Desingularizations of

Conformally-Kähler

Einstein Orbifolds

Claude LeBrun
Stony Brook University

Penrose at 95, and Twistors at 60
St Antony's College,
Oxford, September 11, 2026

For my friend and teacher

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Sir Roger Penrose,

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In celebration of his 95th birthday

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In celebration of his 95th birthday
and his 1966 discovery of twistors.

Most recent results
in this talk:

Joint work with

Joint work with

Tristan Ozuch

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Massachusetts Institute of Technology

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e-print:
arXiv:2601.19215 [math.DG]

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"...the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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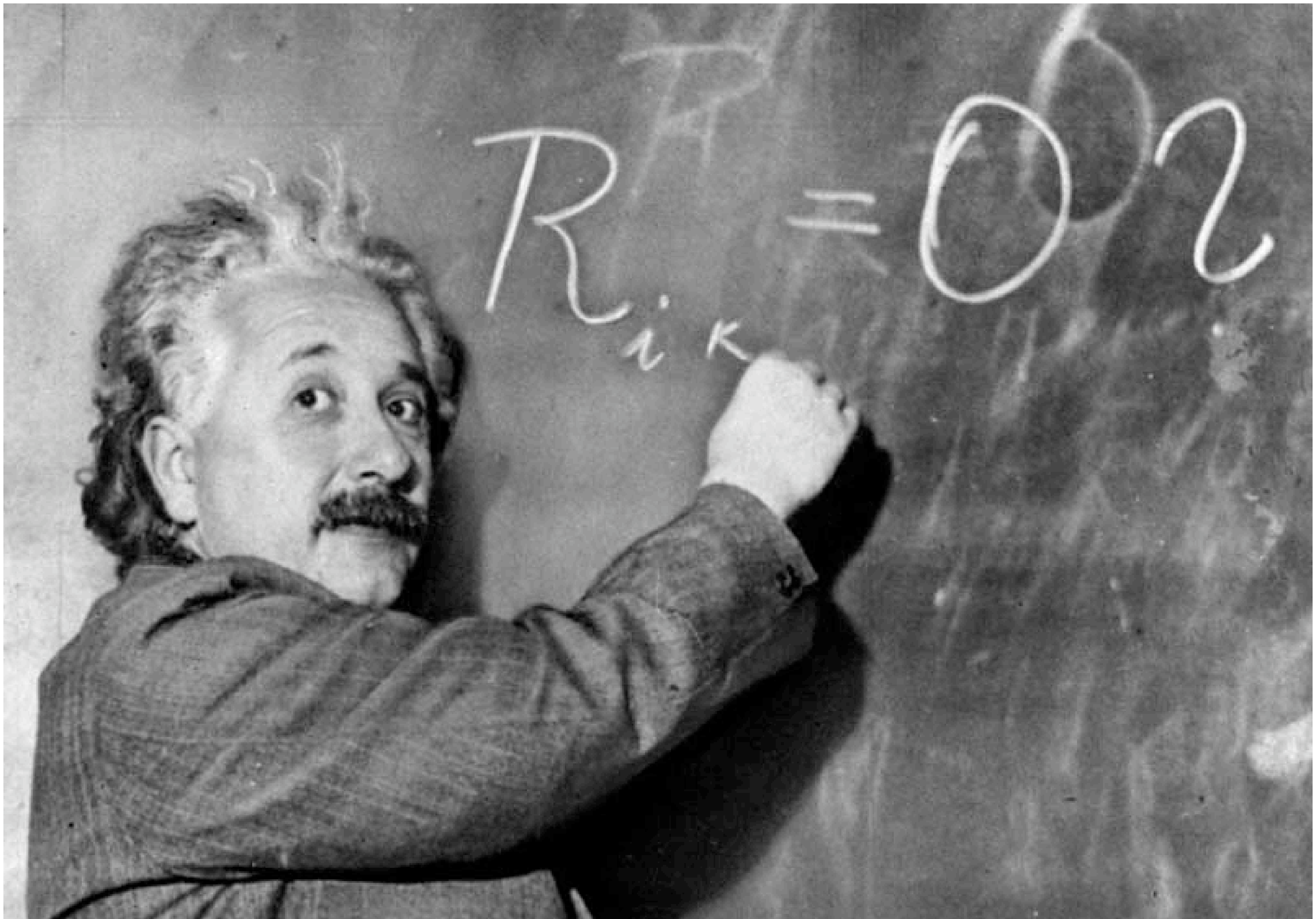

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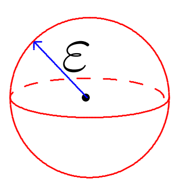
$$\frac{\mathrm{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s\,\frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



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$\Longrightarrow$ Global rigidity results in these low dimensions.

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In high dimensions, the Einstein condition allows for a surprising degree of flexibility!

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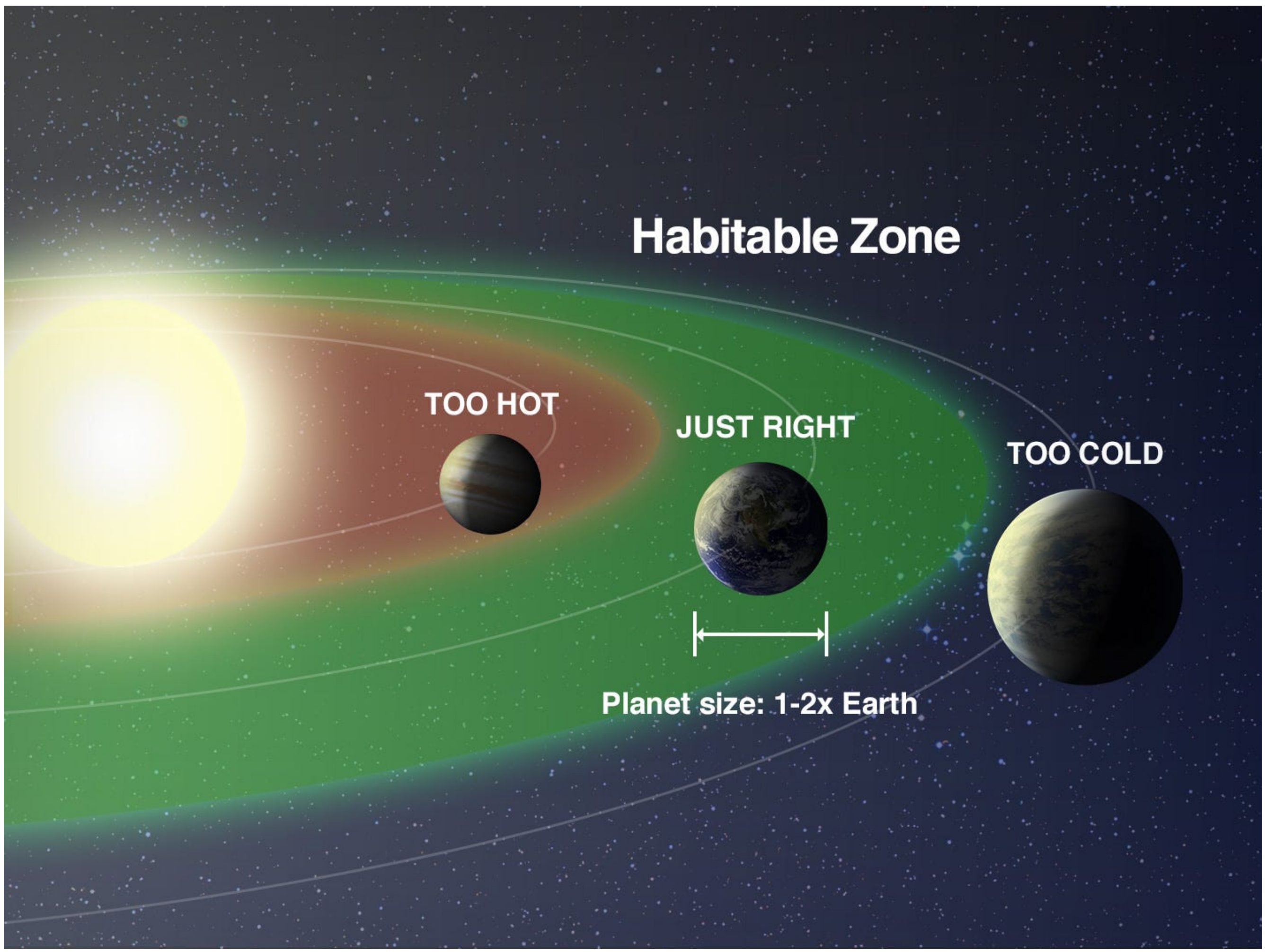

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Λ^+ self-dual 2-forms.

