Desingularizations of

Conformally Kähler,

Einstein Orbifolds

Claude LeBrun
Stony Brook University

"Conformal Geometry, Einstein Manifolds,
and Minimal Submanifolds"
ICM 2026 Satellite Conference,
Pennsylvania State University, July 13, 2026

Most recent results:

Joint work with

Joint work with

Tristan Ozuch

Joint work with

Tristan Ozuch
Massachusetts Institute of Technology

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e-print:
arXiv:2601.19215 [math.DG]

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— A. Einstein, to G. Gamow

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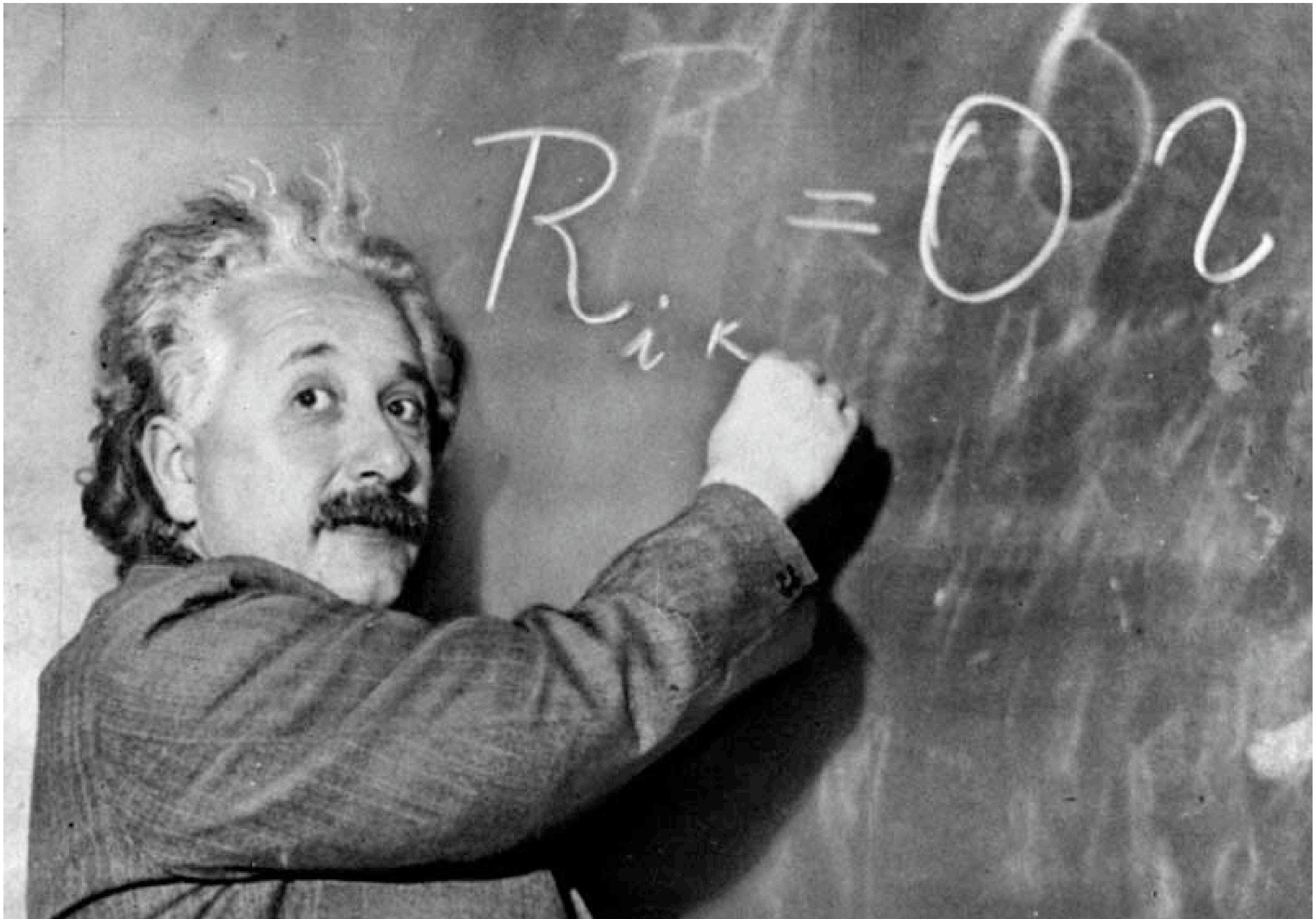

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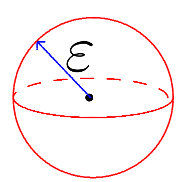

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In high dimensions, the Einstein condition allows for a surprising degree of flexibility!

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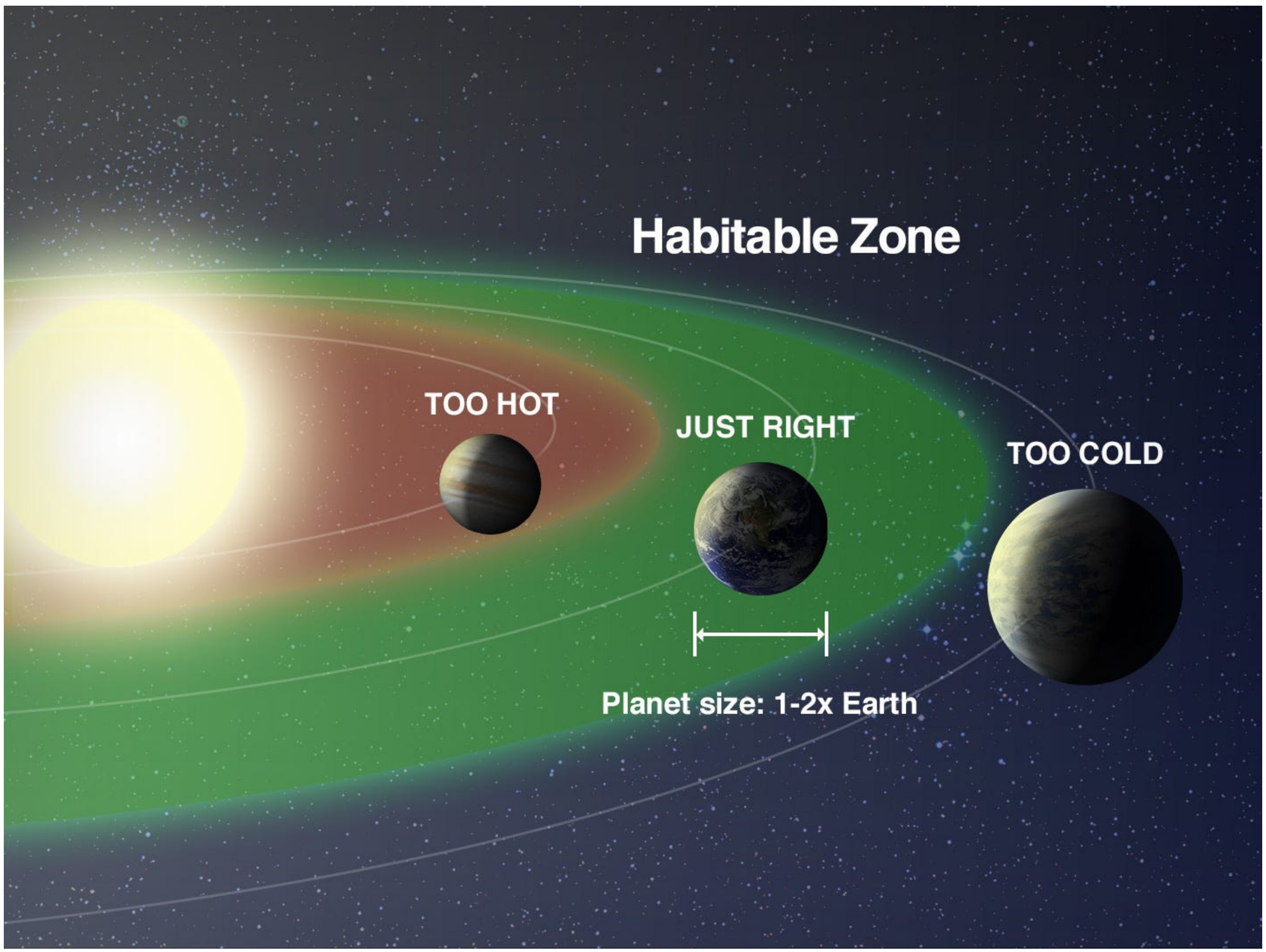

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$$\star^2 = 1.$$

Λ^+ self-dual 2-forms.

