Anti-Self-Dual 4-Manifolds,

Quasi-Fuchsian Groups, &

Almost-Kähler Geometry

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Stony Brook University

Seminario de Geometría Diferencial
Universidad de La Laguna, October 16, 2018

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To appear in Comm. An. Geom.

Facets of Kähler Geometry:

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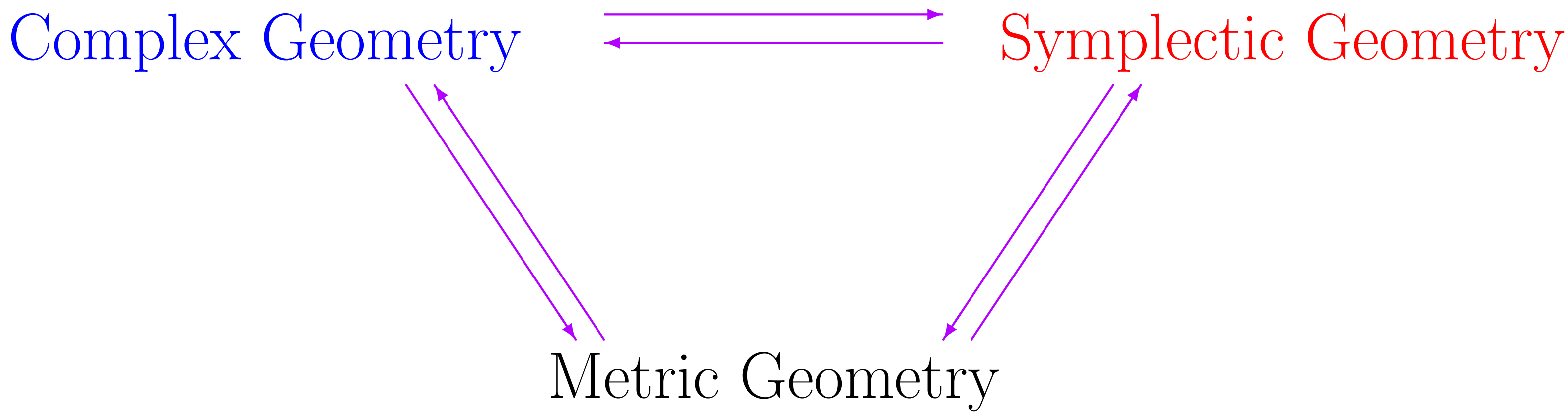

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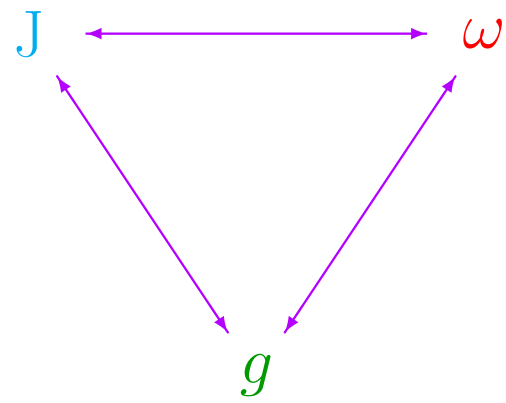

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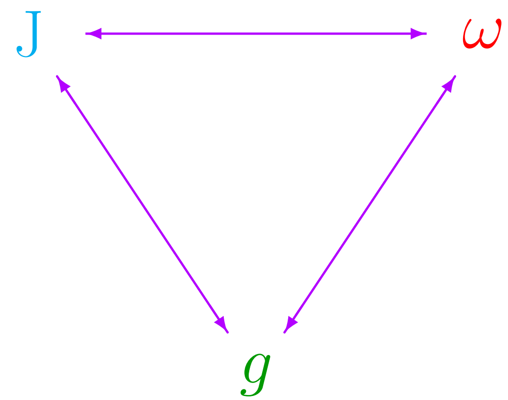

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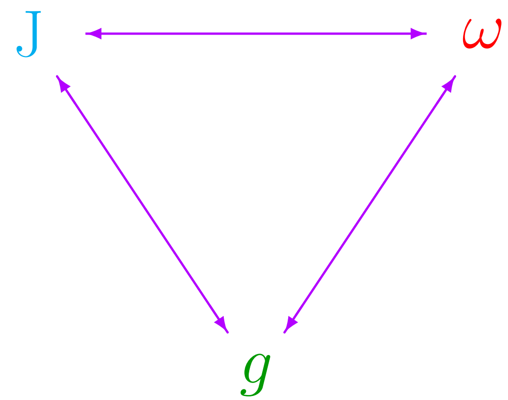

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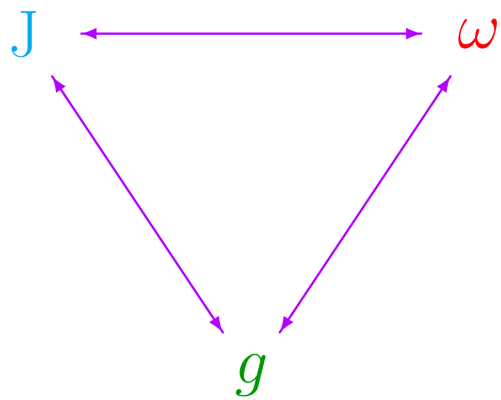

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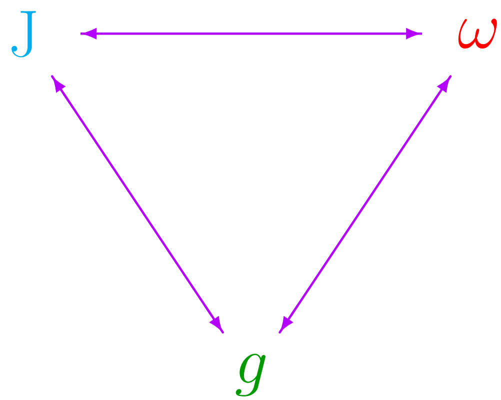

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Higher dimensions are demonstrably different.

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Imitates Kähler geometry in a non-Kähler setting.

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$$\text{Kähler} \Longleftrightarrow J \text{ integrable.}$$

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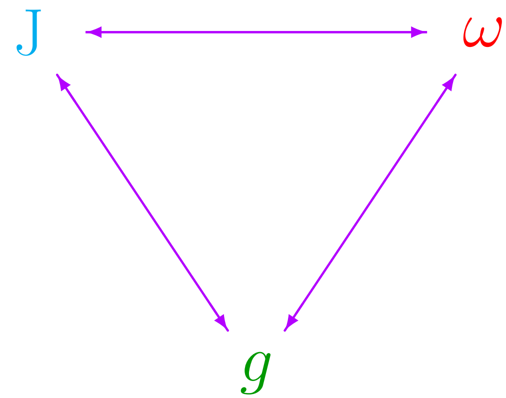

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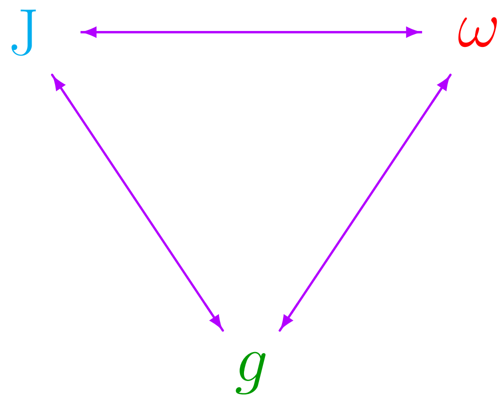

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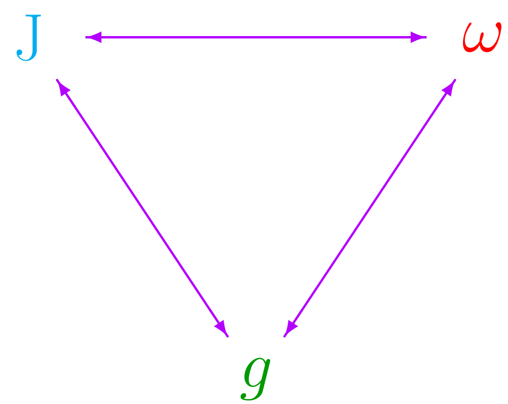

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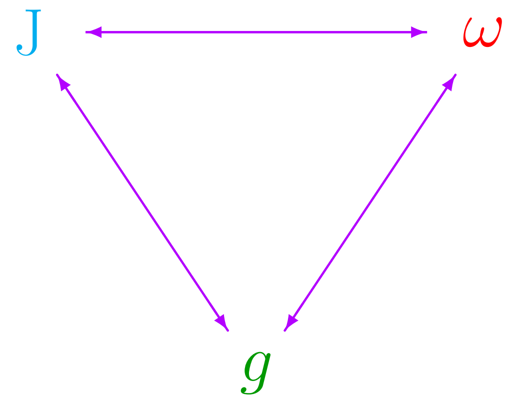
For example, can avoid explicitly mentioning J.



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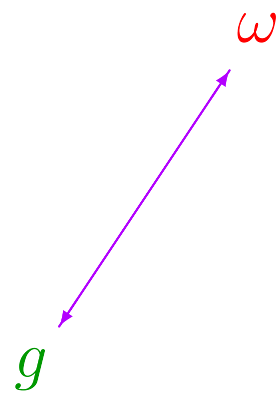
ω
g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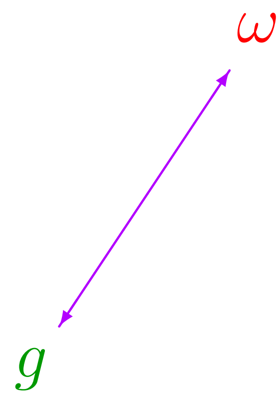
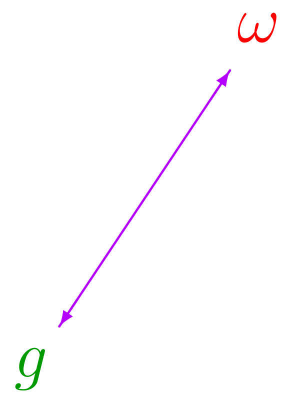
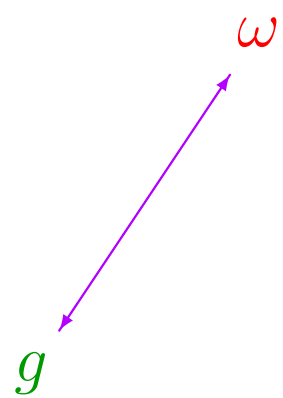


Lemma. *An oriented Riemannian manifold* (M^{2m}, g)



Lemma. *An oriented Riemannian manifold* (M^{2m}, g) *is almost-Kähler*



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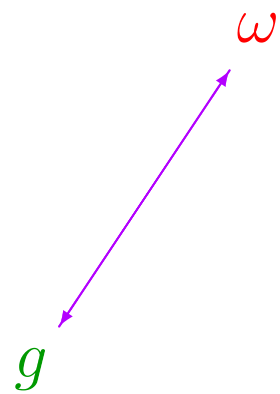

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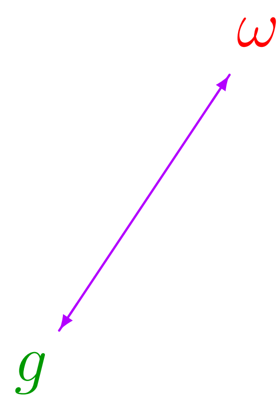