Λ^- anti-self-dual 2-forms.

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Which 4-manifolds admit Einstein metrics?

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$$d\omega = 0, \qquad \lrcorner\,\omega : TM \xrightarrow{\cong} T^*M.$$

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$$\omega = dx \wedge dy + dz \wedge dt$$

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Some Suggestive Questions. *If* (M^4, ω) *is a symplectic* 4*-manifold, when does* M^4 *admit an Einstein metric* g *(a priori unrelated to* ω*)? What if we also require* $\lambda > 0$*?*

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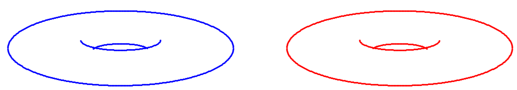

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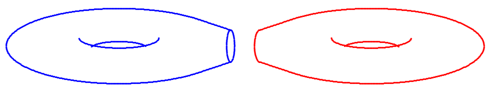

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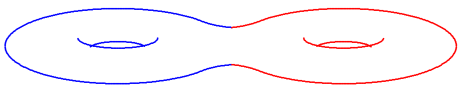

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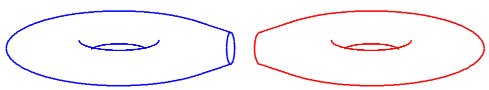

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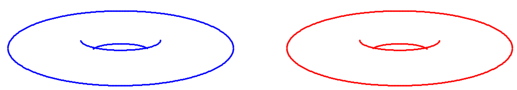

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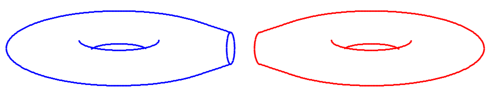

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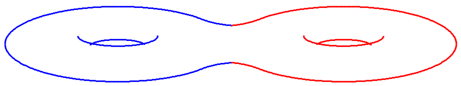

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Diffeotypes: exactly the Del Pezzo surfaces.

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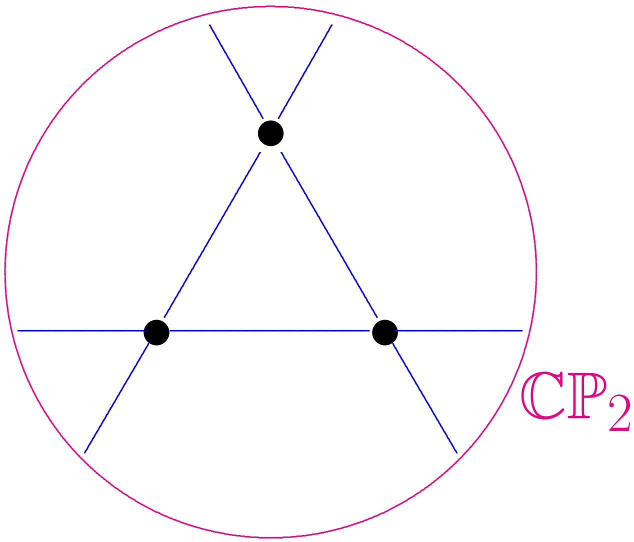

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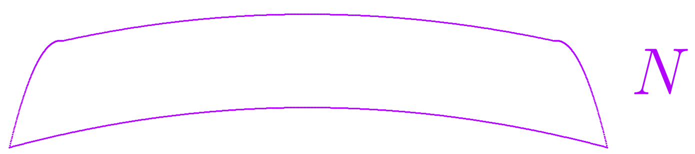

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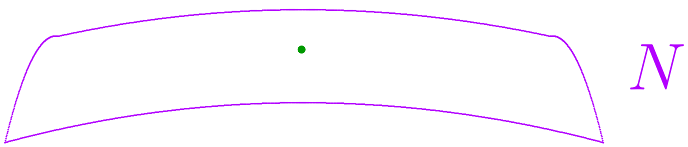

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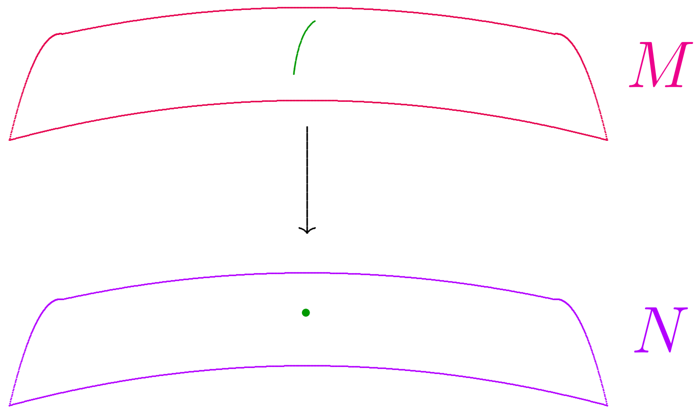

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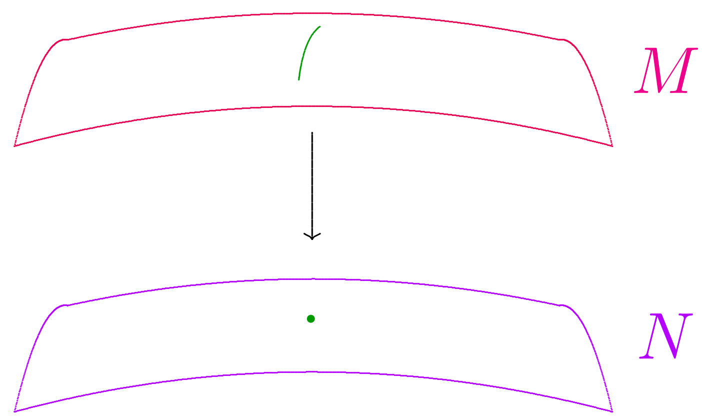

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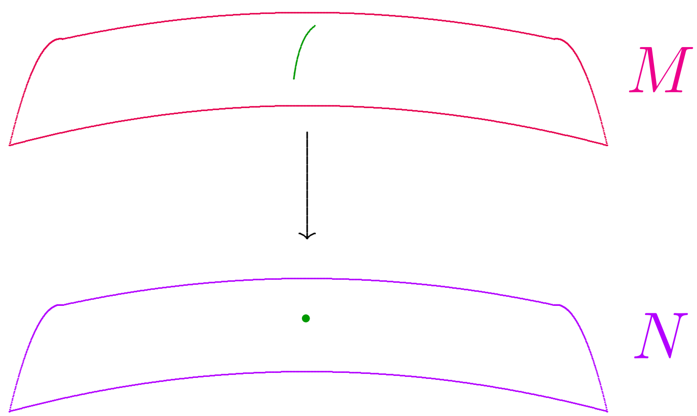