Λ^- anti-self-dual 2-forms.

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Which 4-manifolds admit Einstein metrics?

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$$d\omega = 0, \qquad \lrcorner\omega : TM \xrightarrow{\cong} T^*M.$$

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$$\omega = dx \wedge dy + dz \wedge dt$$

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Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold, when does M^4 admit an Einstein metric g (a priori unrelated to ω)? What if we also require $\lambda > 0$?*

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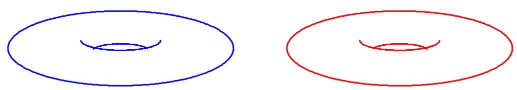

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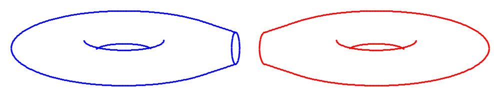

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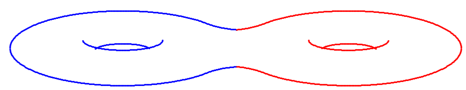

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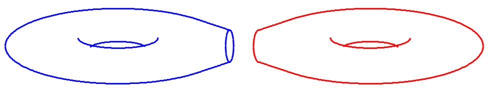

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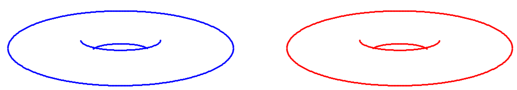

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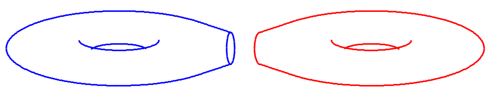

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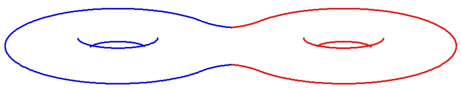

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Diffeotypes: exactly the Del Pezzo surfaces.

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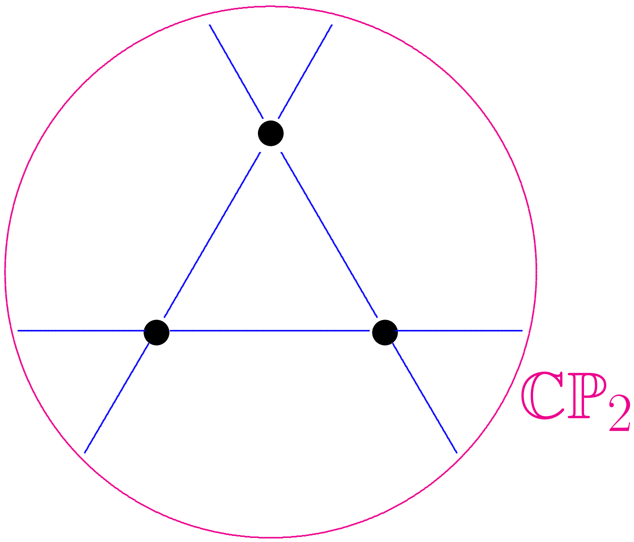

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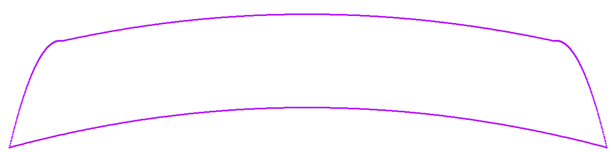

N

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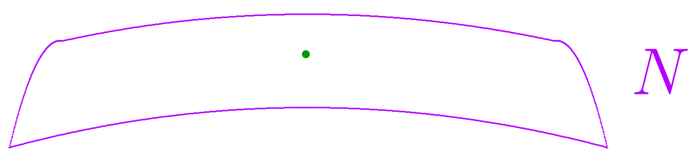

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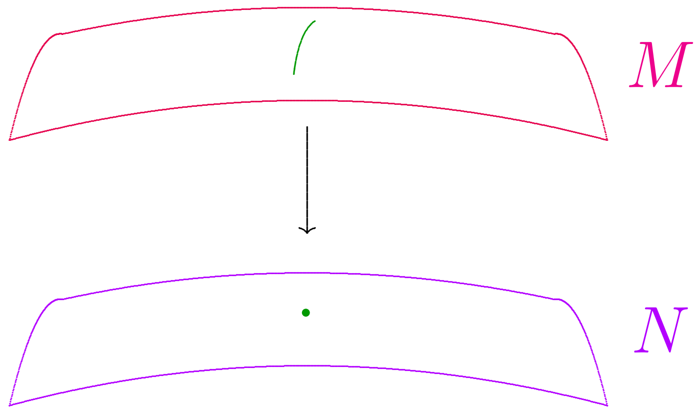

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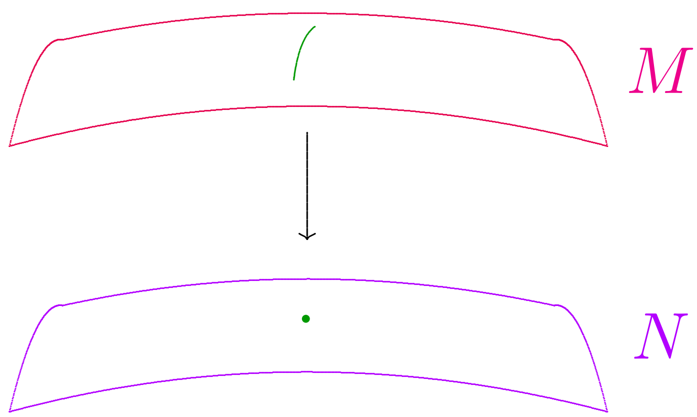

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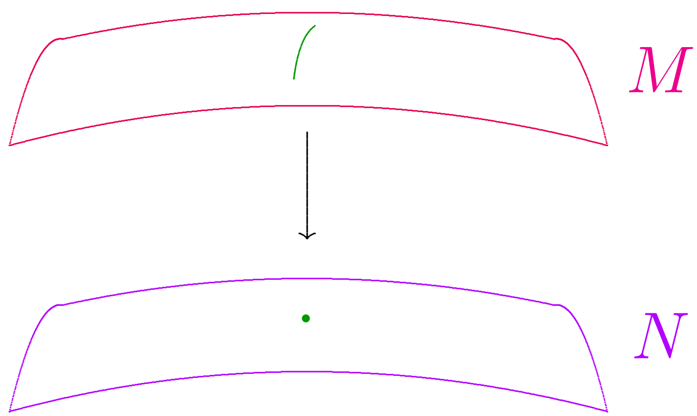

