COMPLEX MANIFOLDS, FALL 2024

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Class 1. Holomorphic functions (August 27)

Introduction. The subject of this course is *complex manifolds*. Recall that a smooth manifold is a space in which some neighborhood of every point is homeomorphic to an open subset of $\mathbb{R}^n$, such that the transitions between those open sets are given by smooth functions. Similarly, a complex manifold is a space in which some neighborhood of every point is homeomorphic to an open subset of $\mathbb{C}^n$, such that the transitions between those open sets are given by holomorphic functions.

Here is a brief overview of what we are going to do this semester. The first few classes will be taken up with studying holomorphic functions in several variables; in some ways, they are similar to the familiar theory of functions in one complex variable, but there are also many interesting differences. Afterwards, we will use that basic theory to define complex manifolds.

The study of complex manifolds has two different subfields:

(1) Function theory: concerned with properties of holomorphic functions on open subsets $D \subseteq \mathbb{C}^n$.
(2) Geometry: concerned with global properties of (for instance, compact) complex manifolds.

In this course, we will be more interested in global results; we will develop the local theory only as needed.

Two special classes of complex manifolds will appear very prominently in this course. The first is *Kähler manifolds*; these are (usually, compact) complex manifolds that are defined by a differential-geometric condition. Their study involves a fair amount of differential geometry, which will be introduced at the right moment. The most important example of a Kähler manifold is complex projective space $\mathbb{P}^n$ (and any submanifold). This space is also very important in algebraic geometry, and we will see many connections with that field as we go along. (Note that no results from algebraic geometry will be assumed, but if you already know something, this course will show you a different and more analytic point of view towards complex algebraic geometry.) Three of the main results that we will prove about compact Kähler manifolds are:

(1) The Hodge theorem. It says that the cohomology groups $H^*(X, \mathbb{C})$ of a compact Kähler manifold have a special structure, with many useful consequences for their geometry and topology.
(2) The Kodaira embedding theorem. It gives necessary and sufficient conditions for being able to embed X into projective space.
(3) Chow's theorem. It says that a complex submanifold of projective space is actually an algebraic variety.

The second class is *Stein manifolds*; here the main example is $\mathbb{C}^n$ (and its submanifolds). Since the 1950s, the main tool for studying Stein manifolds has been the theory of coherent sheaves. Sheaves provide a formalism for passing from local results (about holomorphic functions on small open subsets of $\mathbb{C}^n$, say) to global results, and we will carefully define and study coherent sheaves. Time permitting, we will prove the following two results:

(1) The embedding theorem. It says that a Stein manifold can always be embedded into $\mathbb{C}^n$ for sufficiently large n.
(2) The finiteness theorem. It says that the cohomology groups of a coherent sheaf on a compact complex manifold are finite-dimensional vector spaces; the proof uses the theory of Stein manifolds.

Along the way, we will introduce many useful techniques, and prove many other interesting theorems.

About the course. I am not following a single textbook; instead, I will make notes for each lecture available on my website, at

`https://www.math.stonybrook.edu/~cschnell/mat545/`

References in the notes will be by lecture, meaning that Lemma 3.2, say, occurs in the third lecture. There will be weekly homework assignments, too; each assignment will be handed out on Thursday, and will be due on Thursday of the following week. Homework problems will sometimes be the details of some proof, sometimes more specific examples or questions. You can probably find many of the statements and solutions in various textbooks, but please resist the temptation to look them up.

Holomorphic functions. Our first task is to generalize the notion of holomorphic function from one to several complex variables. There are many equivalent ways of saying that a function $f(z)$ in one complex variable is holomorphic (e.g., the derivative $f'(z)$ exists; f can be locally expanded into a convergent power series; f satisfies the Cauchy-Riemann equations; etc.). Perhaps the most natural definition in several variables is the following:

Definition 1.1. Let D be an open subset of $\mathbb{C}^n$, and let $f\colon D \to \mathbb{C}$ be a complex-valued function on D. Then f is *holomorphic* in D if each point $a \in D$ has an open neighborhood U, such that the function f can be expanded into a power series

$$f(z) = \sum_{k_1,\dots,k_n=0}^{\infty} c_{k_1,\dots,k_n}(z_1 - a_1)^{k_1} \cdots (z_n - a_n)^{k_n} \tag{1.2}$$

which converges for all $z \in U$. We denote the set of all holomorphic functions on D by the symbol $\mathscr{O}(D)$.

More generally, we say that a mapping $f\colon D \to E$ between open sets $D \subseteq \mathbb{C}^n$ and $E \subseteq \mathbb{C}^m$ is *holomorphic* if its m coordinate functions $f_1, \dots, f_m\colon D \to \mathbb{C}$ are holomorphic functions on D.