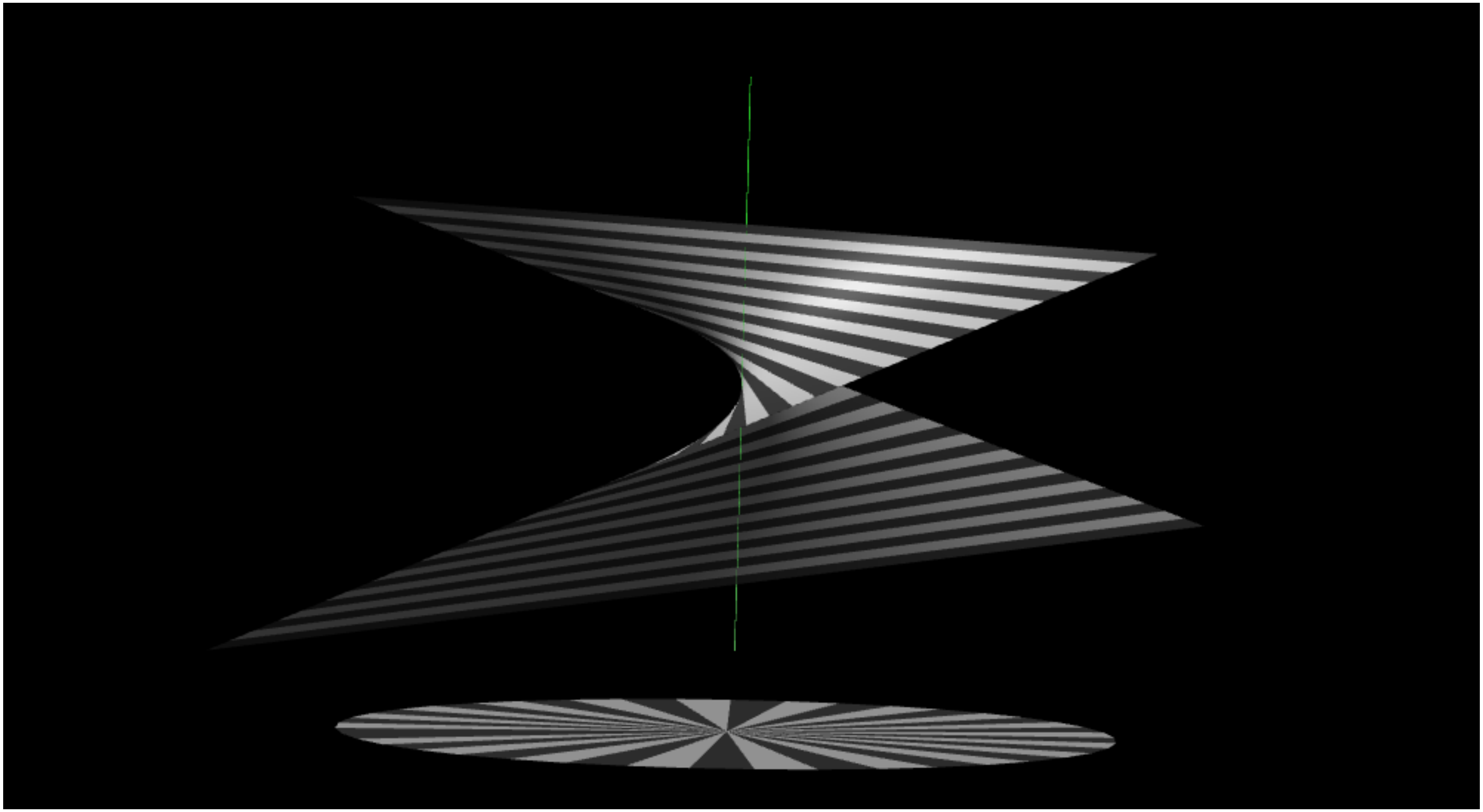

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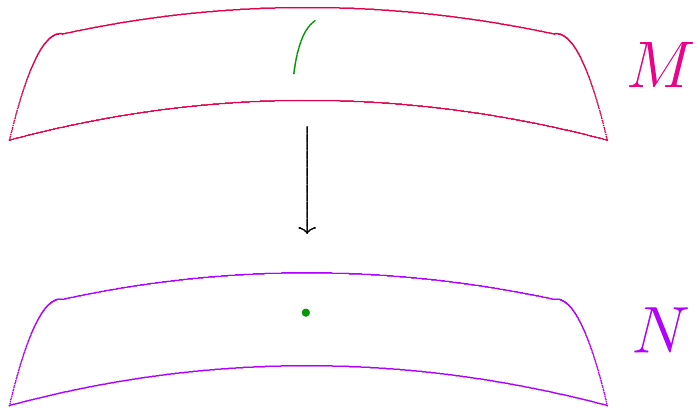