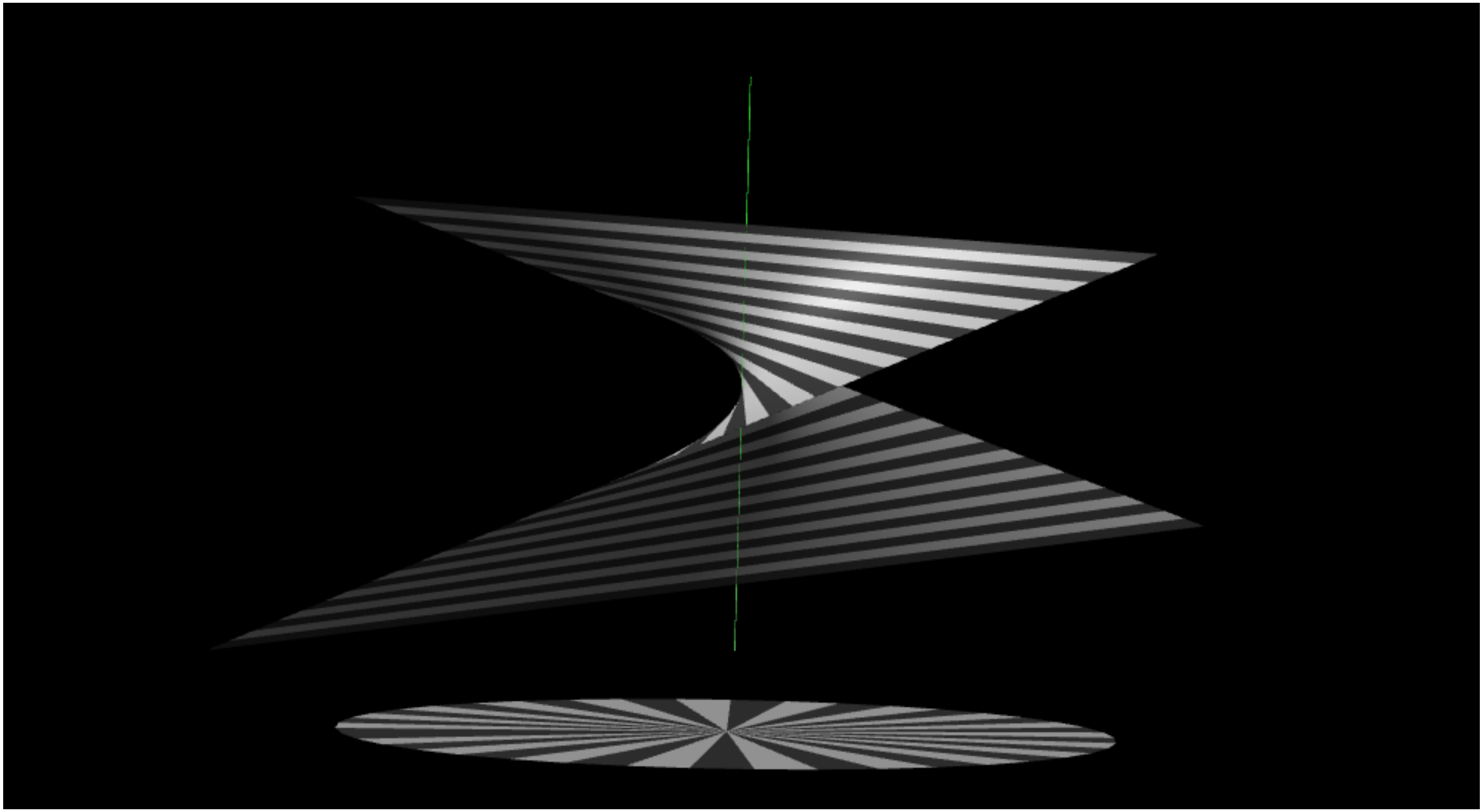

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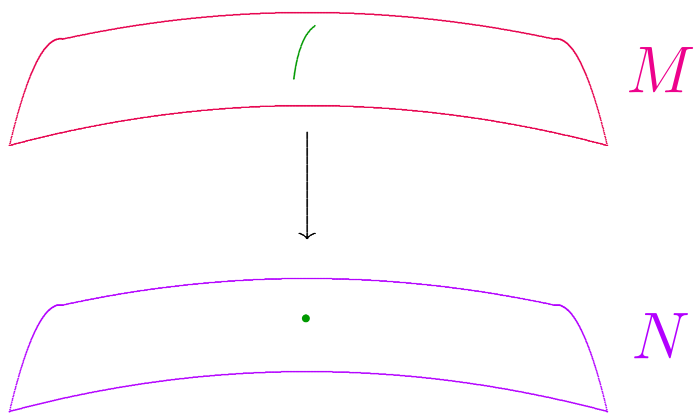