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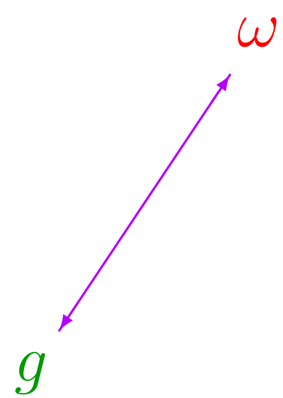

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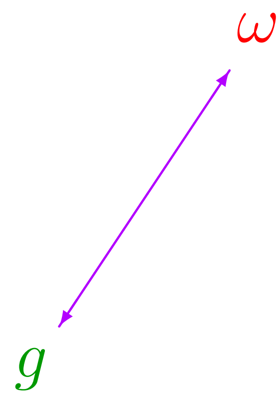

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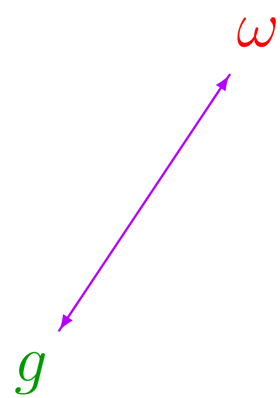

Lemma. *An oriented Riemannian manifold* (M^{2m}, g) *is almost-Kähler w/ respect to the* 2*-form* $\omega \iff$

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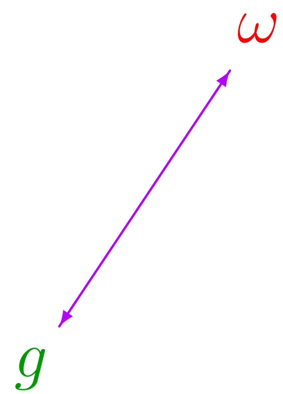
Simplifies dramatically when $m = 2$:



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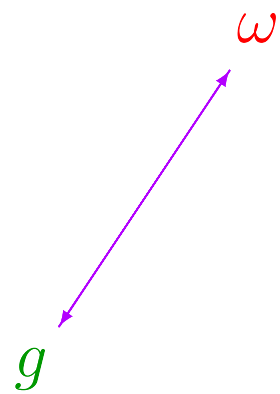

Lemma. *An oriented Riemannian 4-manifold (M, g) is almost-Kähler w/ respect to the 2-form ω* $\Longleftrightarrow$

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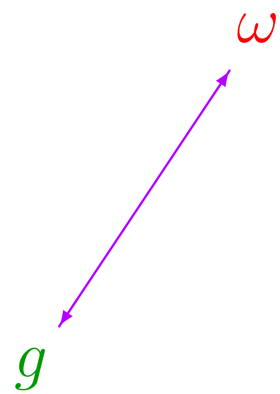

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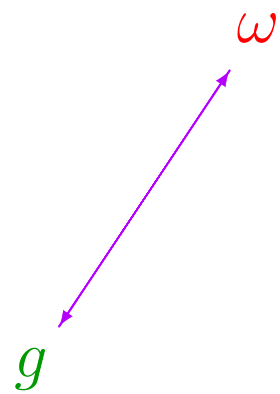

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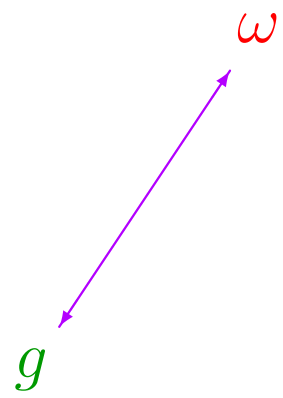

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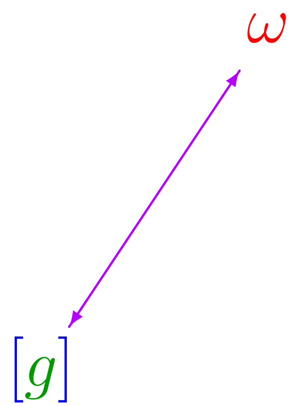

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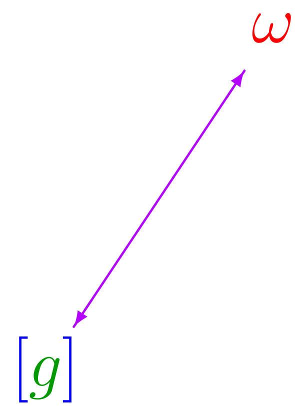

Proposition. *A conformal class $[g]$ on a smooth compact oriented 4-manifold M is represented by an almost-Kähler metric g iff it carries a self-dual harmonic 2-form ω that is $\neq 0$ everywhere.*

Moreover, the set of conformal classes $[g]$ on M that carry such a harmonic form ω is open in the C^2 topology.

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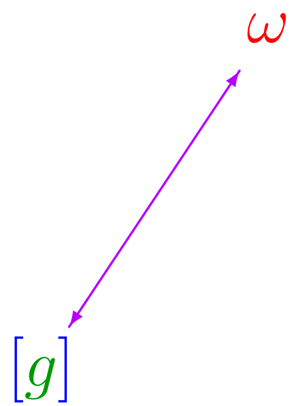

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In particular, the numbers

$$b_\pm(M) = \dim \mathcal{H}^\pm_g$$

are independent of g, and so are invariants of M.

$b_{\pm}(M)$?

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$$\begin{bmatrix} +1 & & & & & \\ & \ddots & & & & \\ & & +1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix}.$$

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$$\left[\begin{array}{cc} \underbrace{\begin{array}{ccc} +1 & & \\ & \ddots & \\ & & +1 \end{array}}_{b_+(M)} & \\ & b_-(M) \left\{ \begin{array}{ccc} -1 & & \\ & \ddots & \\ & & -1 \end{array} \right. \end{array} \right].$$

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$$\tau(M) = b_+(M) - b_-(M)$$

"Signature" of M.

Signature defined in terms of intersection pairing

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(Thom-Hirzebruch Signature Formula)

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Has major consequences in conformal geometry.

On oriented (M^4, g),

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

Riemann curvature of g

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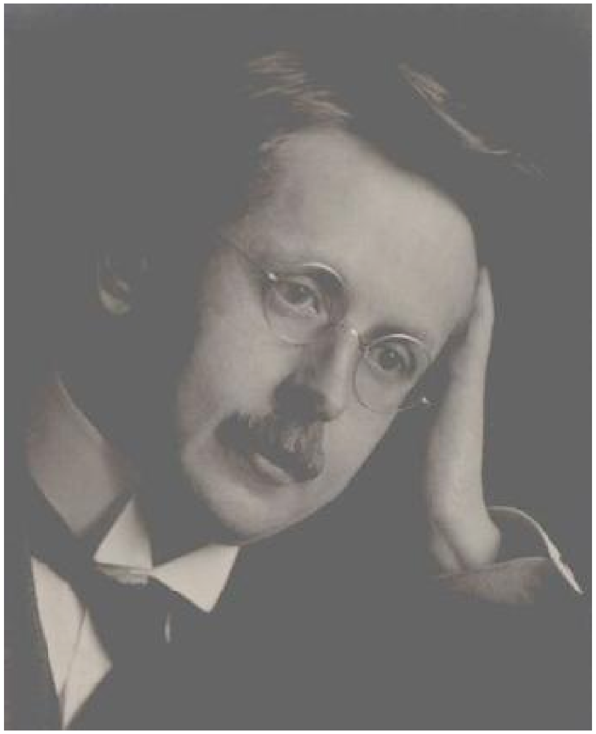

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Basic problems: For given smooth compact M^4,

- What is $\inf \mathscr{W}$?
- Do there exist minimizers?

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But we've already noted that

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So Weyl functional is essentially equivalent to

$$[g] \longmapsto \int_M |W_+|^2 d\mu_g$$

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(ASD)

Twistor picture of anti-self-duality condition:

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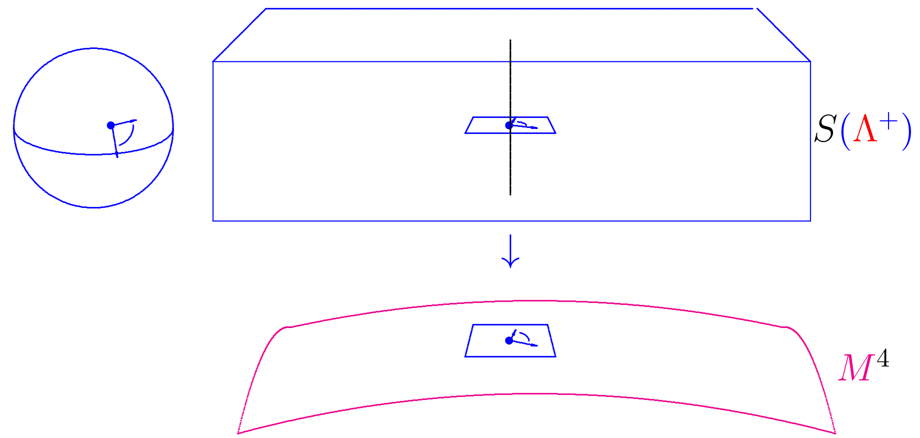

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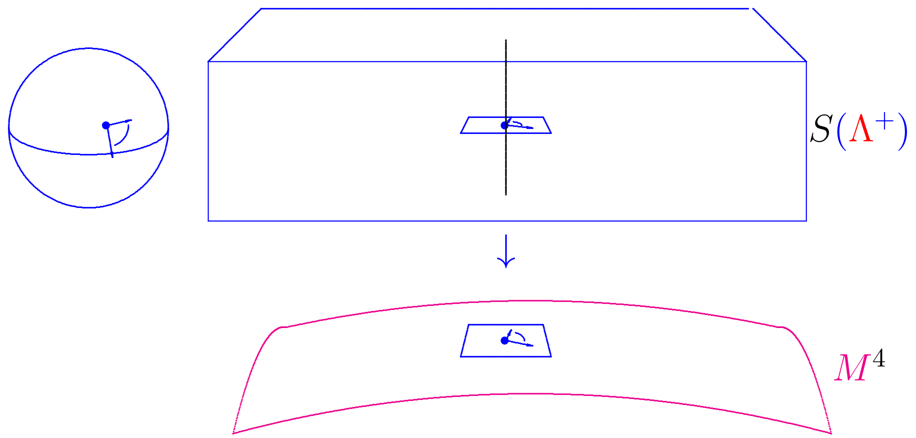

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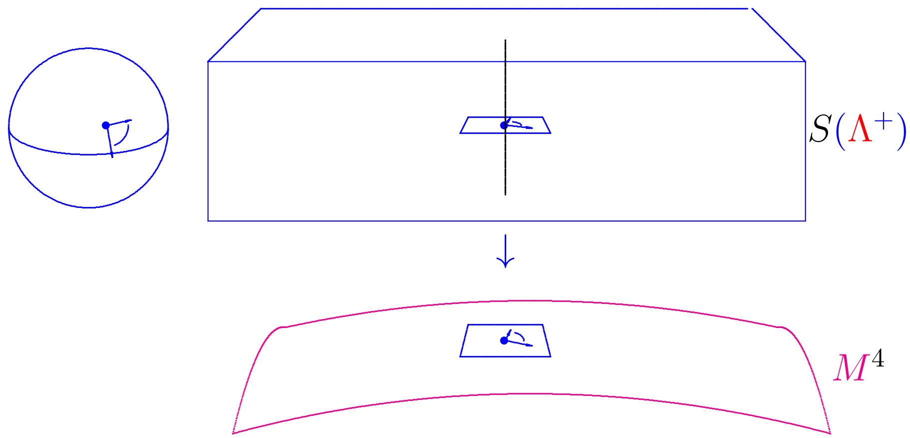

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Reconceptualizes earlier work by Penrose.