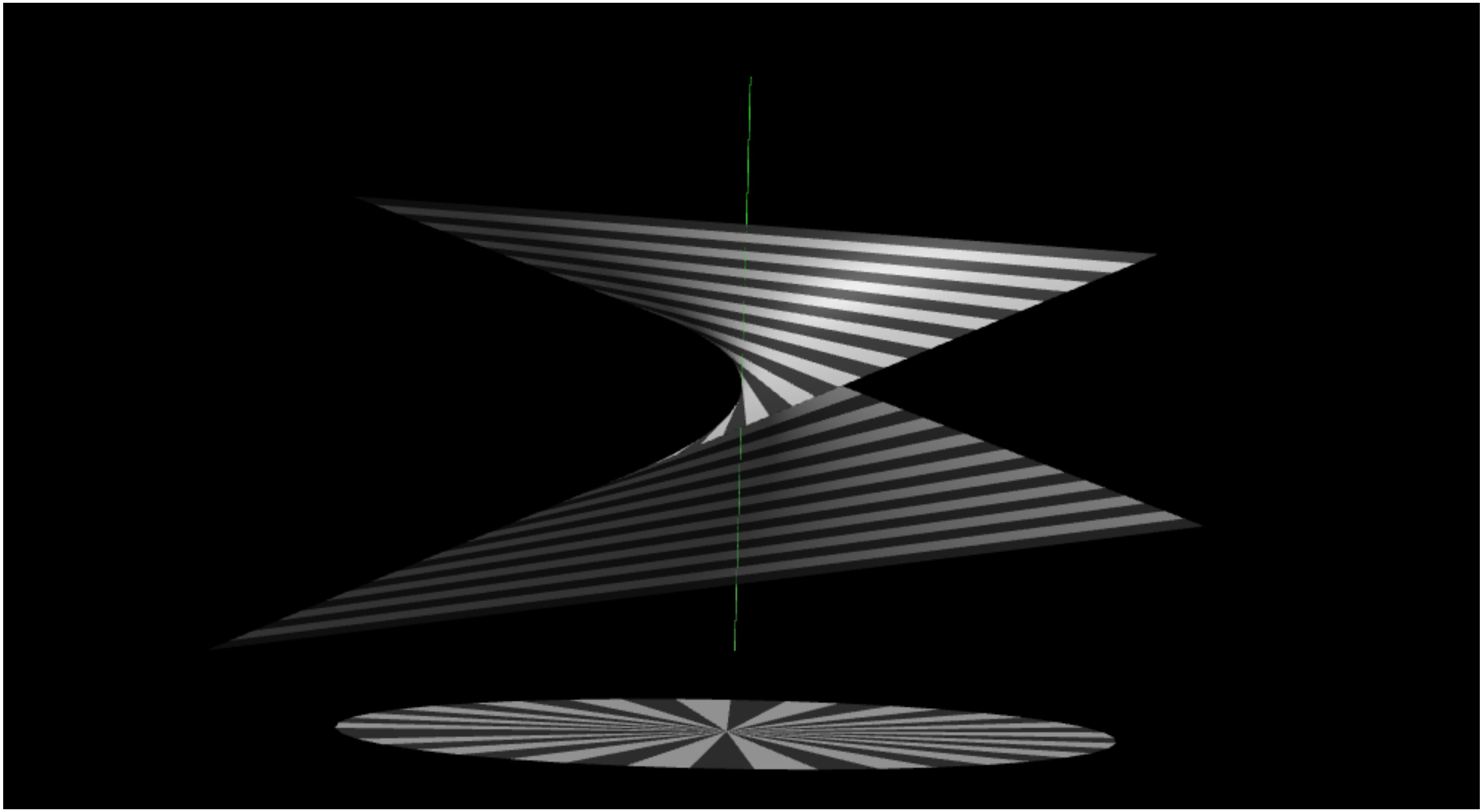

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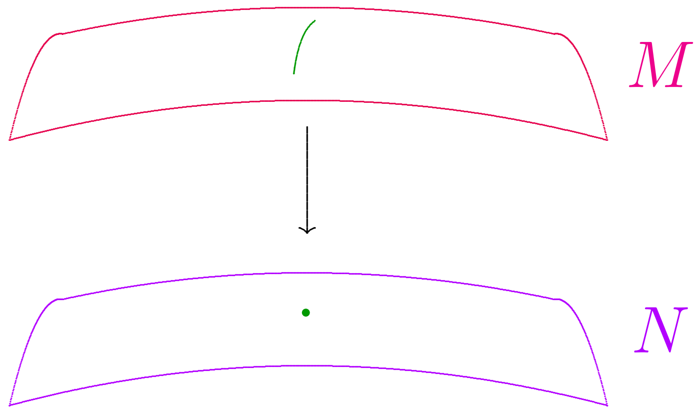
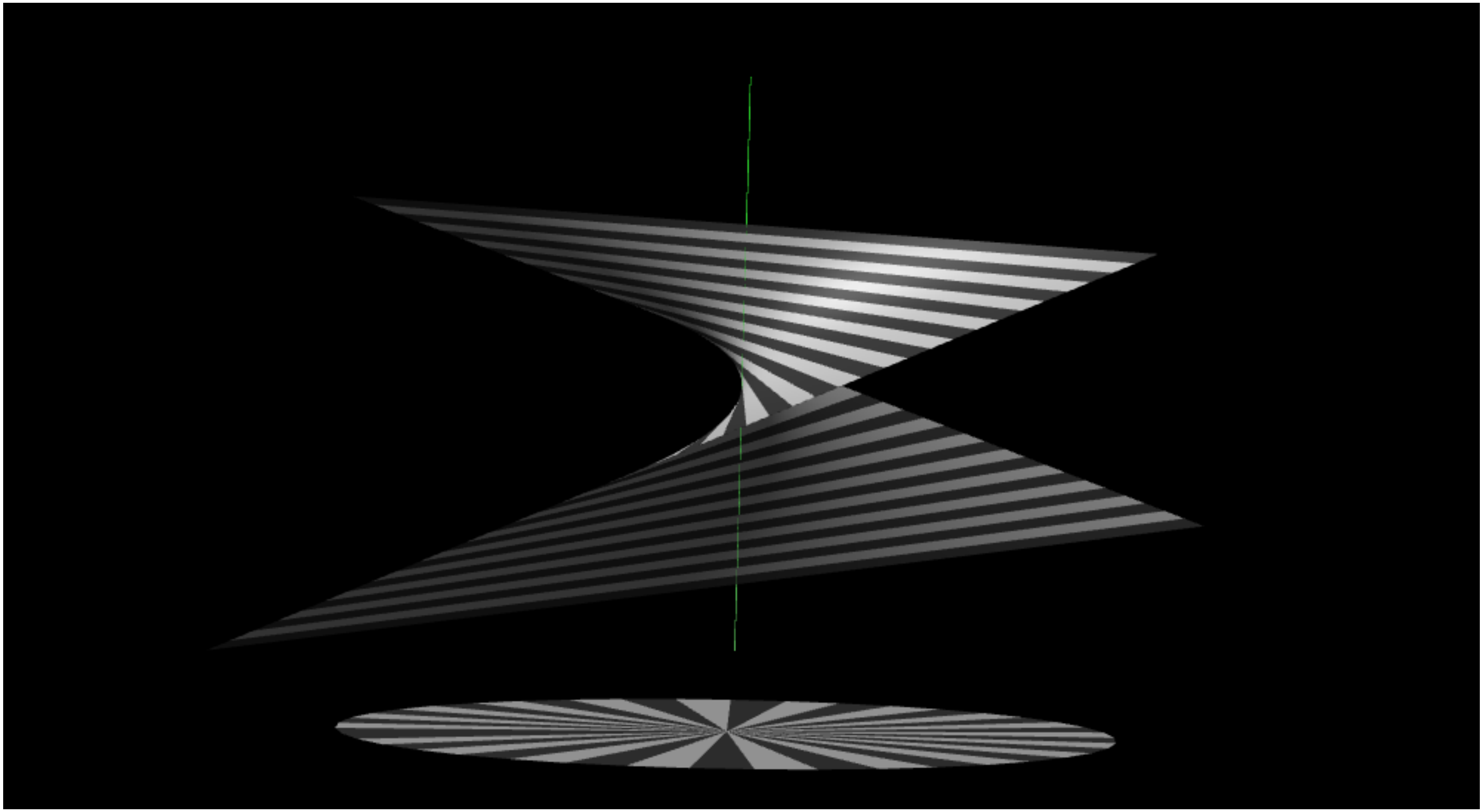


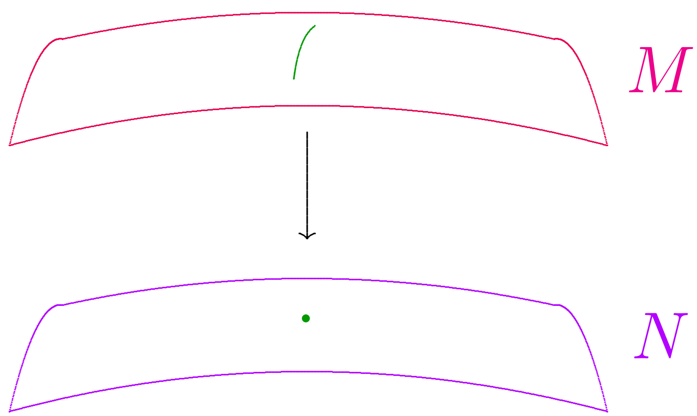


Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

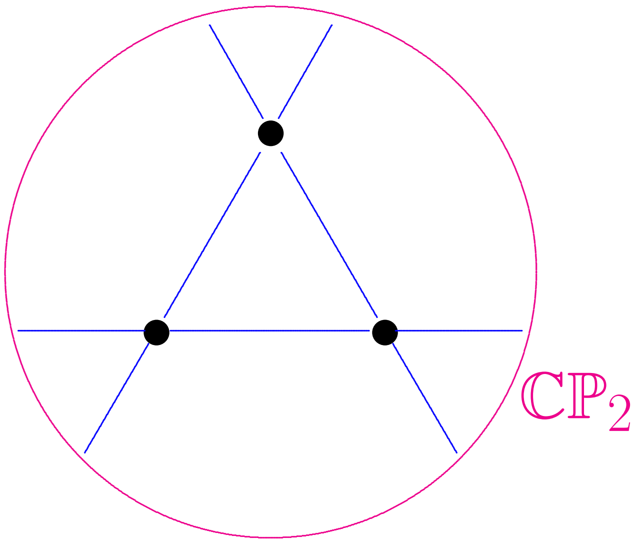
in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