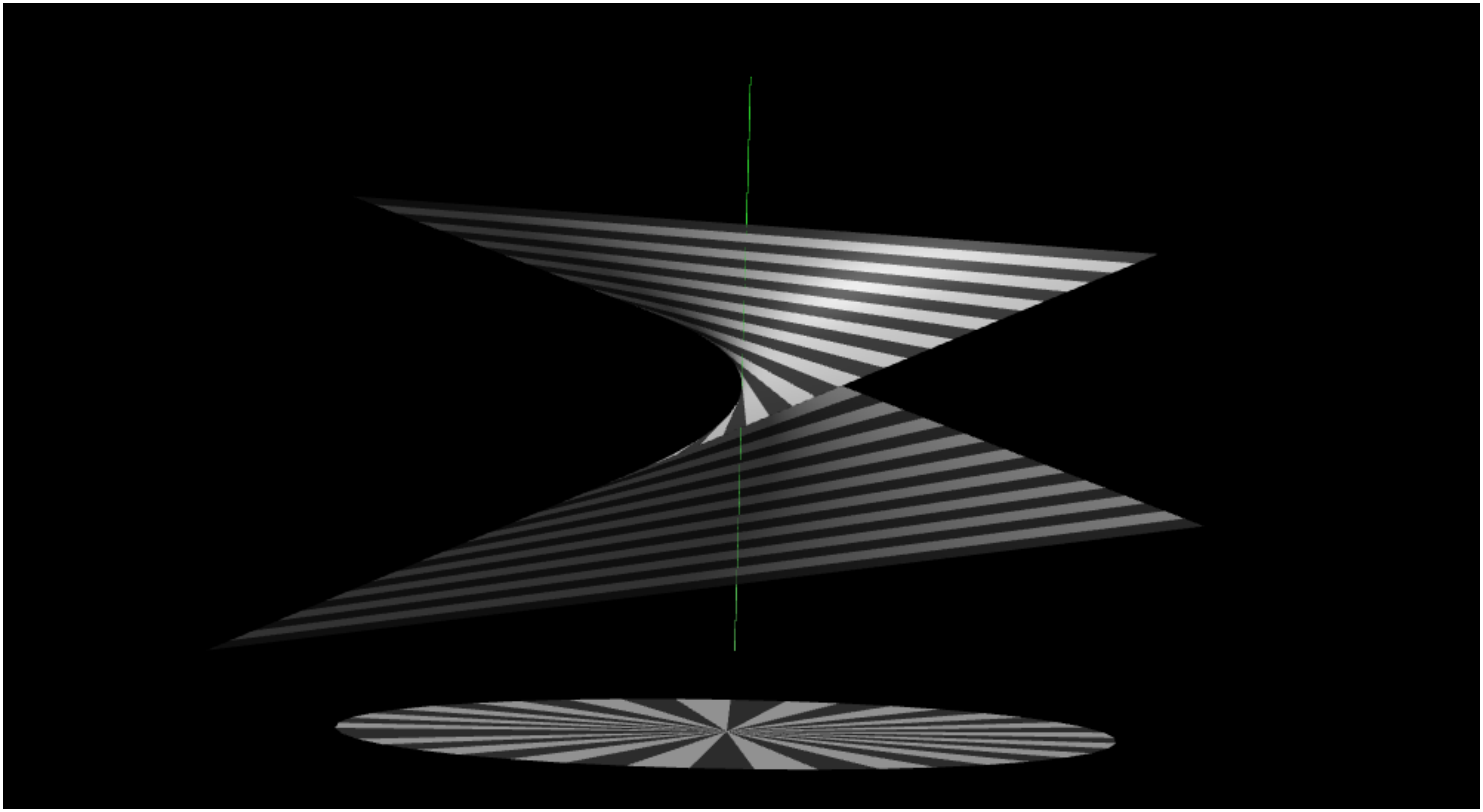

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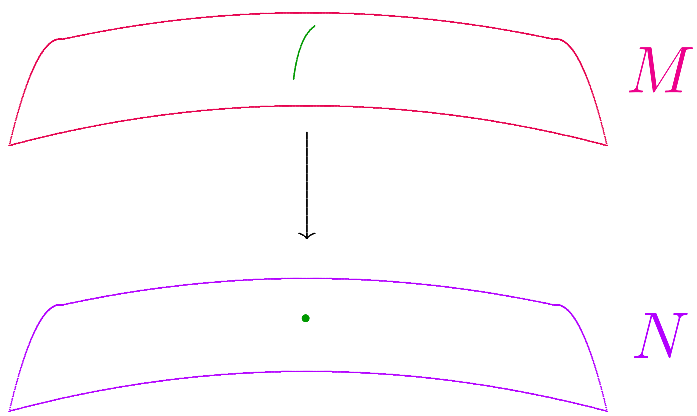

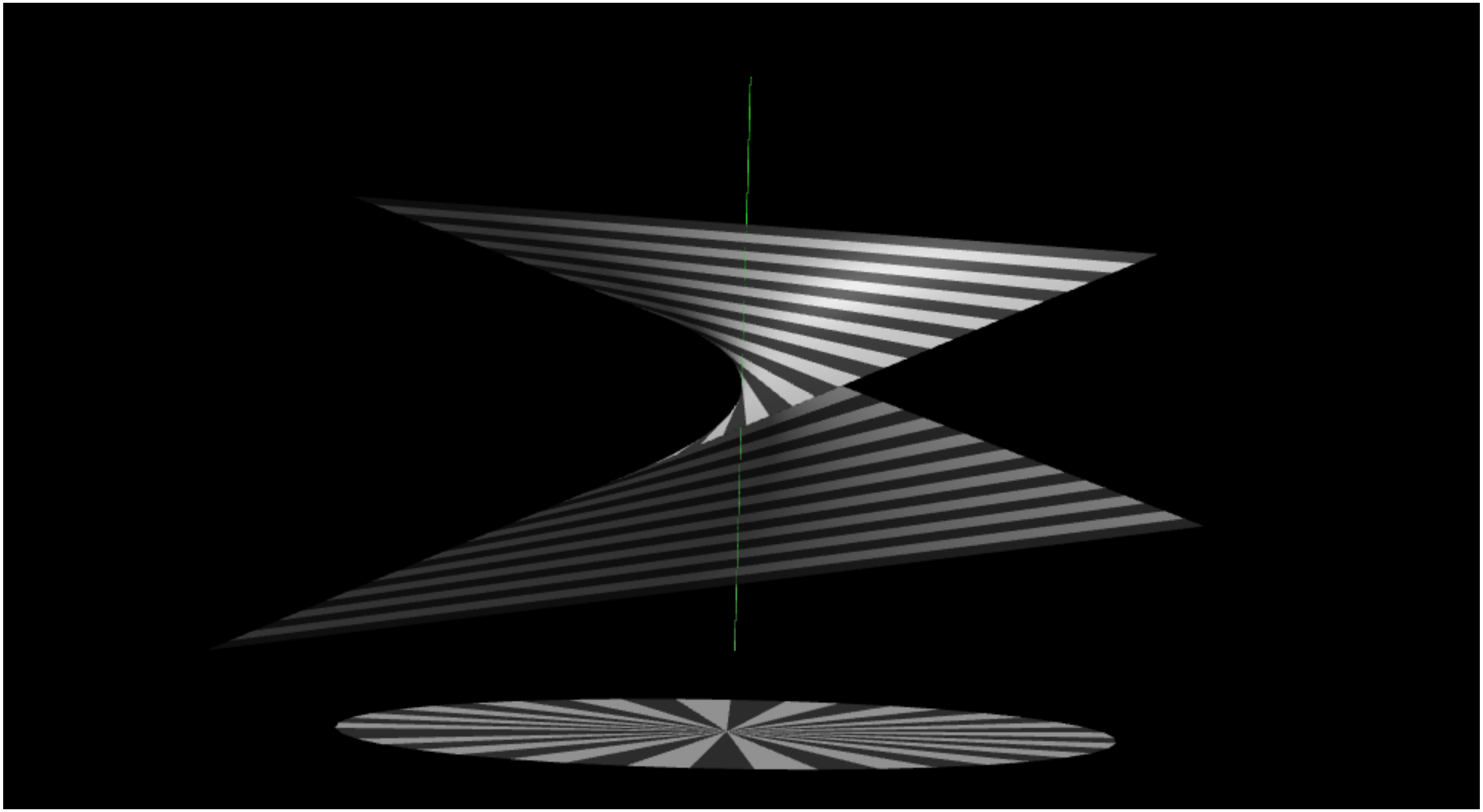

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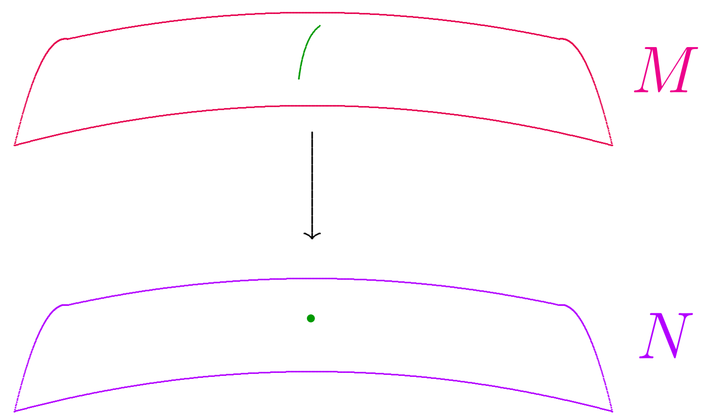
If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

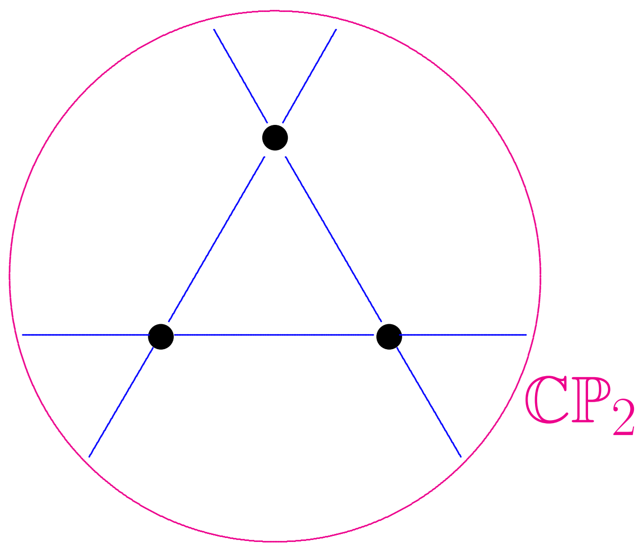
in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