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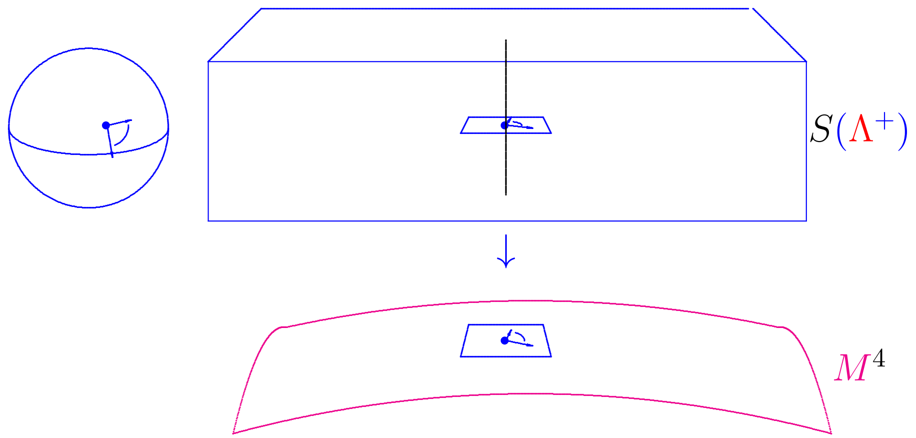

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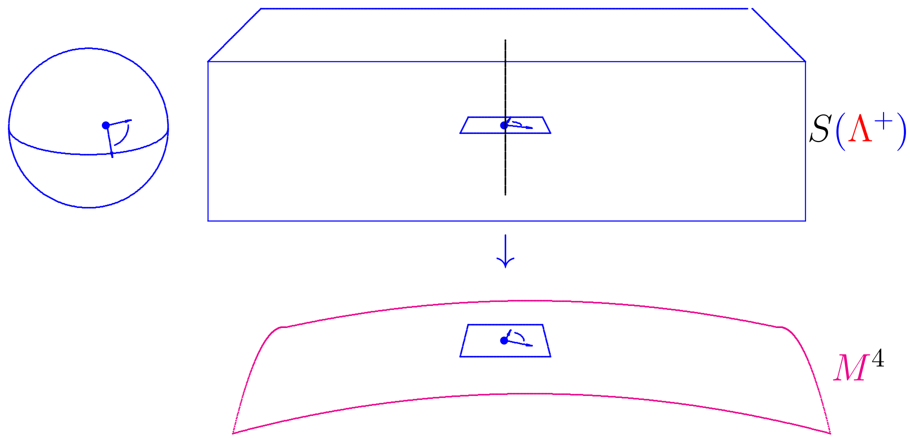

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Motivates study of ASD metrics,
and yields methods for constructing them.

So ASD metrics are linked to complex geometry. . .

A different link with complex geometry:

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special case of cscK manifolds,

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Results proved about SFK in '90s foreshadowed many more recent results about general case.

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  - eight specific finite quotients of these
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  - $\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2$, $k \geq 10$
  -
  -
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$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

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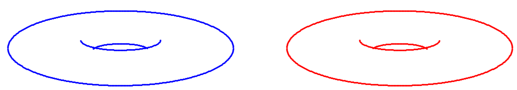

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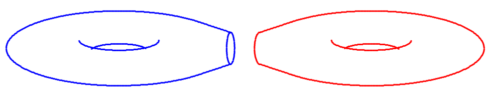

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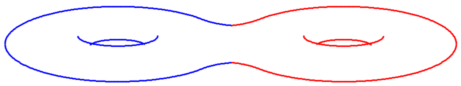

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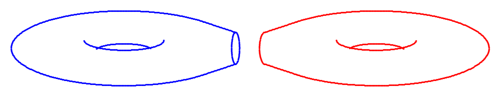

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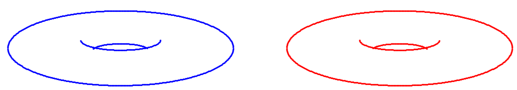

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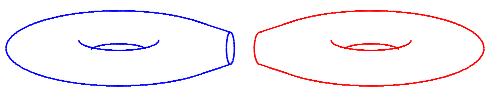

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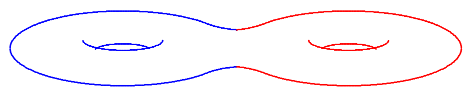

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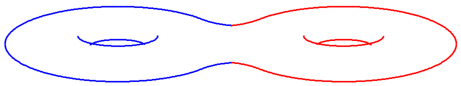

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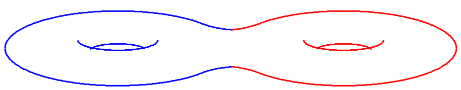

Blowing Up: $M \rightsquigarrow M\#\overline{\mathbb{CP}}_2$

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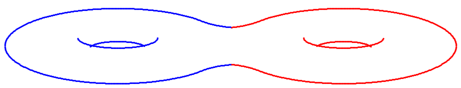

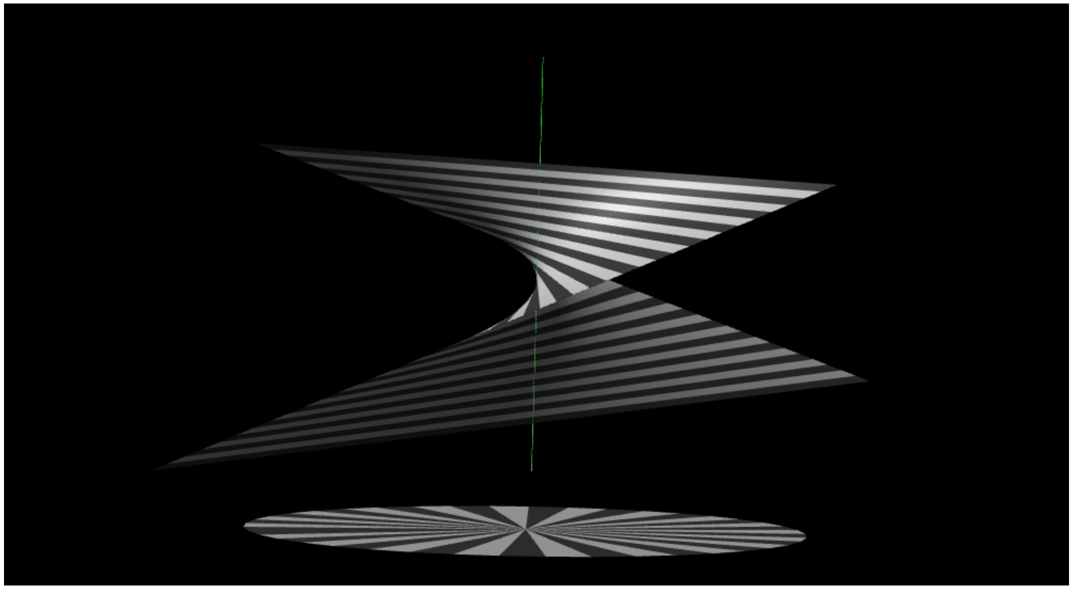

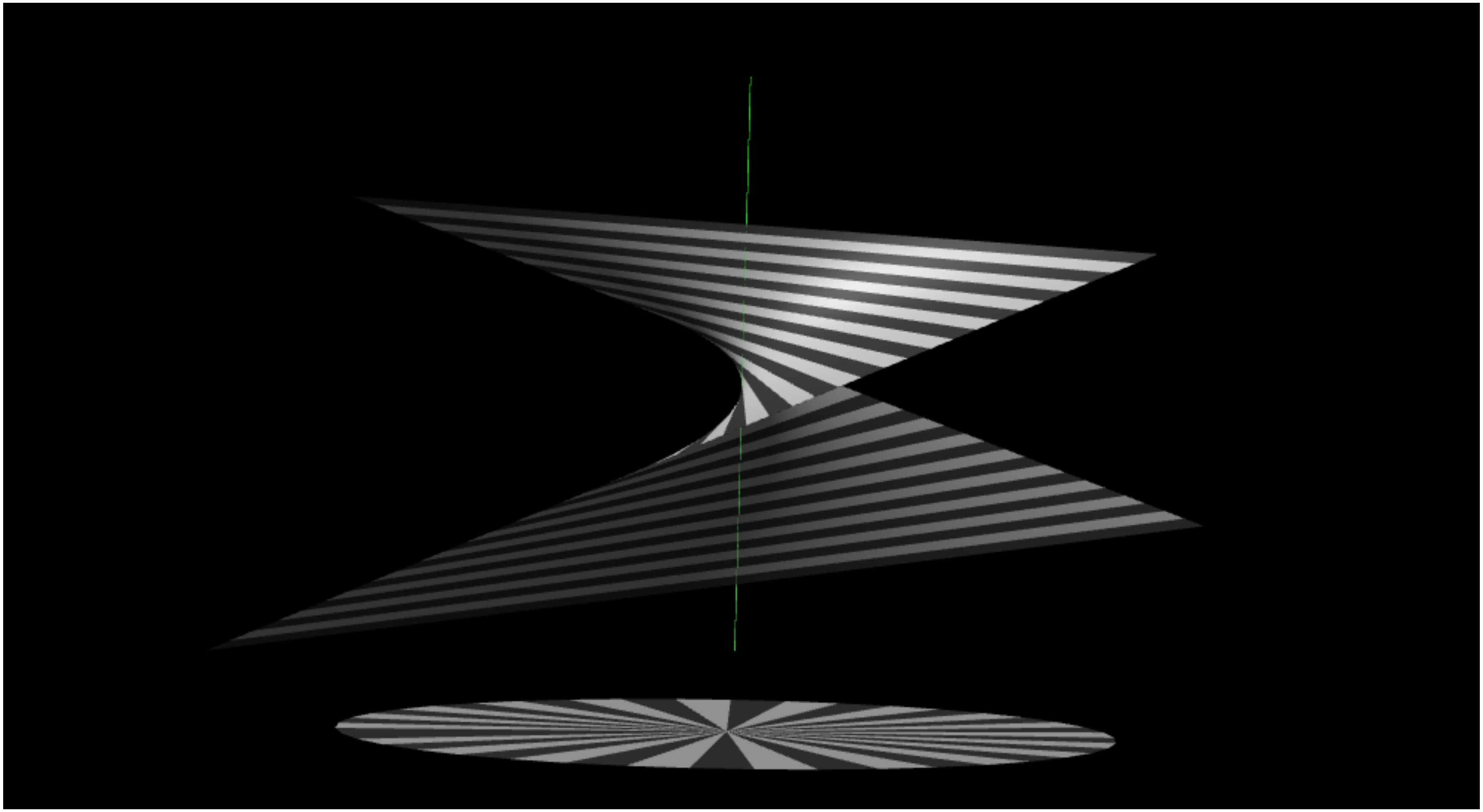

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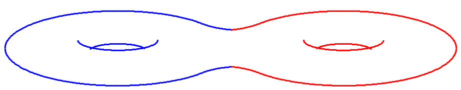