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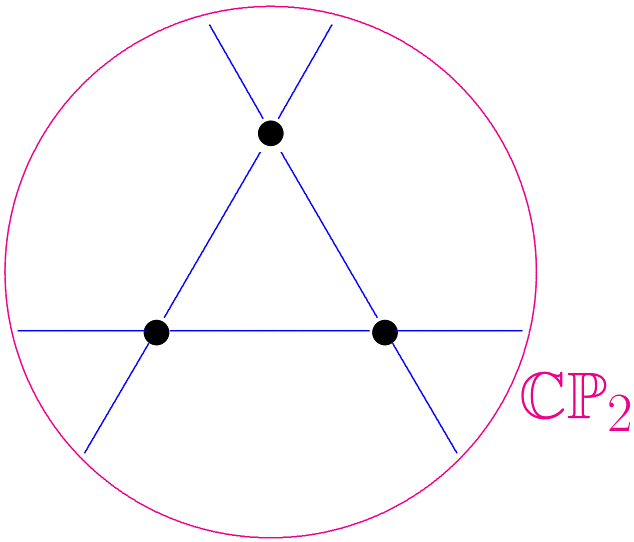

No 3 on a line,

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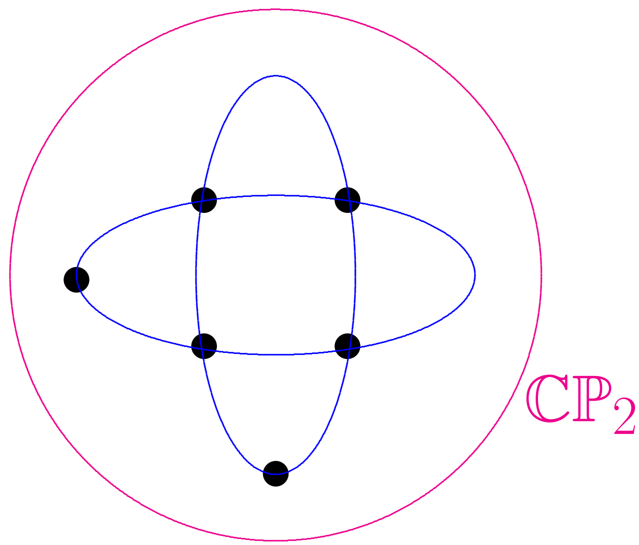

No 3 on a line, no 6 on conic,

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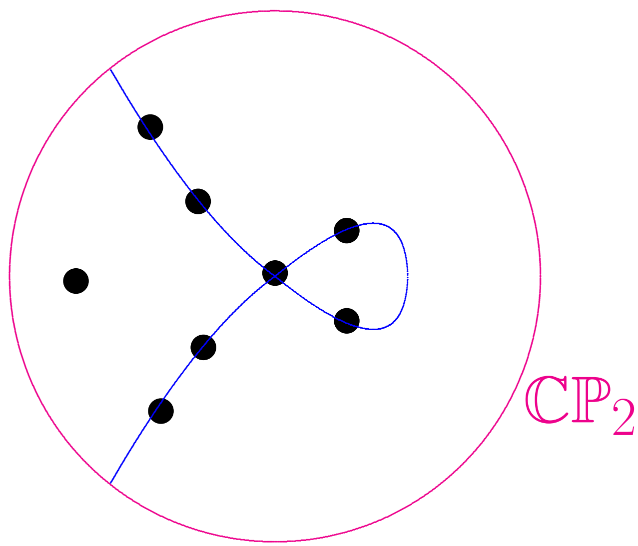

No 3 on a line, no 6 on conic, no 8 on nodal cubic.

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$$g = g(J\cdot, J\cdot)$$

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Main ingredients:

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Compact version:

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pre-compactness theorem for closed Riem'n n-manifolds

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Theorem. *Let $\{(M_j, g_j)\}$ be a sequence of cpt connected Einstein 4-manifolds with fixed Einstein constant $\lambda > 0$, fixed Euler characteristic χ, and $Vol(M_j, g_j)$ bounded away from zero.*

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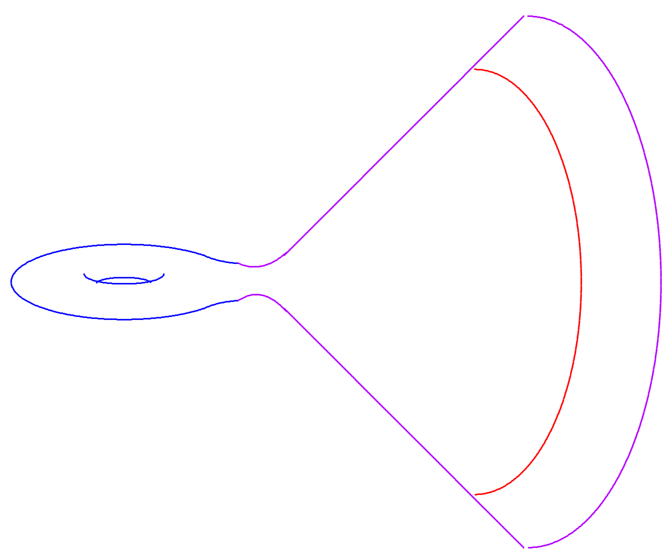

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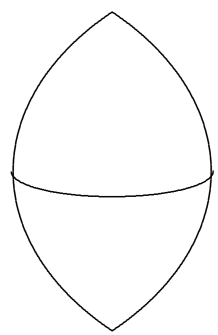

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Goal: Show that this doesn't change anything!

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We avoid this question by means of a definition!

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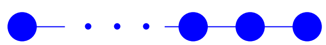
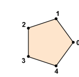
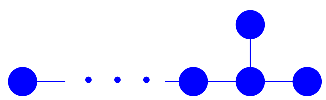
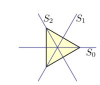
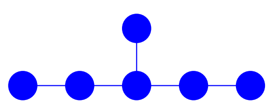
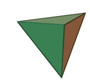
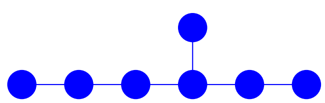
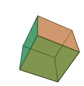
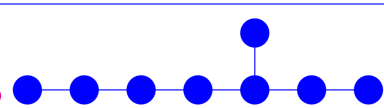
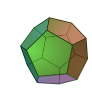
A_k 	*Cyclic*	
D_k 	*Binary Dihedral*	
E_6 	*Binary Tetrahedral*	
E_7 	*Binary Octahedral*	
E_8 	*Binary Icosahedral*	

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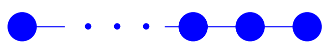
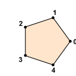
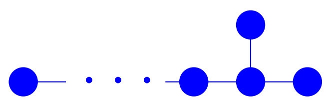
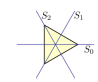
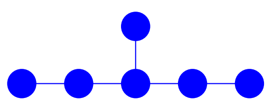
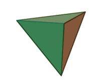
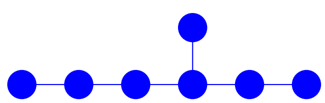
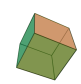
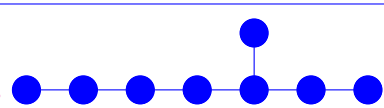
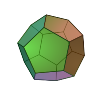
A_k		$\mathbb{Z}_{k+1}$	
D_k		Dih^*_{k-2}	
E_6		*Binary Tetrahedral*	
E_7		*Binary Octahedral*	
E_8		*Binary Icosahedral*	