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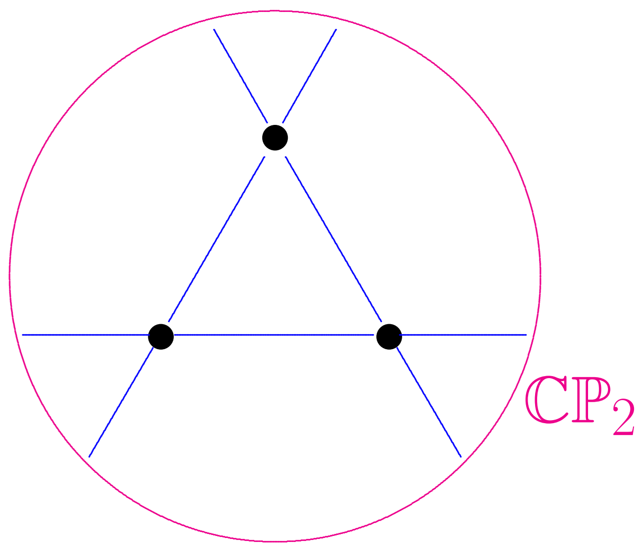

No 3 on a line,

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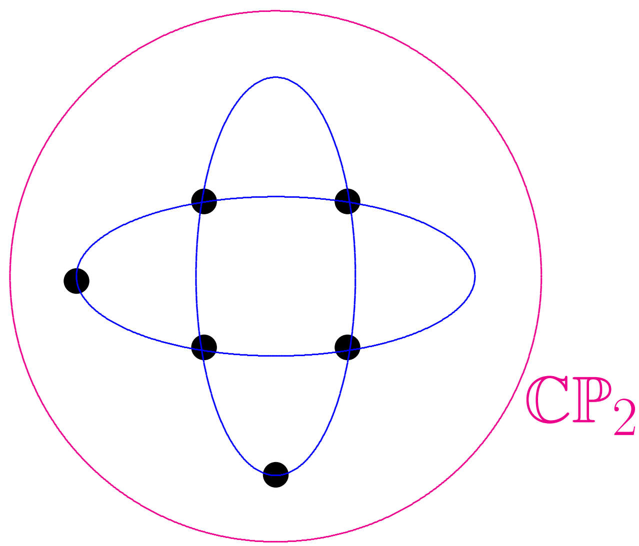

No 3 on a line, no 6 on conic,

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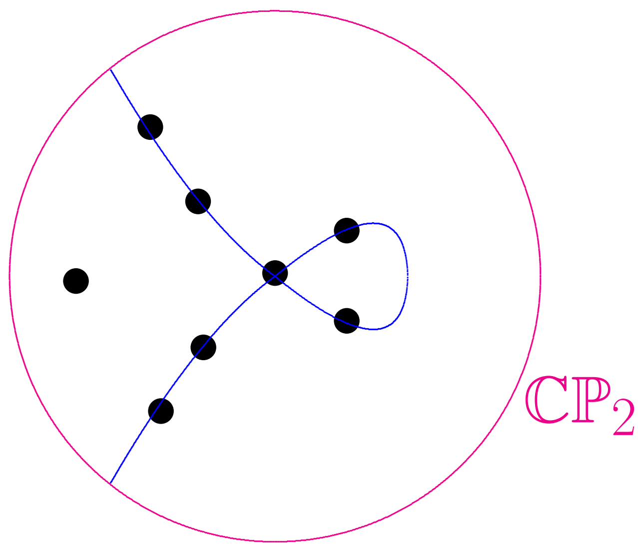

No 3 on a line, no 6 on conic, no 8 on nodal cubic.

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$$g = g(J\cdot, J\cdot)$$

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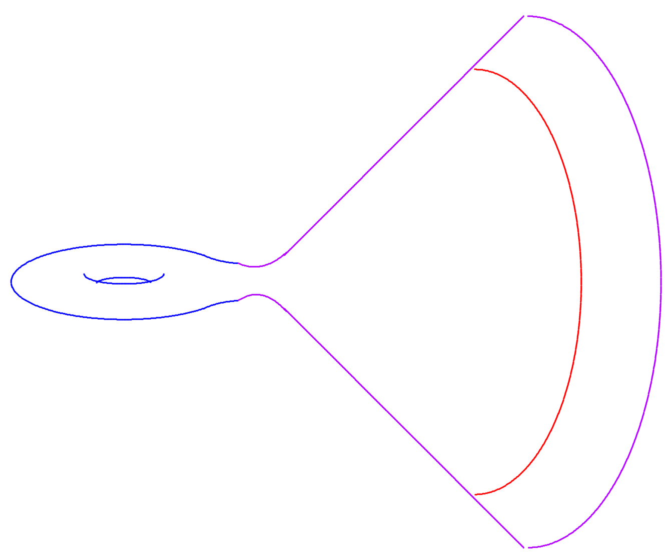

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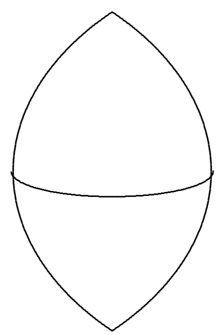

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Goal: Show that this doesn't change anything!

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We avoid this question by means of a definition!

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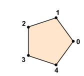
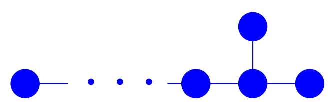
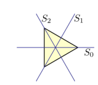
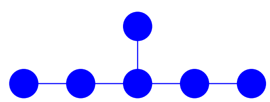
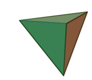
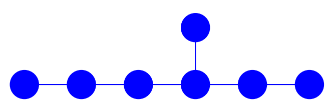
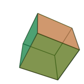
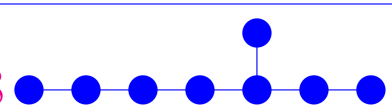
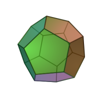
A_k 	*Cyclic*	
D_k 	*Binary Dihedral*	
E_6 	*Binary Tetrahedral*	
E_7 	*Binary Octahedral*	
E_8 	*Binary Icosahedral*	

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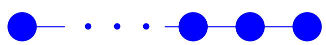
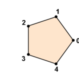
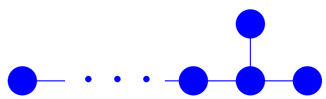
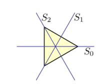
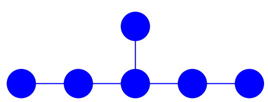
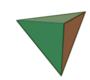
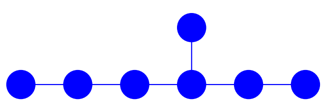
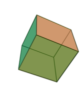
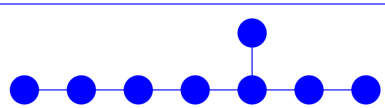
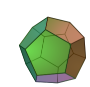
A_k		$\mathbb{Z}_{k+1}$	
D_k		Dih^*_{k-2}	
E_6		*Binary Tetrahedral*	
E_7		*Binary Octahedral*	
E_8		*Binary Icosahedral*	