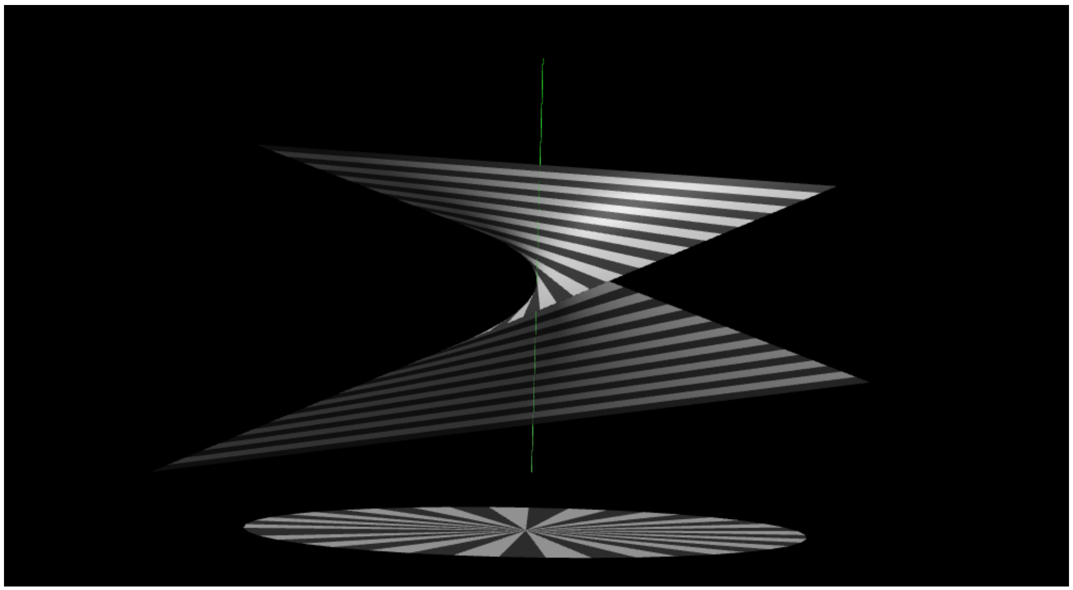

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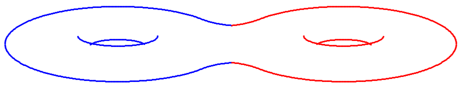

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with equality iff g conformal to Kähler-Einstein.

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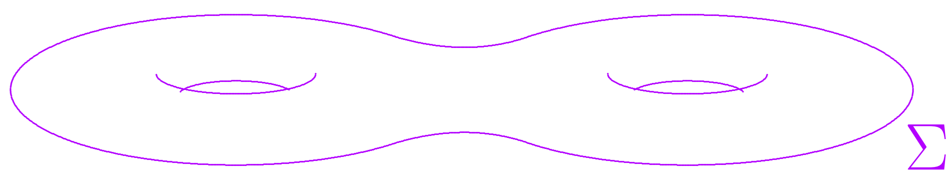

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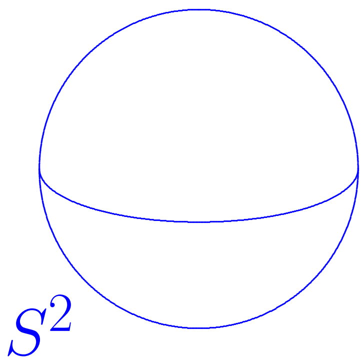

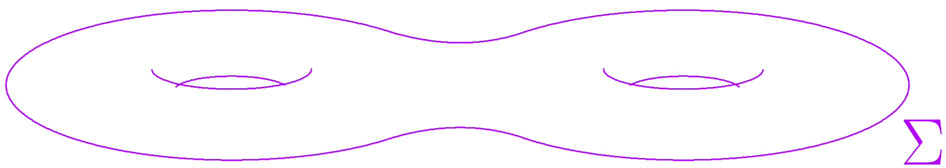

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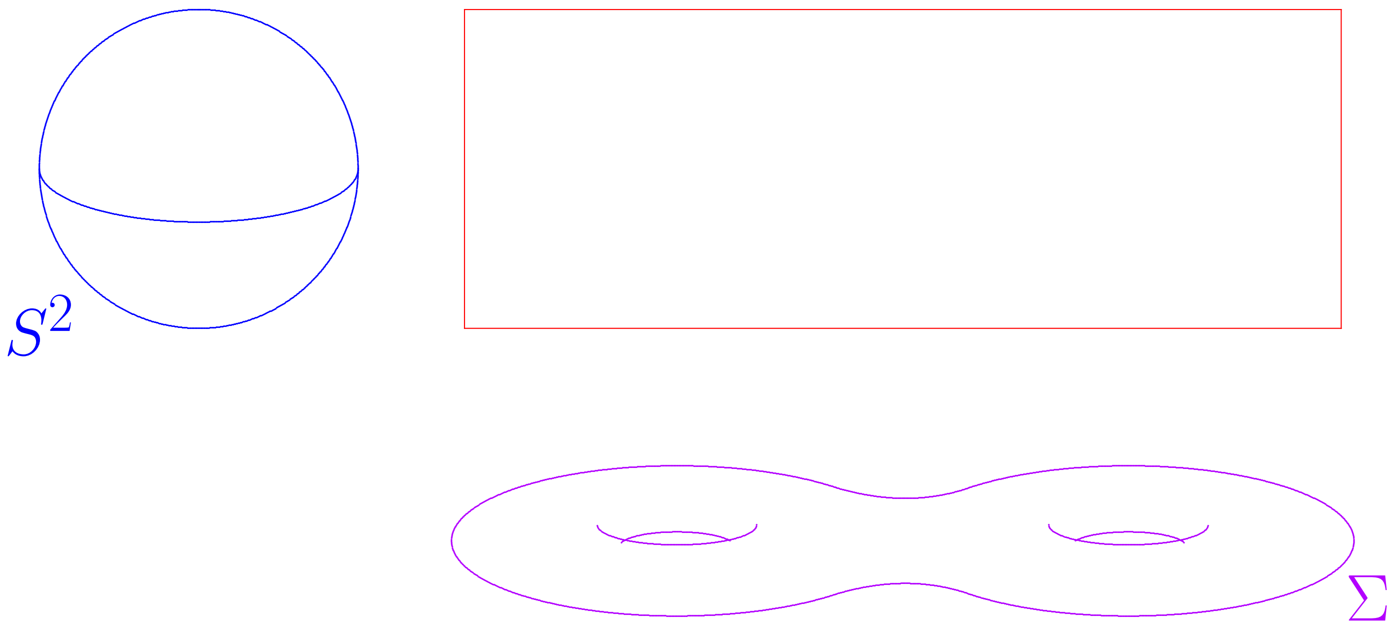

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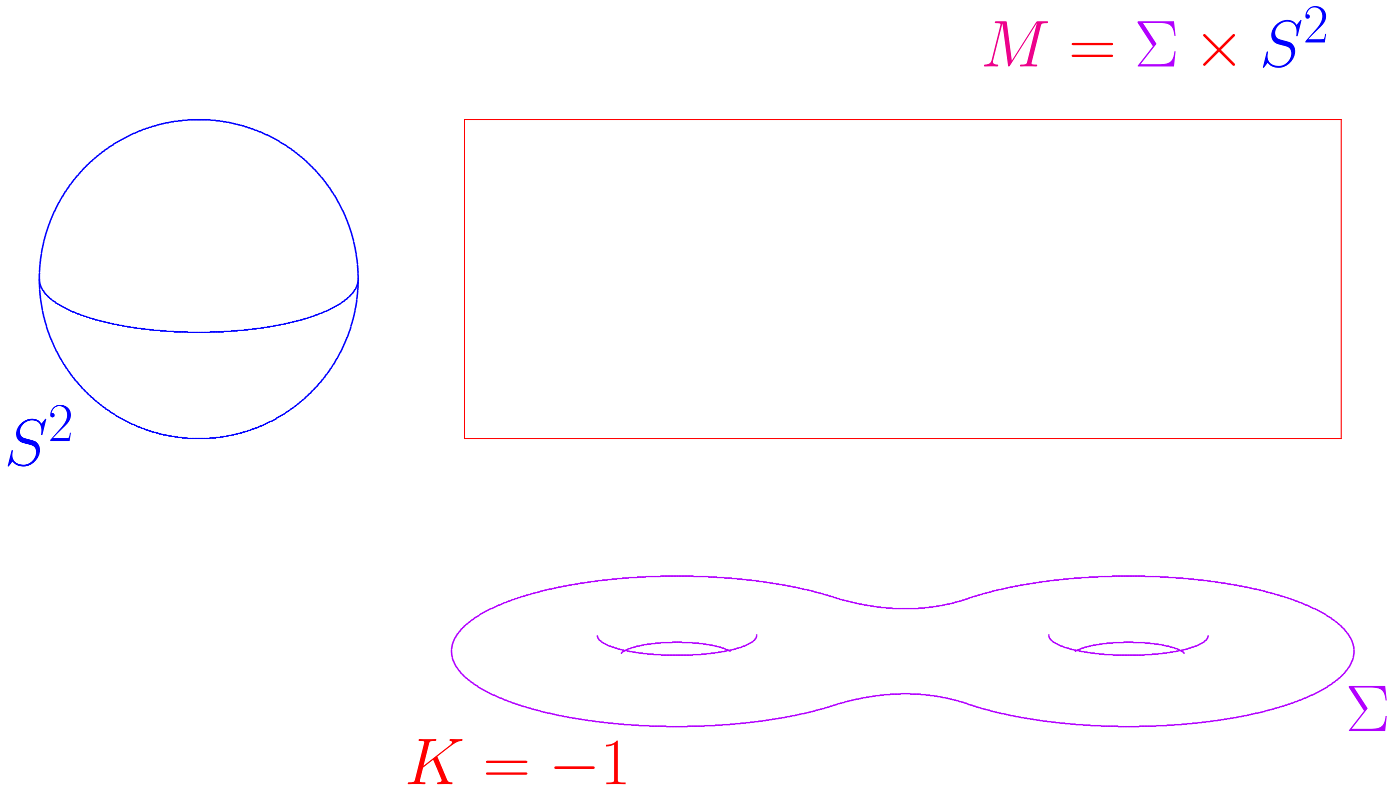