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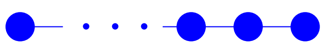
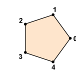
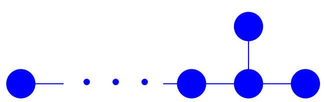
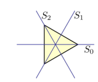
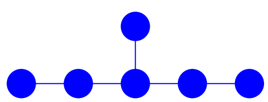
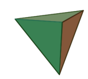
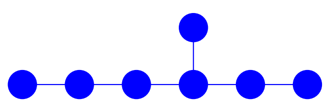
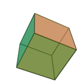
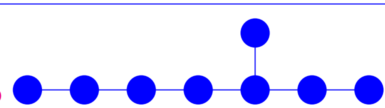
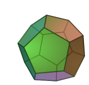
A_k 	*Cyclic*	
D_k 	*Binary Dihedral*	
E_6 	*Binary Tetrahedral*	
E_7 	*Binary Octahedral*	
E_8 	*Binary Icosahedral*	

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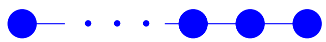
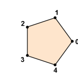
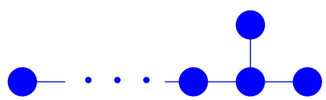
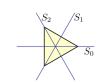
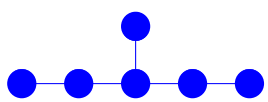
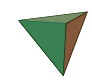
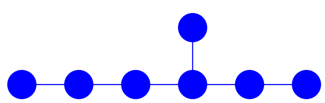
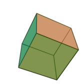
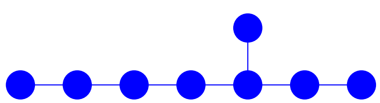
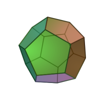
A_k		*Cyclic*	
D_k		*Binary Dihedral*	
E_6		*Binary Tetrahedral*	
E_7		*Binary Octahedral*	
E_8		*Binary Icosahedral*	

The corresponding gravitational instantons are exactly the hyper-Kähler ALE manifolds.

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Standard shorthand: $\frac{1}{\ell m^2}(1, \ell m n - 1)$.

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These examples form part of a systematic picture. . .

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- $\mathbb{CP}_2 \# 7 \overline{\mathbb{CP}}_2$: only $A_1, \ldots, A_4$ or $\frac{1}{4}(1,1)$ arise.

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- $\mathbb{CP}_2 \# 7 \overline{\mathbb{CP}}_2$: only $A_1, \ldots, A_4$ or $\frac{1}{4}(1,1)$ arise.
- $\mathbb{CP}_2 \# 8 \overline{\mathbb{CP}}_2$: only (!) $A_1, \ldots, A_{10}$, D_4, $\frac{1}{4}(1,1)$, $\frac{1}{8}(1,3)$, or $\frac{1}{9}(1,2)$ singularities arise.

Example.

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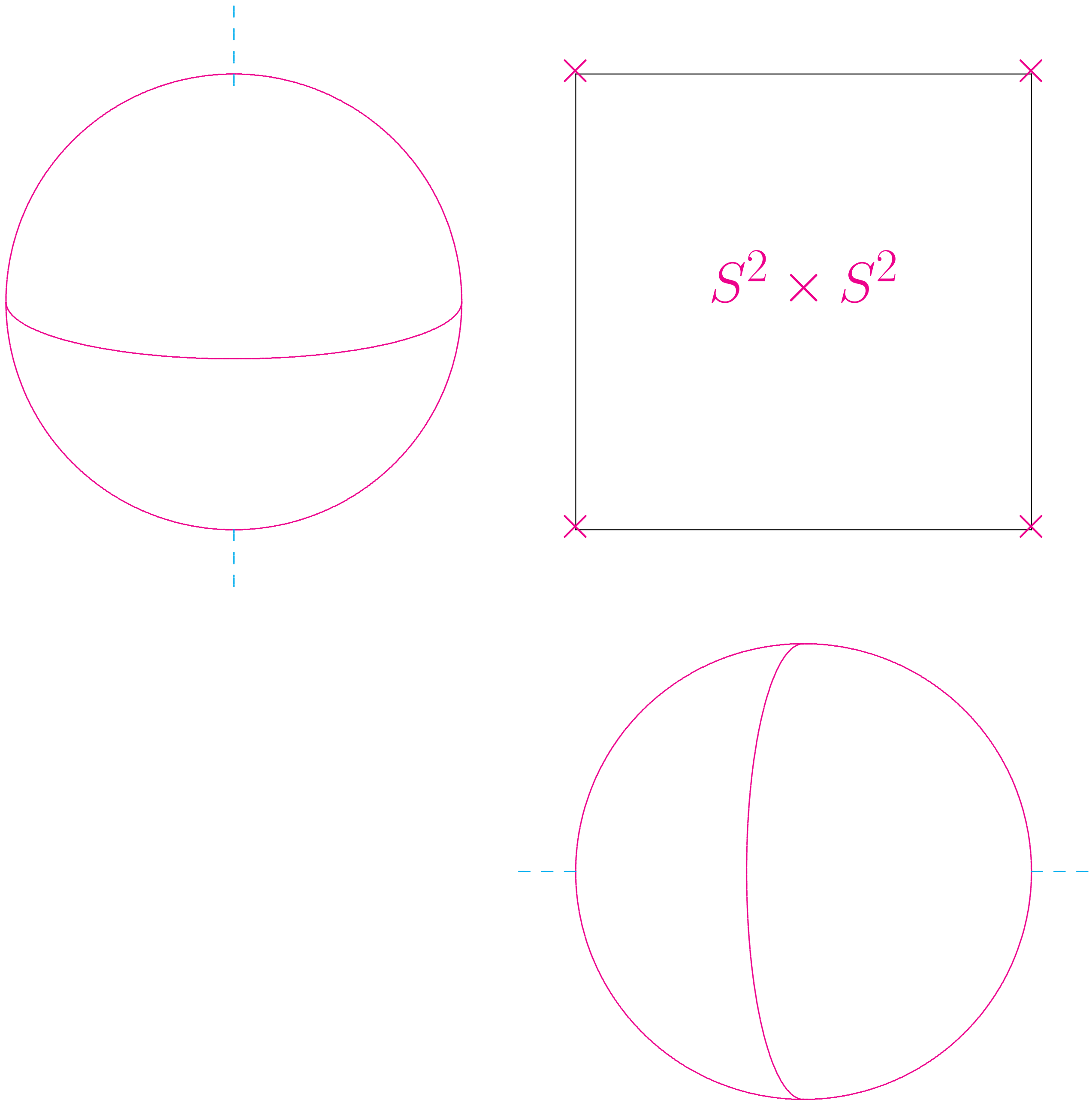