It is often convenient to use multi-index notation with formulas in several variables: for $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ and $z \in \mathbb{C}^n$, we let $z^k = z_1^{k_1} \cdots z_n^{k_n}$; we can then write the formula in (1.2) more compactly as

$$f(z) = \sum_{k \in \mathbb{N}^n} c_k (z-a)^k.$$

The familiar convergence results from one complex variable carry over to this setting (with the same proofs). For example, if the series (1.2) converges at a point $b \in \mathbb{C}^n$, then it converges absolutely and uniformly on the open polydisk

$$\Delta(a; r) = \left\{ z \in \mathbb{C}^n \;\middle|\; |z_j - a_j| < r_j \right\},$$

where $r_j = |b_j - a_j|$ for $j = 1, \dots, n$. In particular, a holomorphic function f is automatically continuous, being the uniform limit of continuous functions. A second consequence is that the series (1.2) can be rearranged arbitrarily; for instance, we may give certain values $b_1, \dots, b_{j-1}, b_{j+1}, \dots, b_n$ to the coordinates $z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n$, and then (1.2) can be rearranged into a convergent power series in $z_j - a_j$ alone. This means that a holomorphic function $f \in \mathscr{O}(D)$ is holomorphic in each variable separately, in the sense that $f(b_1, \dots, b_{j-1}, z, b_{j+1}, \dots, b_n)$ is a holomorphic function of z, provided only that $(b_1, \dots, b_{j-1}, z, b_{j+1}, \dots, b_n) \in D$.

Those observations have a partial converse, known as Osgood's lemma; it is often useful for proving that some function is holomorphic.

Lemma 1.3 (Osgood's lemma)**.** *Let f be a complex-valued function on an open subset $D \subseteq \mathbb{C}^n$. If f is continuous and holomorphic in each variable separately, then it is holomorphic on D.*

Proof. Let $a \in D$ be an arbitrary point, and choose a closed polydisk

$$\overline{\Delta}(a; r) = \left\{ z \in \mathbb{C}^n \;\middle|\; |z_j - a_j| \leq r_j \right\}$$

contained in D. On an open neighborhood of $\Delta(a; r)$, the function f is holomorphic in each variable separately. We may therefore apply Cauchy's integral formula for functions of one complex variable repeatedly, until we arrive at the formula

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_1 - a_1| = r_1} \cdots \int_{|\zeta_n - a_n| = r_n} f(\zeta_1, \dots, \zeta_n) \frac{d\zeta_n}{\zeta_n - z_n} \cdots \frac{d\zeta_1}{\zeta_1 - z_1},$$

valid for any $z \in \Delta(a; r)$. For fixed z, the integrand is a continuous function on the compact set

$$S(a, r) = \left\{ \zeta \in \mathbb{C}^n \;\middle|\; |\zeta_j - a_j| = r_j \right\},$$

and so Fubini's theorem allows us to replace the iterated integral above by

$$f(z) = \frac{1}{(2\pi i)^n} \int_{S(a,r)} \frac{f(\zeta_1, \dots, \zeta_n) d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}. \tag{1.4}$$

Now for any point $z \in \Delta(a; r)$, the power series

$$\frac{1}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} = \sum_{k_1, \dots, k_n = 0}^{\infty} \frac{(z_1 - a_1)^{k_1} \cdots (z_n - a_n)^{k_n}}{(\zeta_1 - a_1)^{k_1 + 1} \cdots (\zeta_n - a_n)^{k_{n+1}}}$$

converges absolutely and uniformly on S. We may therefore substitute this series expansion into (1.4); after interchanging summation and integration, and reordering the series, it follows that $f(z)$ has a convergent series expansion as in (1.2) on $\Delta(a; r)$, where

$$c_{k_1, \dots, k_n} = \frac{1}{(2\pi i)^n} \int_{S(a,r)} \frac{f(\zeta_1, \dots, \zeta_n) d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - a_1)^{k_1 + 1} \cdots (\zeta_n - a_n)^{k_{n+1}}}$$

This concludes the proof. □

In fact, Lemma 1.3 remains true without the assumption that f is continuous; this is the content of Hartog's theorem, which we do not prove here.

The formula in (1.4) generalizes the Cauchy integral formula to holomorphic functions of several complex variables. But, different from the one-variable case, the integral in (1.4) is not taken over the entire boundary of the polydisk $\Delta(a; r)$, but only over the n-dimensional subset $S(a, r)$.

Cauchy-Riemann equations. In one complex variable, holomorphic functions are characterized by the Cauchy-Riemann equations: a continuously differentiable function $f = u + iv$ in the variable $z = x + iy$ is holomorphic iff $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$. With the help of the two operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

these equations can be written more compactly as $\partial f/\partial \bar{z} = 0$. Osgood's lemma shows that this characterization holds in several variables as well: a continuously differentiable function $f \colon D \to \mathbb{C}$ is holomorphic iff it satisfies

$$\frac{\partial f}{\partial \bar{z}_1} = \cdots = \frac{\partial f}{\partial \bar{z}_n} = 0. \tag{1.5}$$

Indeed, such a function f is continuous and holomorphic in each variable separately, and therefore holomorphic by Lemma 1.3.

