, September 8.