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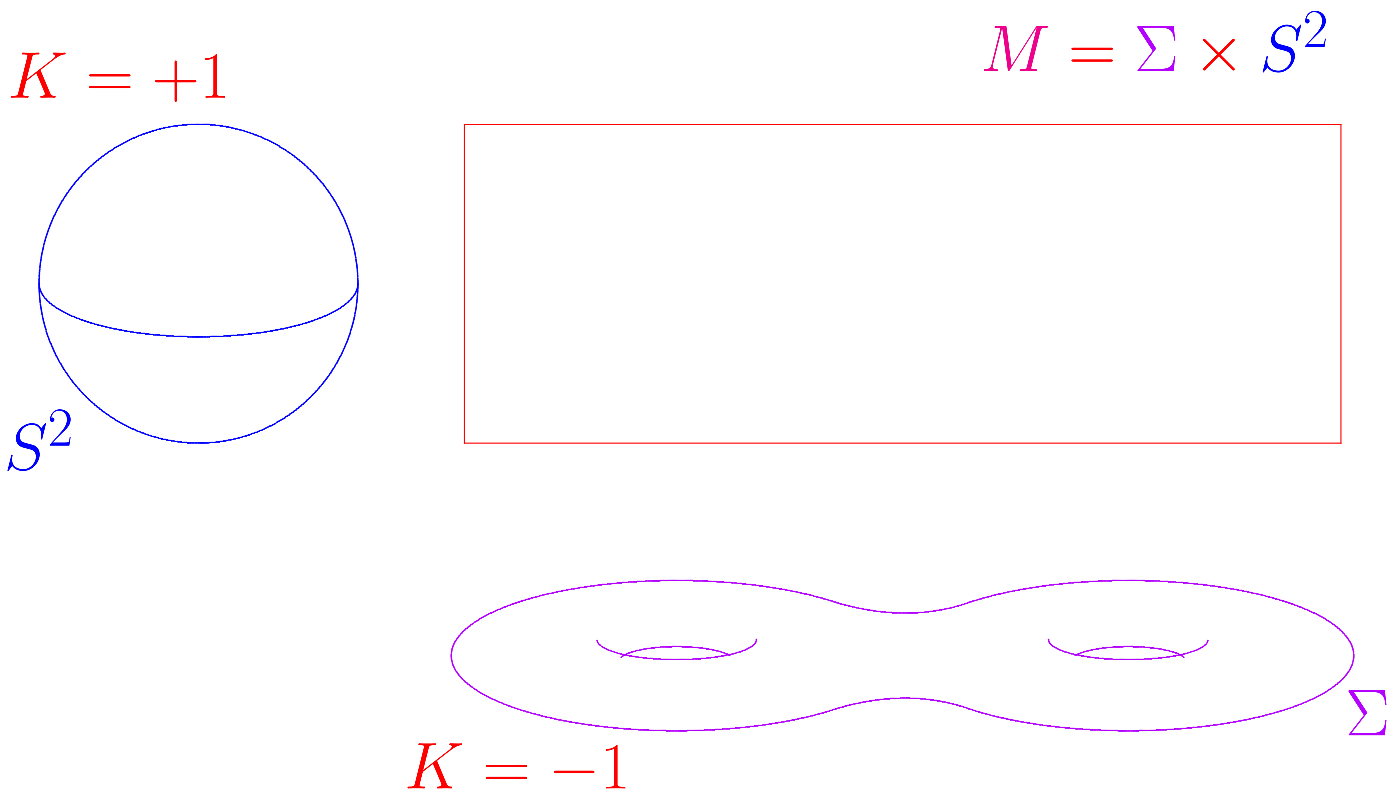

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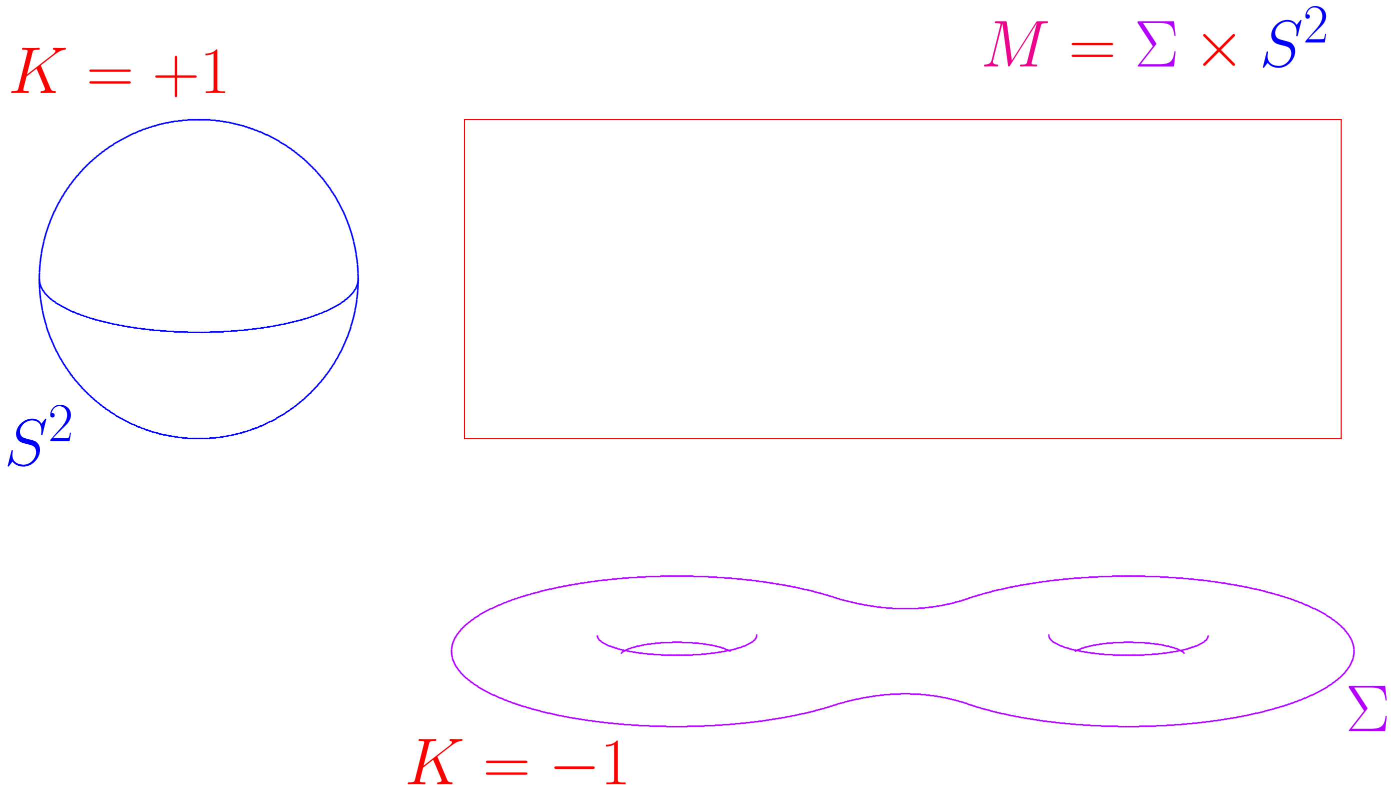

Product is scalar-flat

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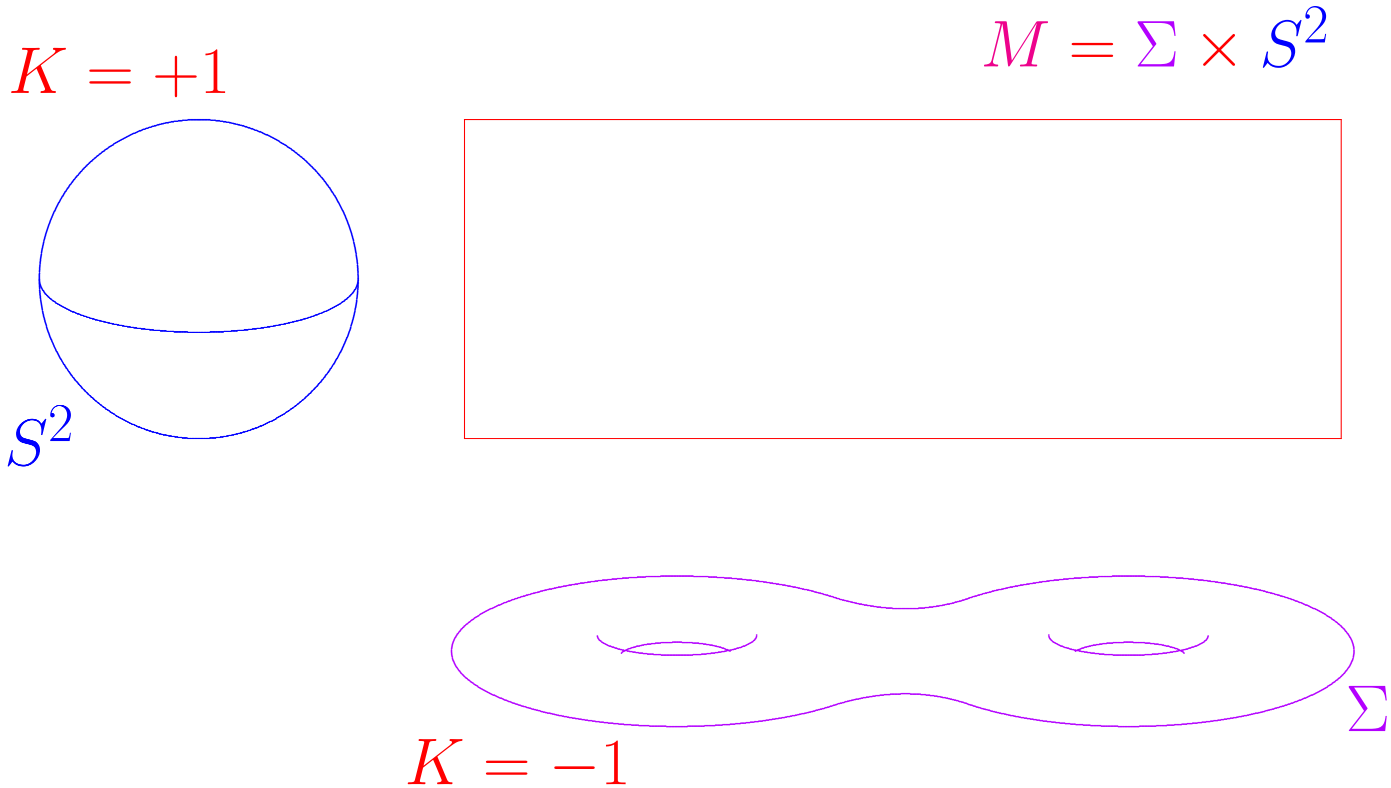