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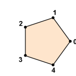
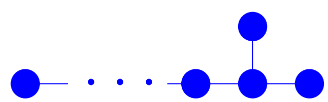
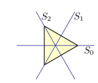
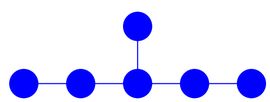
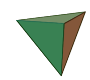
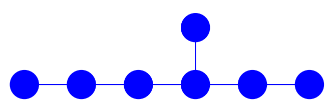
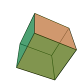
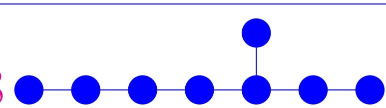
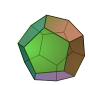
A_k 	*Cyclic*	
D_k 	*Binary Dihedral*	
E_6 	*Binary Tetrahedral*	
E_7 	*Binary Octahedral*	
E_8 	*Binary Icosahedral*	

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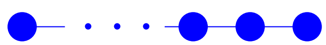
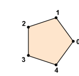
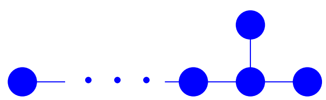
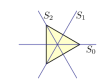
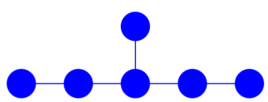
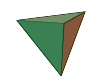
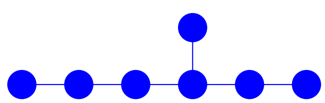
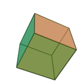
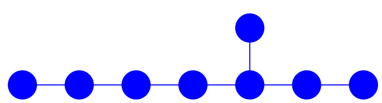
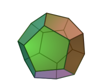
Dynkin Diagram	$\Gamma < SU(2)$	*Classical Figure*
A_k 	*Cyclic*	
D_k 	*Binary Dihedral*	
E_6 	*Binary Tetrahedral*	
E_7 	*Binary Octahedral*	
E_8 	*Binary Icosahedral*	

The corresponding gravitational instantons are exactly the hyper-Kähler ALE manifolds.

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Standard shorthand: $\frac{1}{\ell m^2}(1, \ell m n - 1)$.

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These examples form part of a systematic picture. . .

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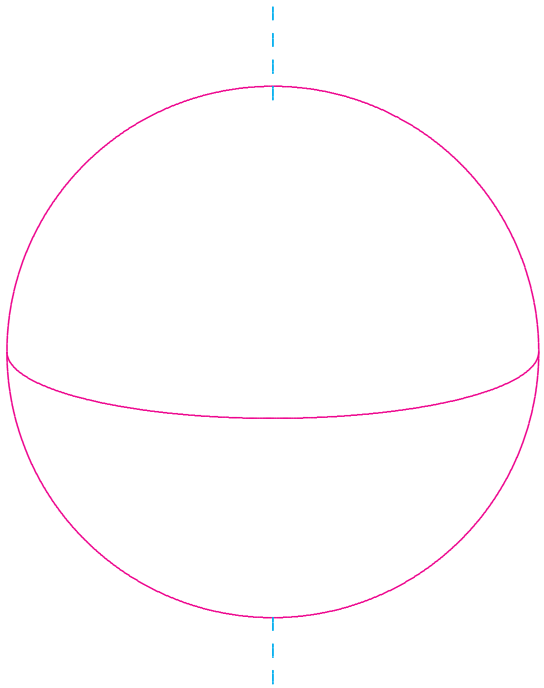