The operators $\partial/\partial z_j$ and $\partial/\partial \bar{z}_j$ are very useful in studying holomorphic functions. It is easy to see that

$$\frac{\partial z_j}{\partial \bar{z}_k} = \frac{\partial \bar{z}_j}{\partial z_k} = 0 \quad \text{while} \quad \frac{\partial z_j}{\partial z_k} = \frac{\partial \bar{z}_j}{\partial \bar{z}_k} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

This allows us to express the coefficients in the power series (1.2) in terms of f: termwise differentiation proves the formula

$$c_{k_1,\dots,k_n} = \frac{1}{(k_1!)\cdots(k_n!)} \cdot \frac{\partial^{k_1+\cdots+k_n} f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}(a). \tag{1.6}$$

As another application of the differential operators $\partial/\partial z_j$ and $\partial/\partial \bar{z}_j$, let us show that the composition of holomorphic mappings is holomorphic. It clearly suffices to show that if $f \colon D \to E$ is a holomorphic mapping between open subsets $D \subseteq \mathbb{C}^n$ and $E \subseteq \mathbb{C}^m$, and $g \in \mathscr{O}(E)$, then $g \circ f \in \mathscr{O}(D)$. Let $z = (z_1, \dots, z_n)$ denote the coordinates on D, and $w = (w_1, \dots, w_m)$ those on E; then $w_j = f_j(z_1, \dots, z_n)$. By the chain rule, we have

$$\frac{\partial(g \circ f)}{\partial \bar{z}_k} = \sum_j \left(\frac{\partial g}{\partial w_j}\frac{\partial f_j}{\partial \bar{z}_k} + \frac{\partial g}{\partial \bar{w}_j}\frac{\partial \bar{f}_j}{\partial \bar{z}_k}\right) = 0,$$

because $\partial f_j/\partial \bar{z}_k = 0$ and $\partial g/\partial \bar{w}_j = 0$.

Actually, the property of preserving holomorphic functions completely characterizes holomorphic mappings.

Lemma 1.7. *A mapping $f \colon D \to E$ between open subsets $D \subseteq \mathbb{C}^n$ and $E \subseteq \mathbb{C}^m$ is holomorphic iff $g \circ f \in \mathscr{O}(D)$ for every holomorphic function $g \in \mathscr{O}(E)$.*

Proof. One direction has already been proved; the other is trivial, since $f_j = w_j \circ f$, where w_j are the coordinate functions on E. □

Basic properties. Before undertaking a more careful study of holomorphic functions, we prove a few basic results that are familiar from the function theory of one complex variable. The first is the identity theorem.

Theorem 1.8. *Let D be a connected open subset of $\mathbb{C}^n$. If f and g are holomorphic functions on D, and if $f(z) = g(z)$ for all points z in a nonempty open subset $U \subseteq D$, then $f(z) = g(z)$ for all $z \in D$.*

Proof. By looking at $f - g$, we are reduced to considering the case where $g = 0$. Since f is continuous, the set of points $z \in D$ where $f(z) = 0$ is relatively closed in D; let E be its interior. By assumption, E is nonempty; to prove that $E = D$, it suffices to show that E is relatively closed in D, because D is connected. To that end, let $a \in D$ be any point in the closure of E, and choose a polydisk $\Delta(a; r) \subseteq D$. Since $a \in \bar{E}$, there is a point $b \in E \cap \Delta(a; r/2)$, and then $a \in \Delta(b; r/2) \subseteq D$. Now f can be expanded into a power series

$$f(z) = \sum_{k \in \mathbb{N}^n} c_k (z - b)^k$$

that converges on $\Delta(b; r/2)$; on the other hand, f is identically zero in a neighborhood of the point b, and so we have $c_k = 0$ for all $k \in \mathbb{N}^k$ by (1.6). It follows that $\Delta(b; r/2) \subseteq E$, and hence that $a \in E$, proving that E is relatively closed in D. □

The second is the following generalization of the maximum principle.

Theorem 1.9. *Let D be a connected open subset of $\mathbb{C}^n$, and $f \in \mathscr{O}(D)$. If there is a point $a \in D$ with $|f(a)| \geq |f(z)|$ for all $z \in D$, then f is constant.*

Proof. Choose a polydisk $\Delta(a; r) \subseteq D$. For any choice of $b \in \Delta(a; r)$, the one-variable function $g(t) = f\big(a + t(b - a)\big)$ is holomorphic on a neighborhood of the unit disk in $\mathbb{C}$, and $|g(0)| \geq |g(t)|$. By the maximum principle, g has to be constant, and so $f(b) = g(1) = g(0) = f(a)$. Thus f is constant on $\Delta(a; r)$; since D is connected, we conclude from Theorem 1.8 that $f(z) = f(a)$ for all $z \in D$. □