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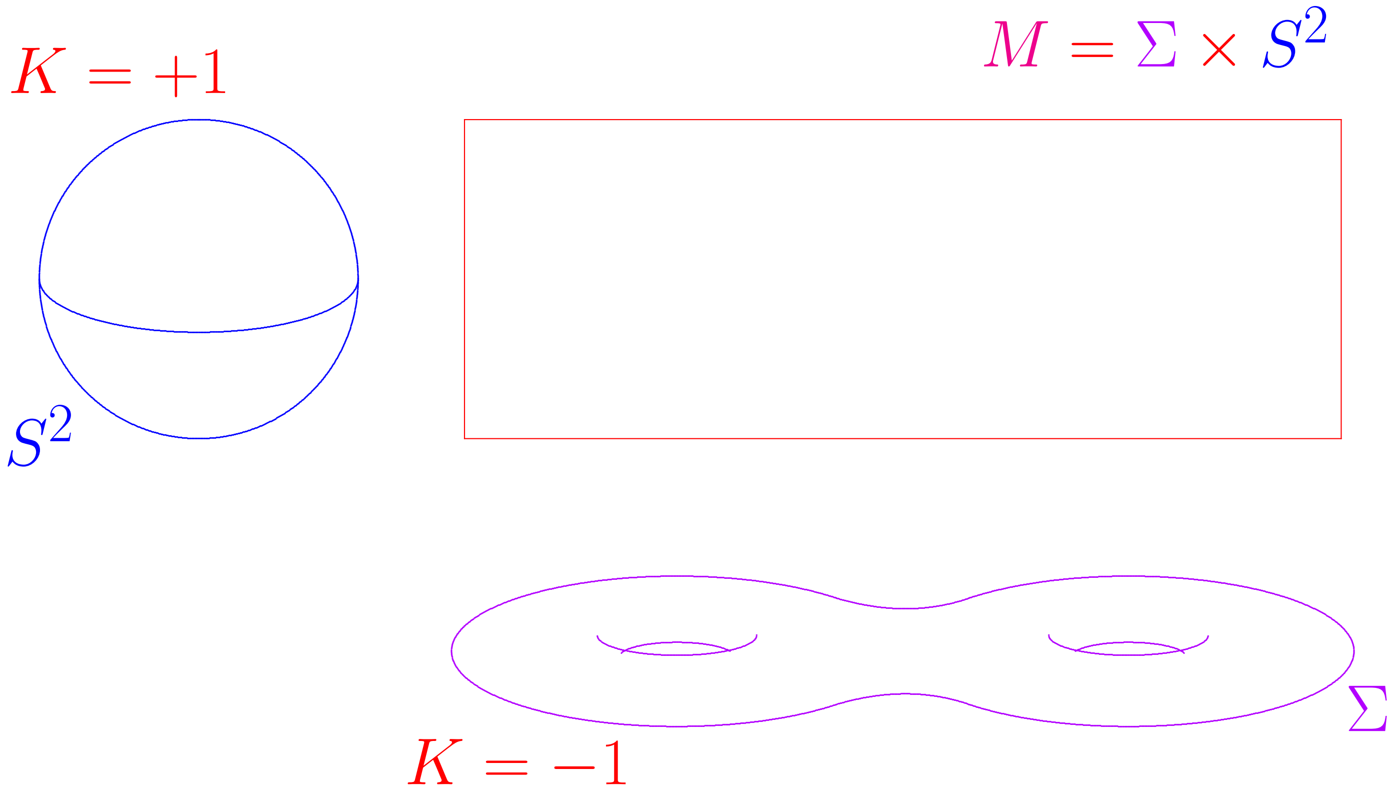

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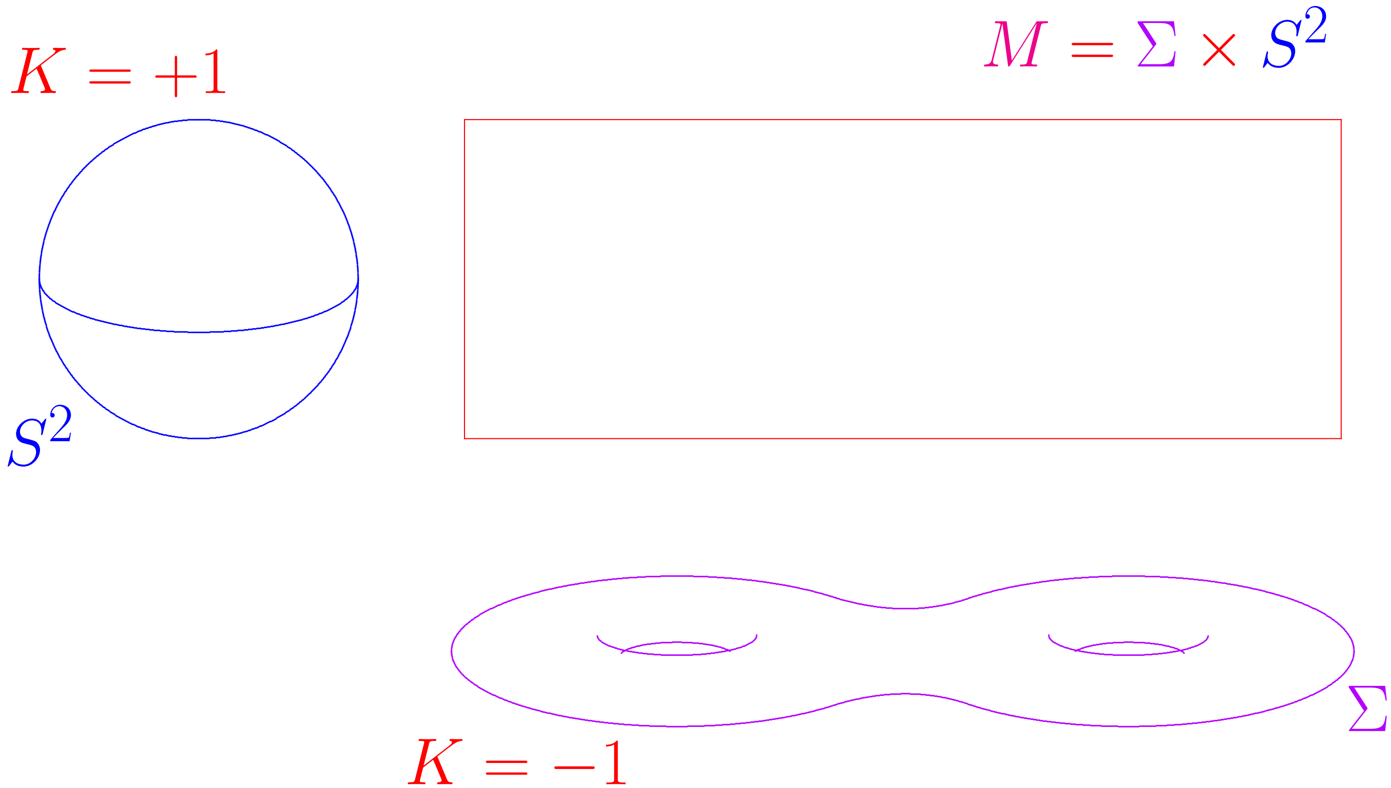

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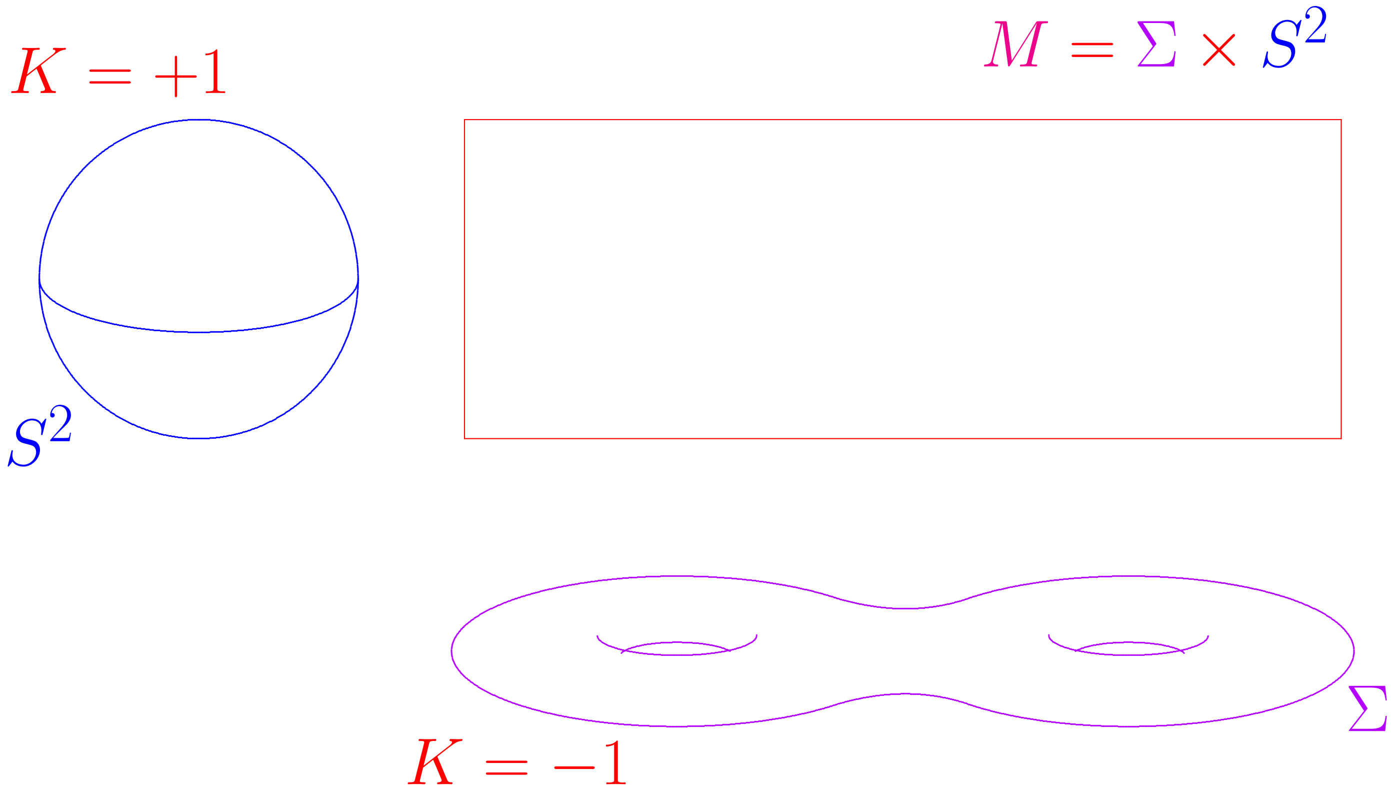

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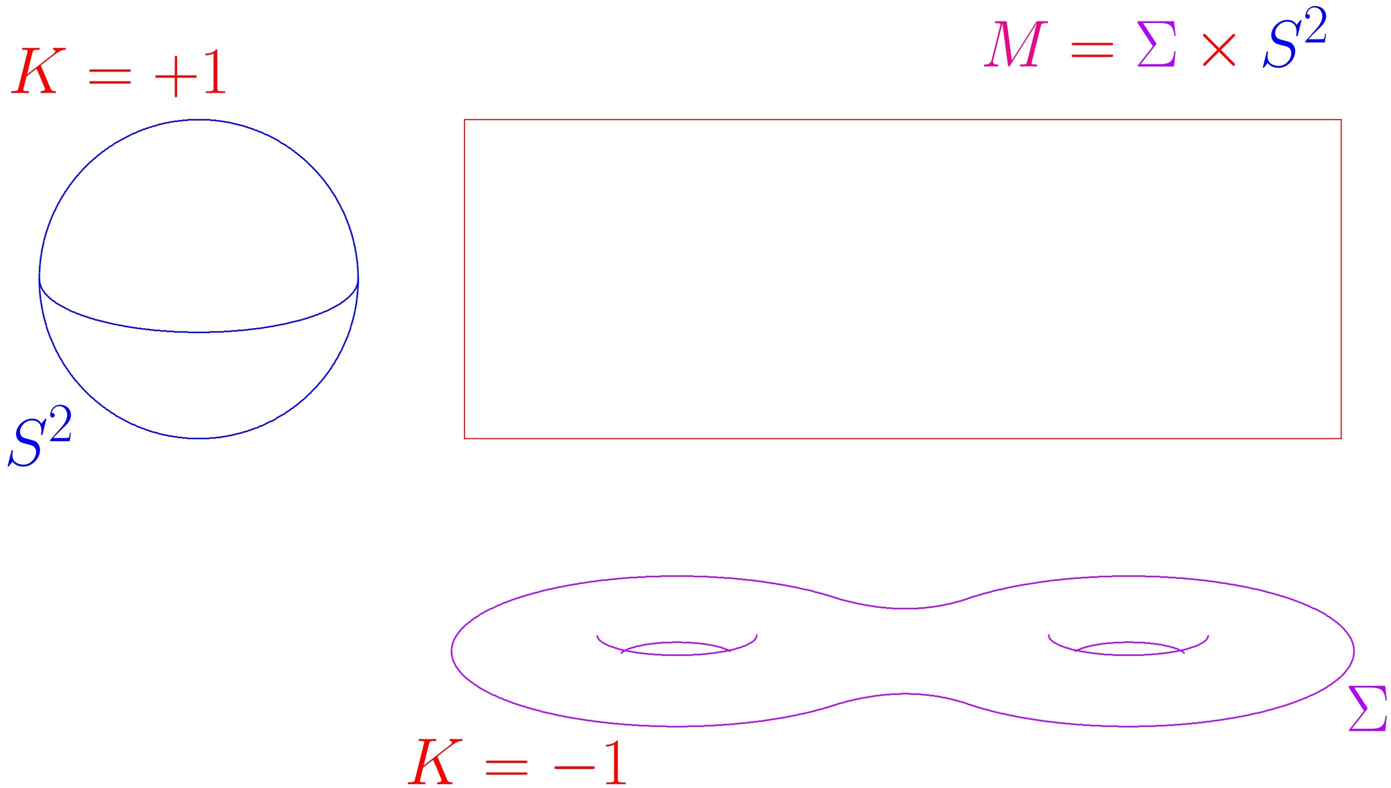

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Locally conformally flat!