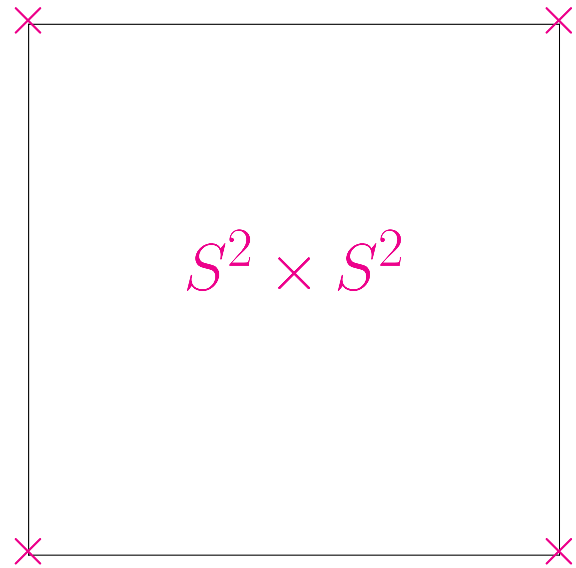

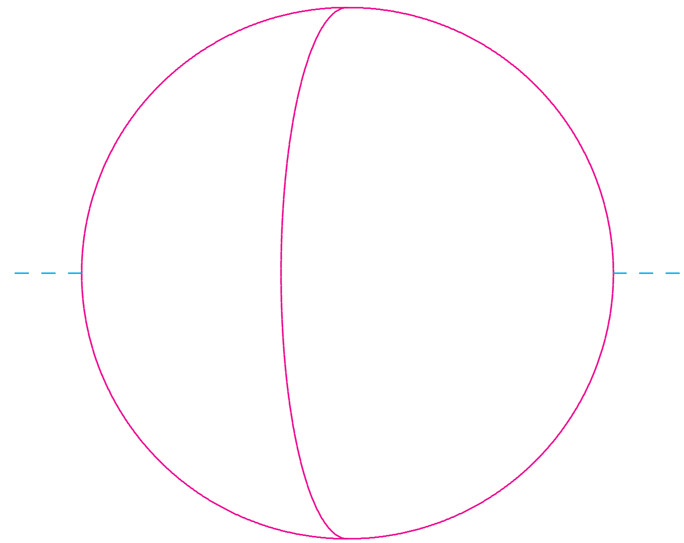

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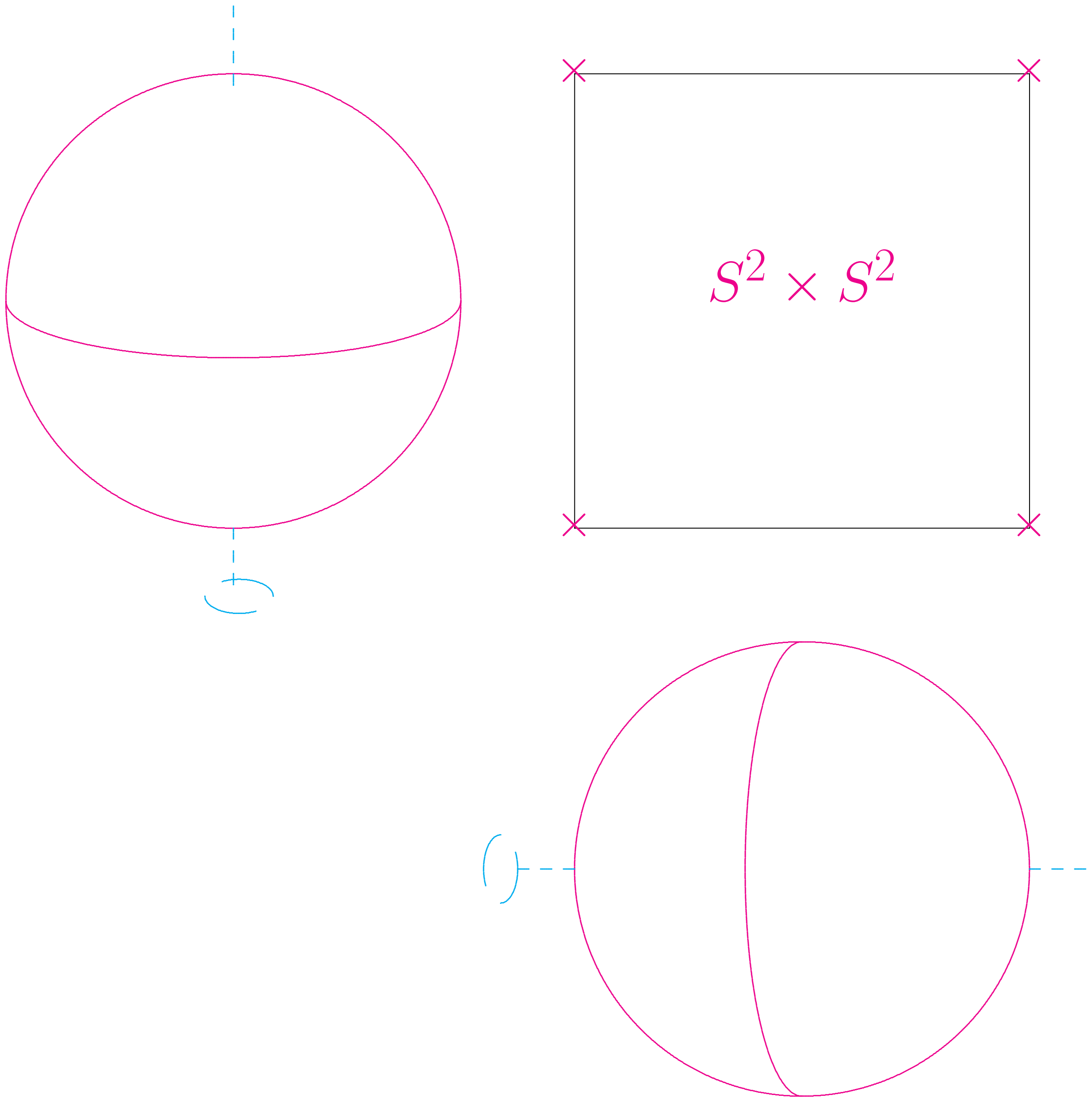

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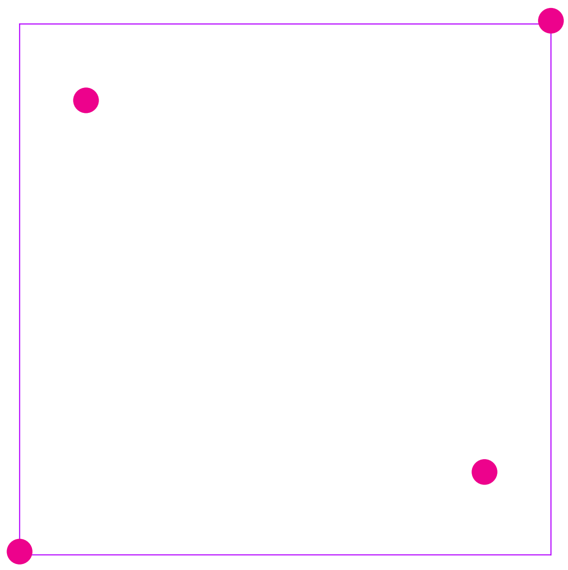

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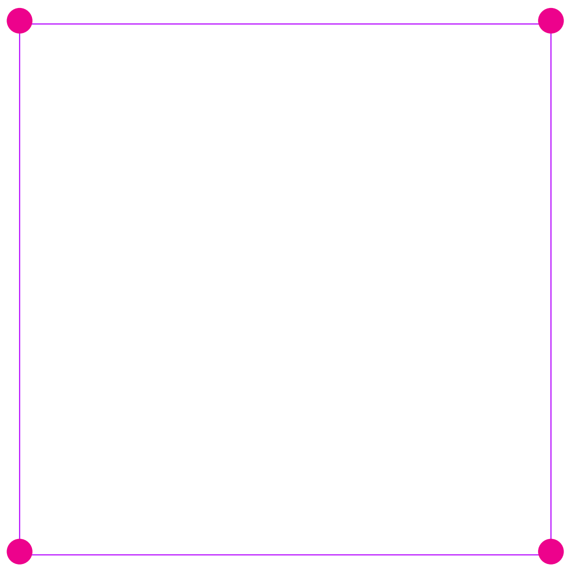

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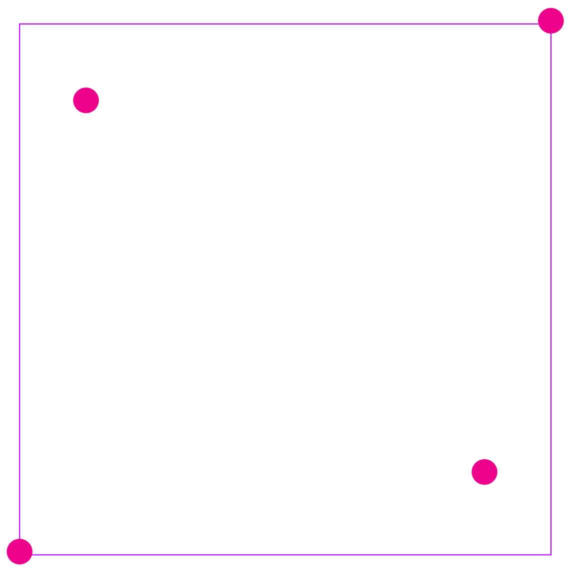

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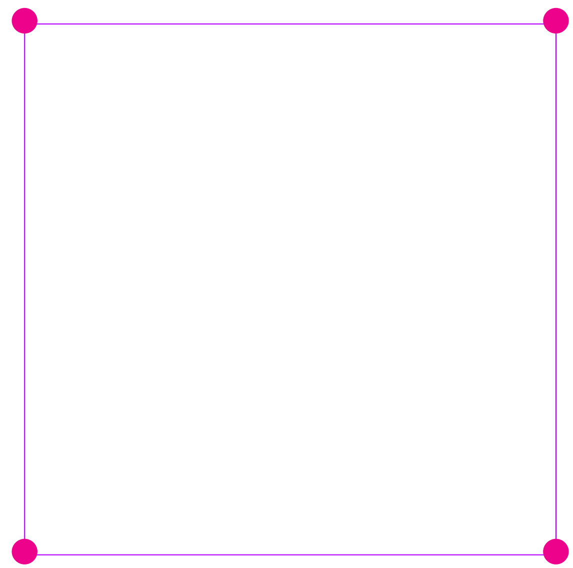

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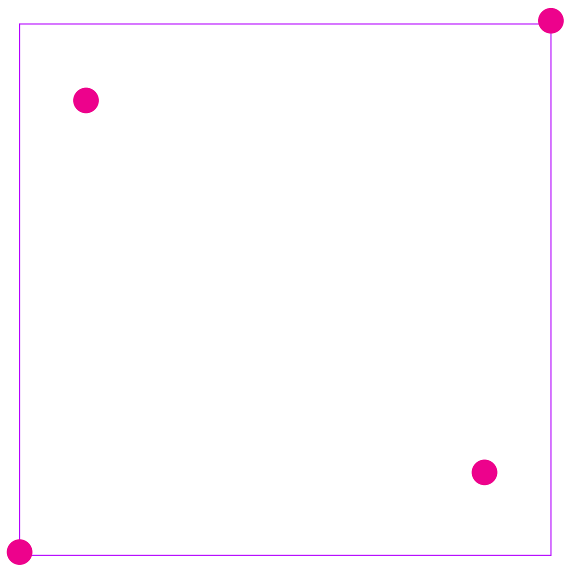