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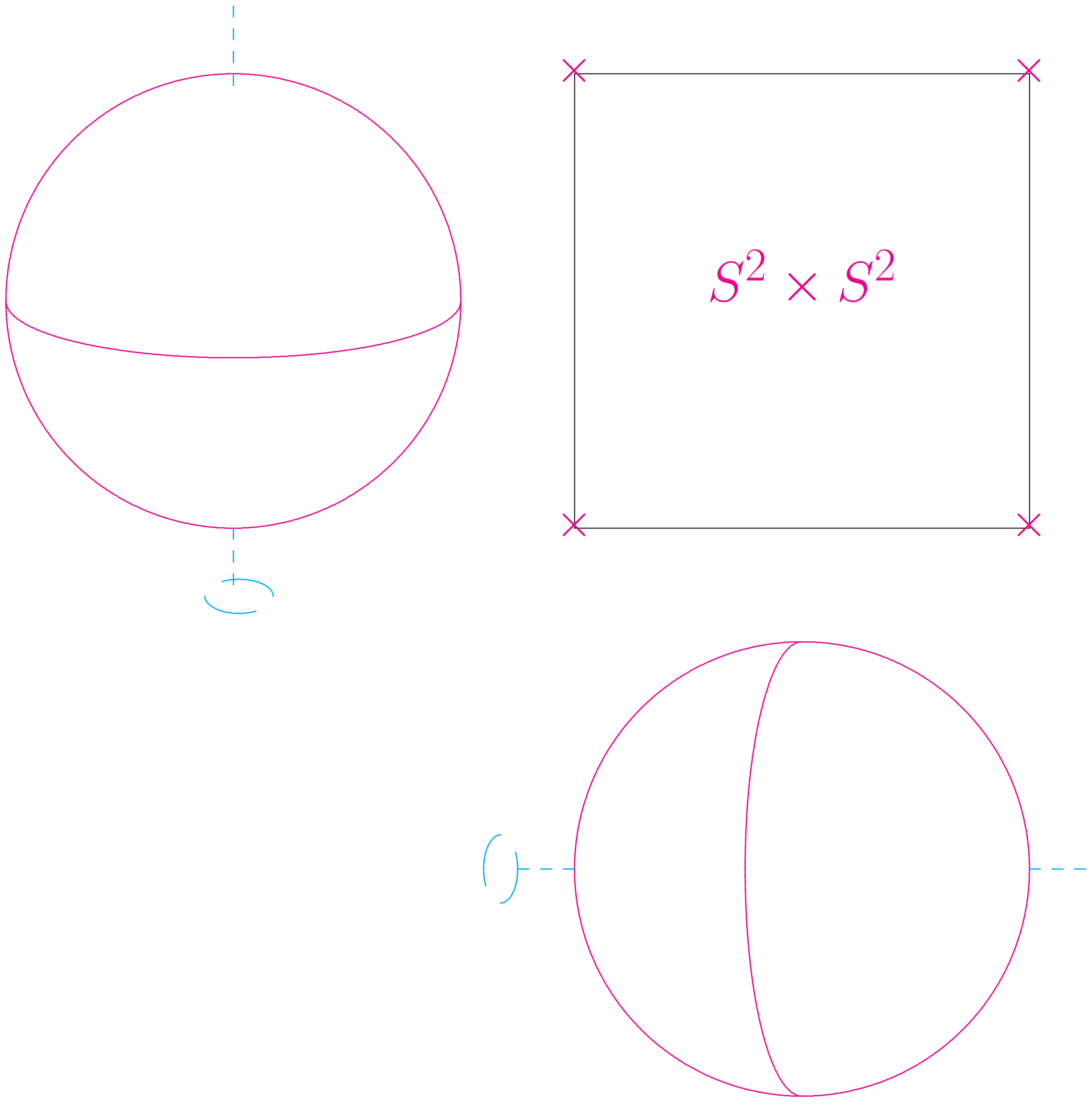

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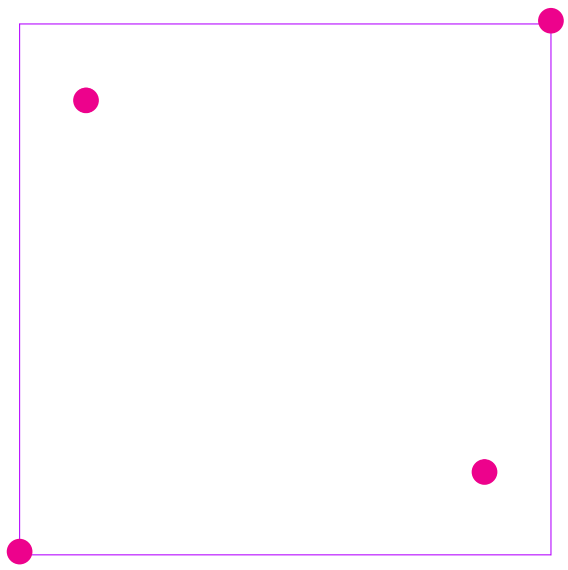

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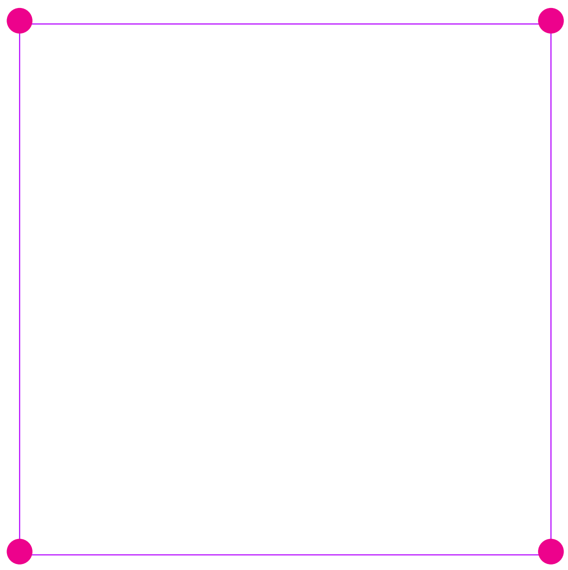

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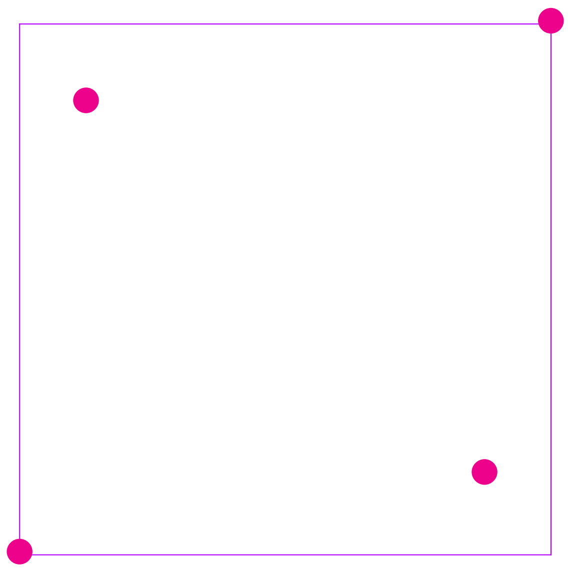

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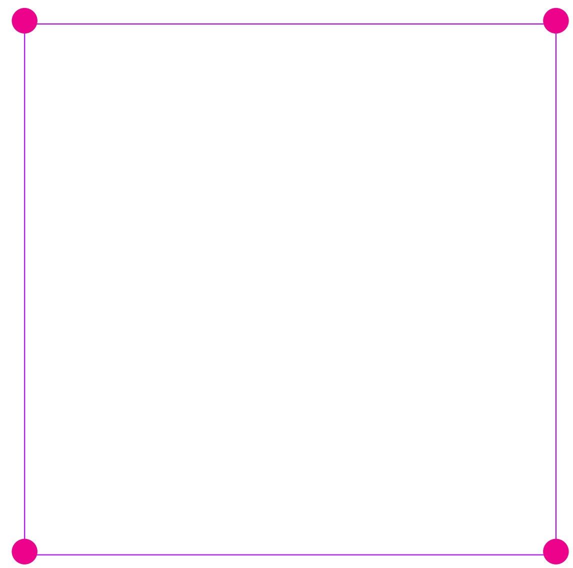