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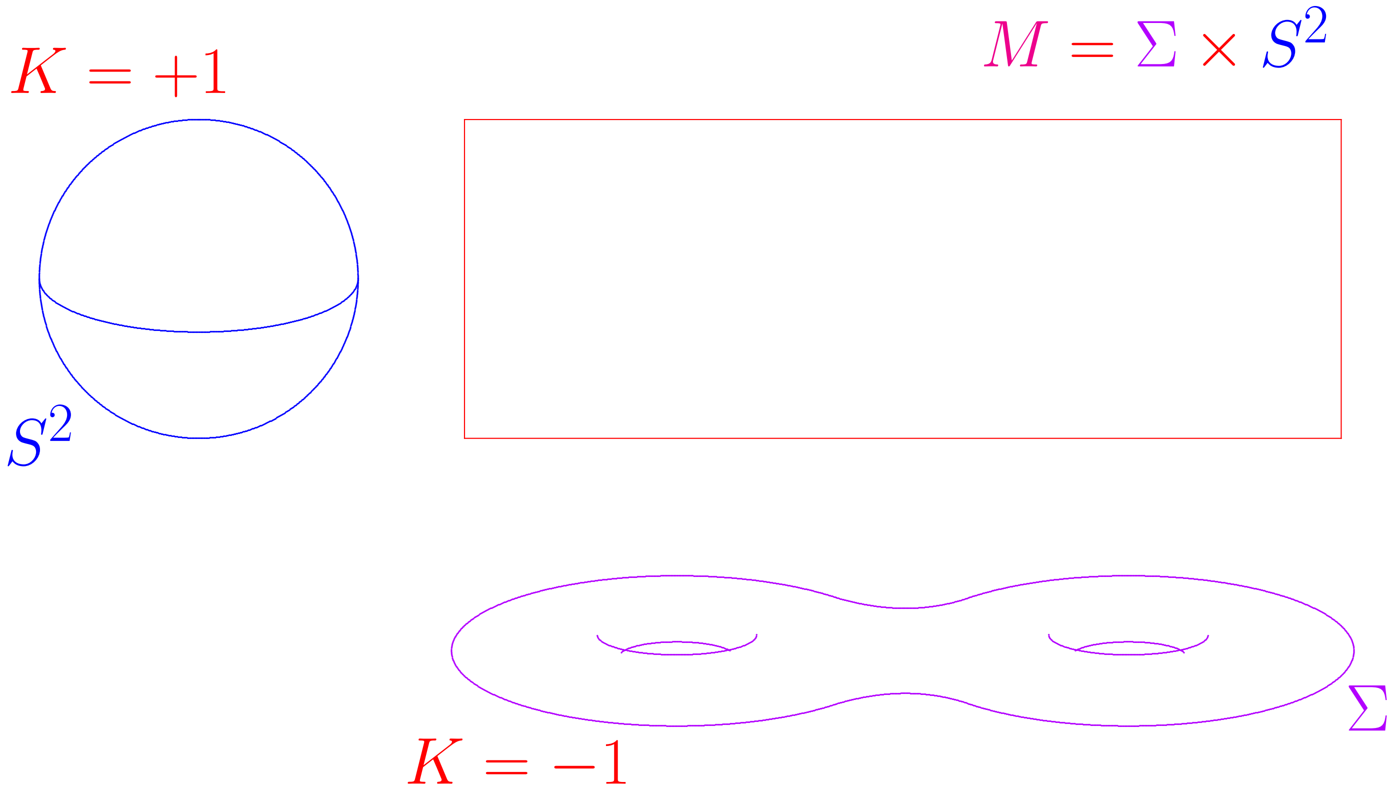

$$\widetilde{M} = \mathcal{H}^2 \times S^2$$

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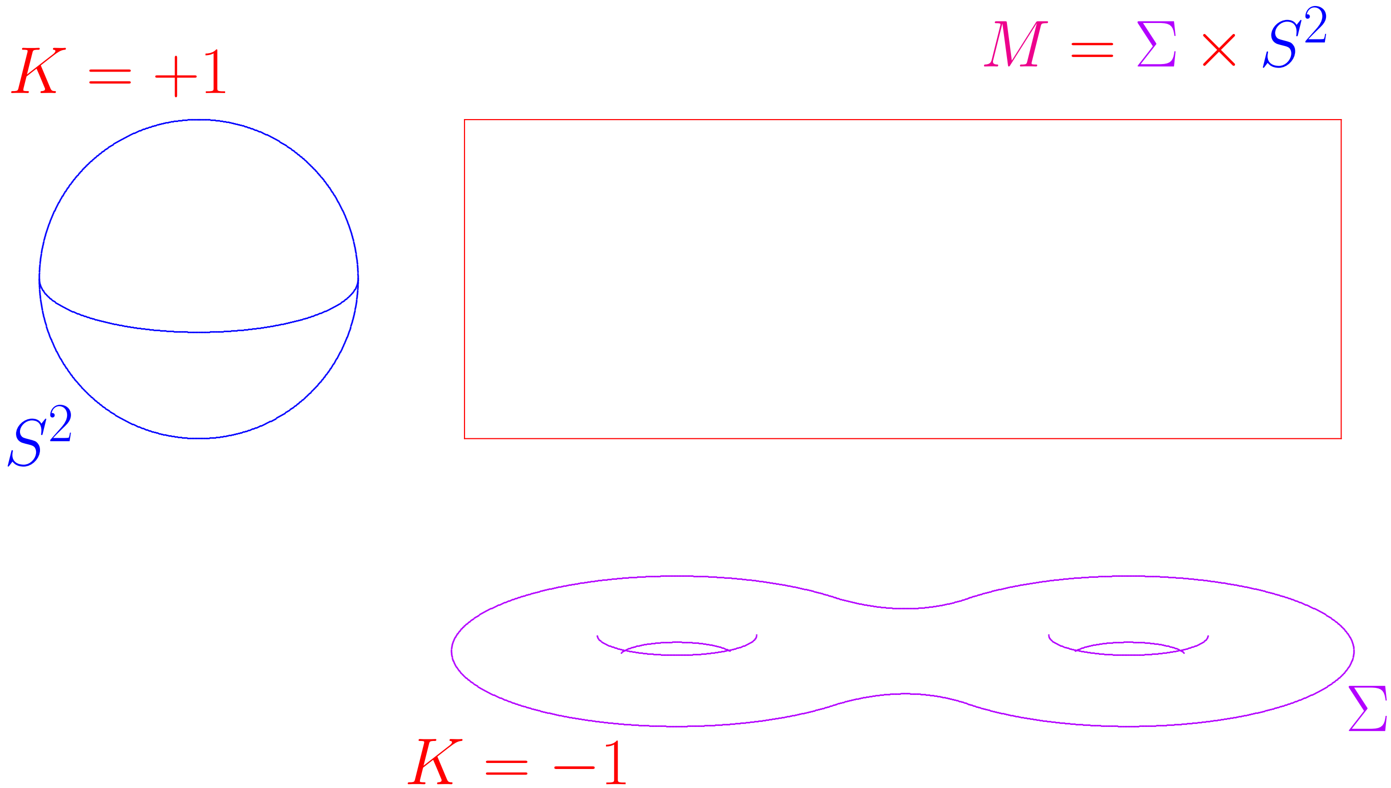

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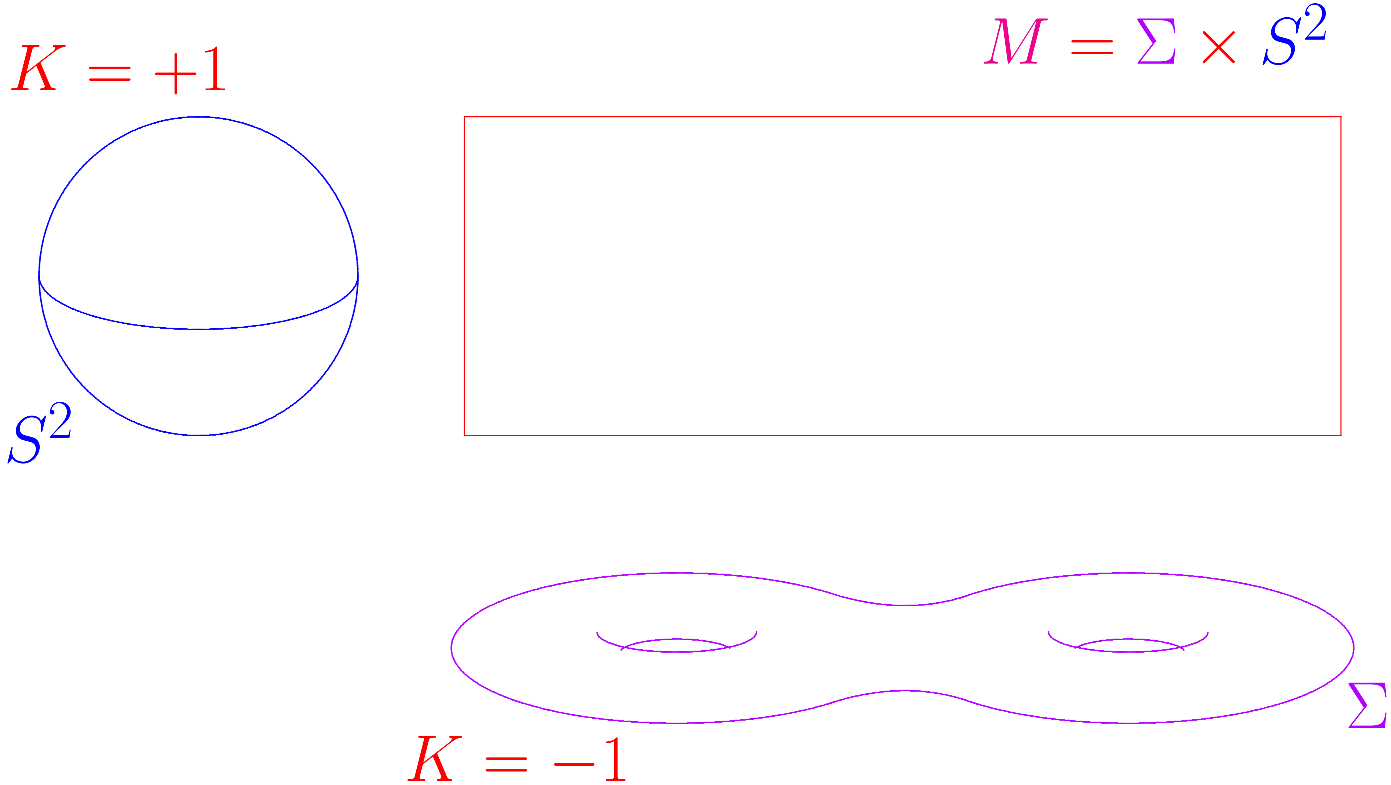