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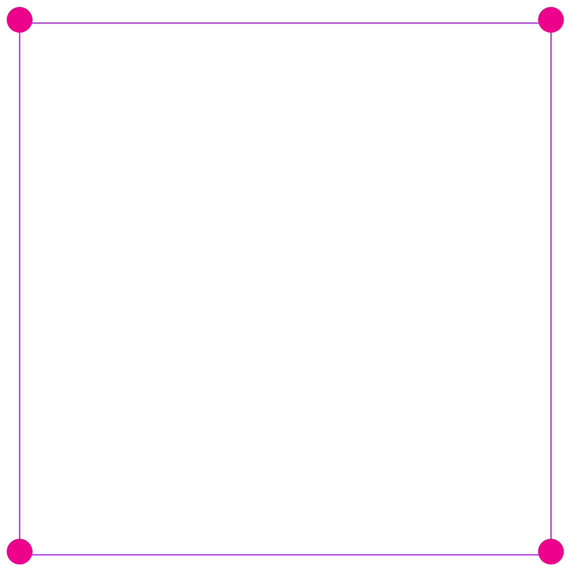

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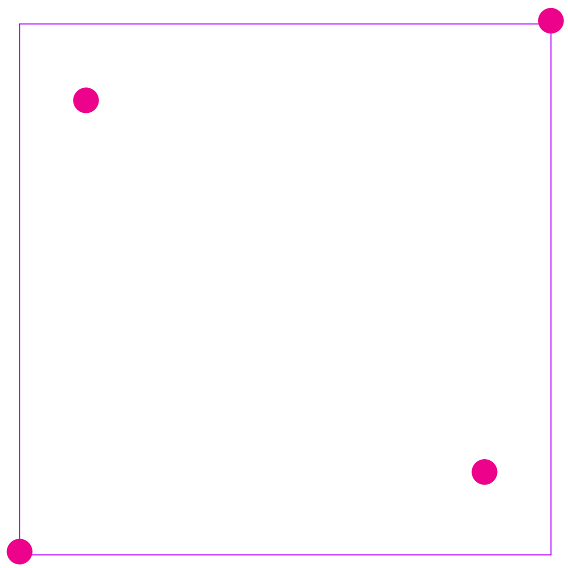

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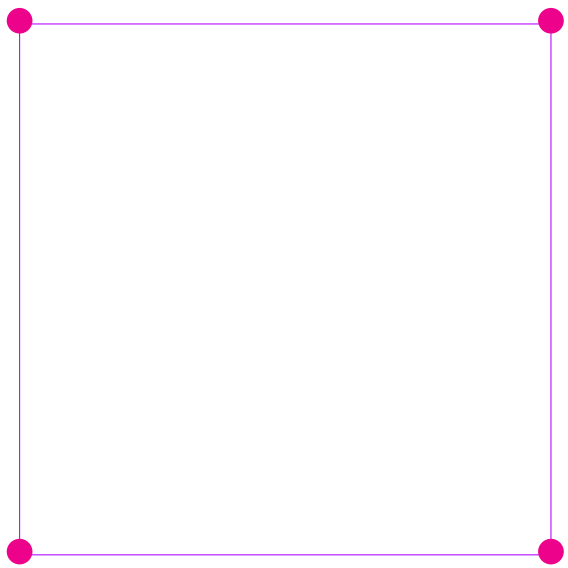

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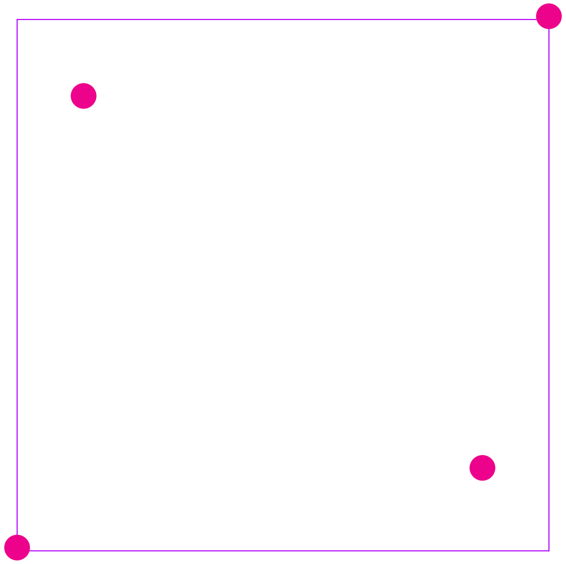

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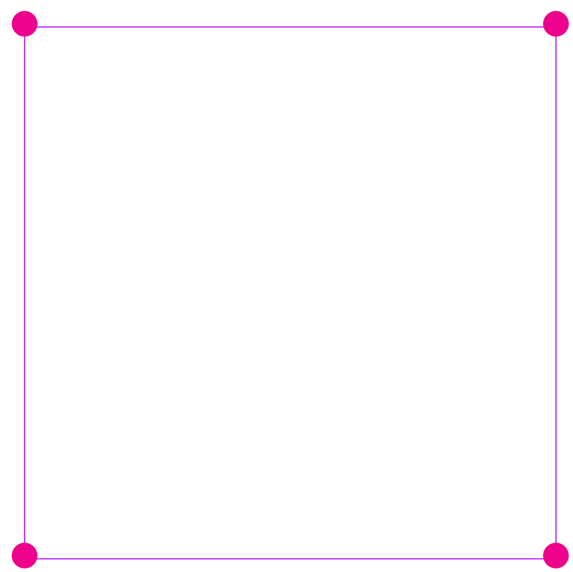

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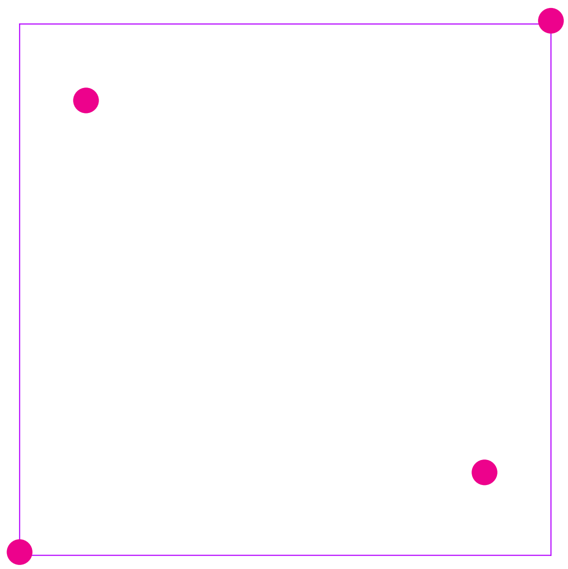

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But what about limits of general Einstein metrics?

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By the Riemannian Goldberg-Sachs Theorem, the Hermitian assumption is equivalent to assuming that the orbifold Einstein metric g_∞ is conformally Kähler.

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This assertion is peculiar to dimension 4.

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It is false in all higher dimensions!

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A more transparent proof was then given in L '21.

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In most cases, this implies that g is Kähler-Einstein. There are two exceptions, but these are both rigid, and thus never lead to non-trivial G-H limits. Conversely, $\lambda > 0$ K-E $\Longrightarrow \det(W^+) > 0$.

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Technically hardest when curvature accumulates on many different length-scales, giving rise to a complicated bubble tree.

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Then show that the given Einstein metrics g_j belong to this universal family for all $j \gg 0$.

And hence that they are actually conformally Kähler!

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