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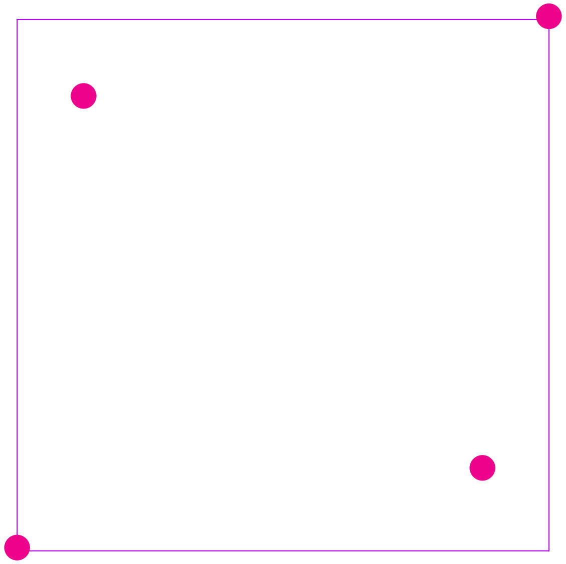

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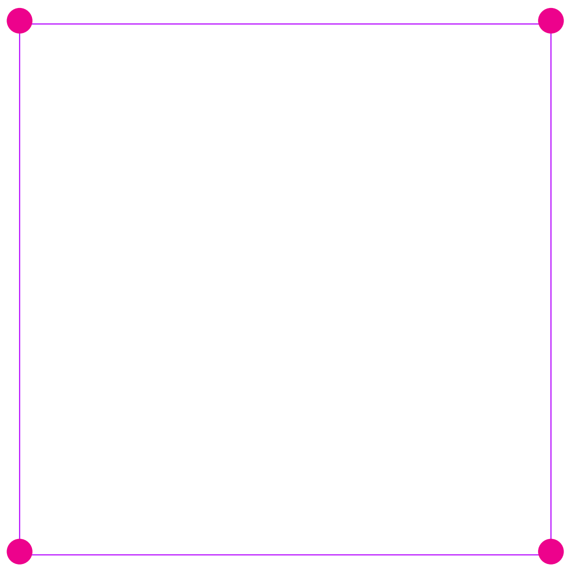

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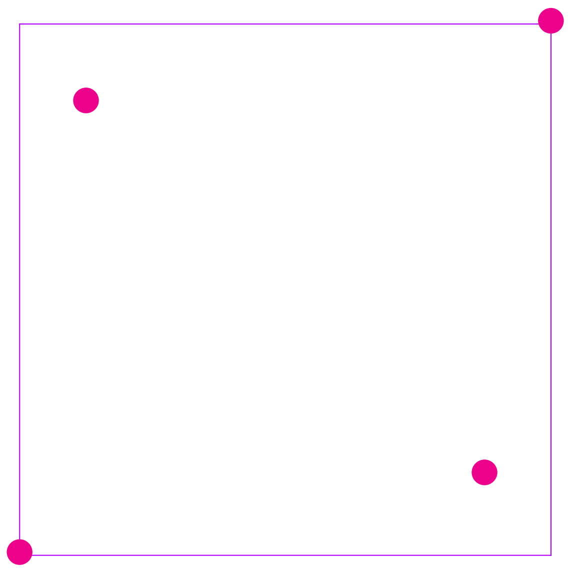

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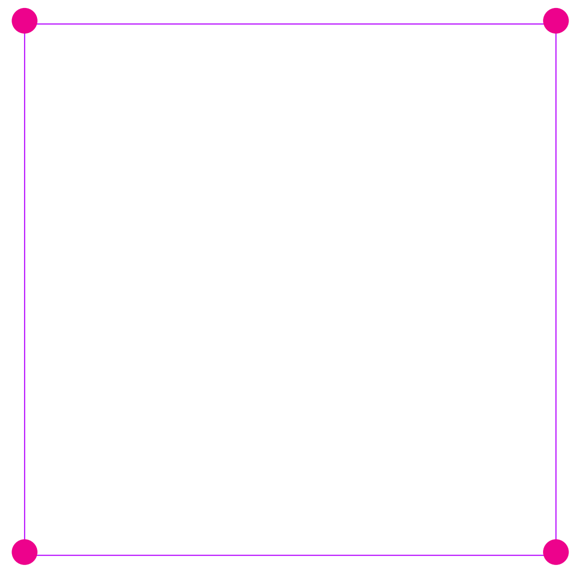

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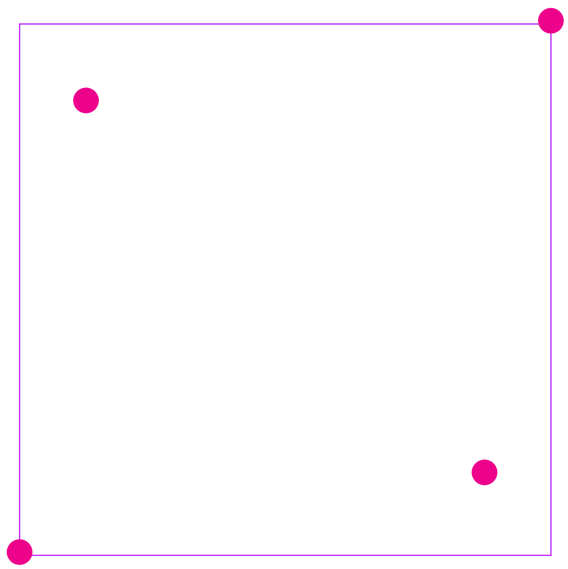

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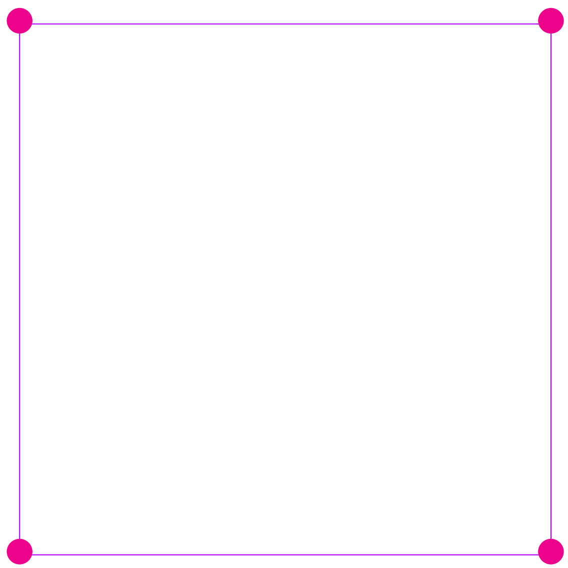

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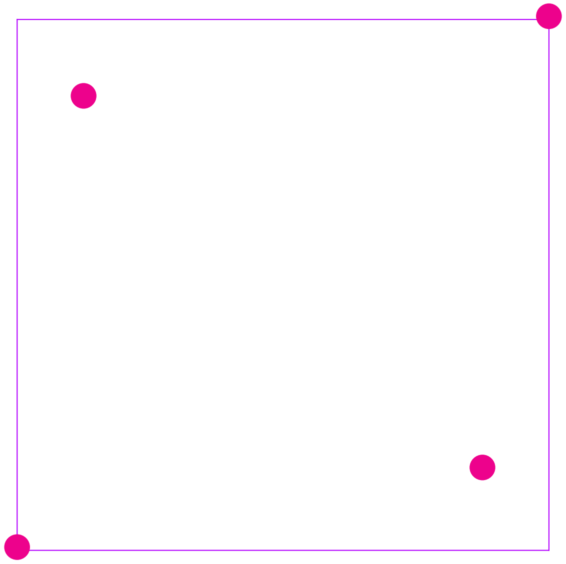

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But what about limits of general Einstein metrics?

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A key tool used in the proofs of the main theorems is based on a criterion discovered by Peng Wu '21.

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A more transparent proof was then given in L '21.

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Technically hardest when curvature accumulates on many different length-scales, giving rise to a complicated bubble tree.

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Then show that the given Einstein metrics g_j below to this universal family for all $j \gg 0$.

And hence that they are actually conformally Kähler!

Happy Birthday, Roger!

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And Many Happy Returns!