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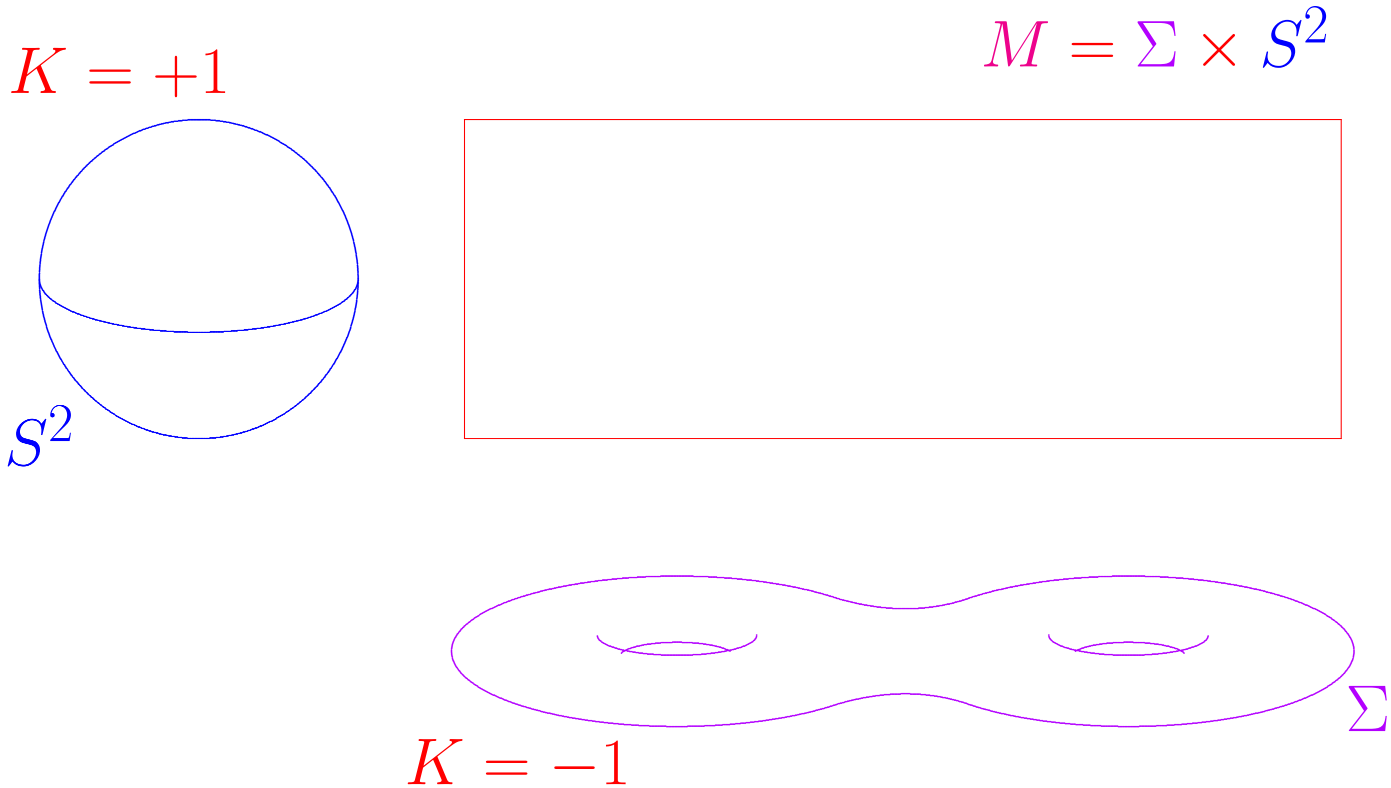

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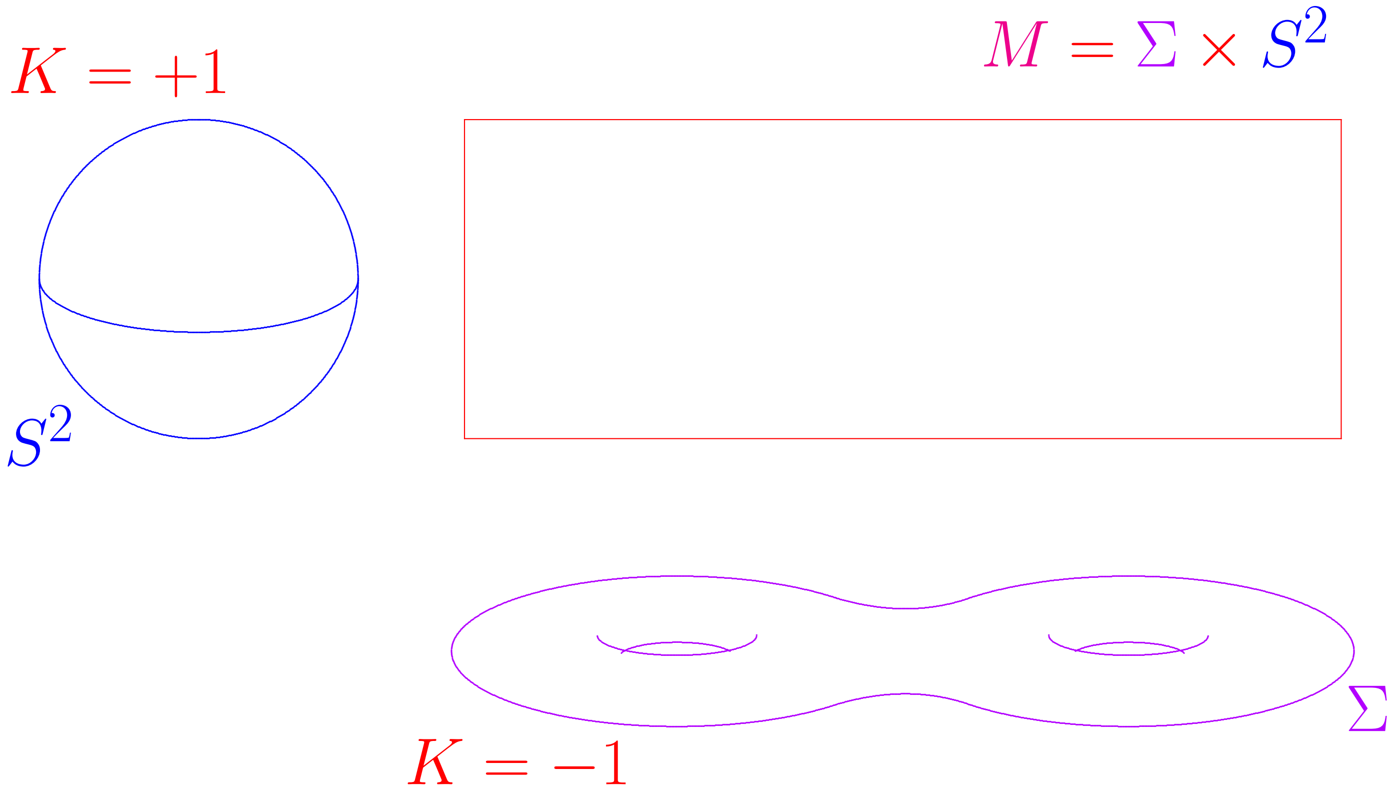

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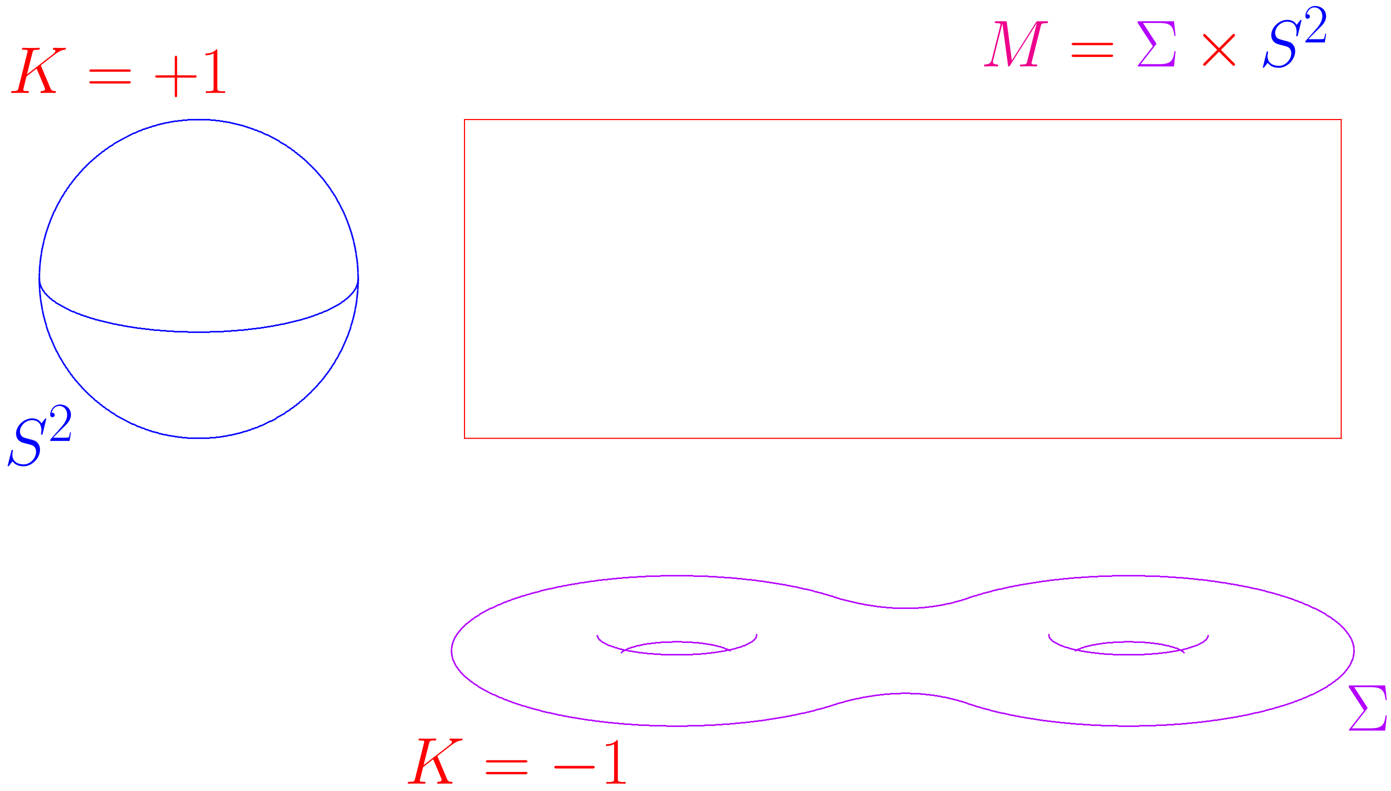

Scalar-flat Kähler deformations: $12(g - 1)$ moduli

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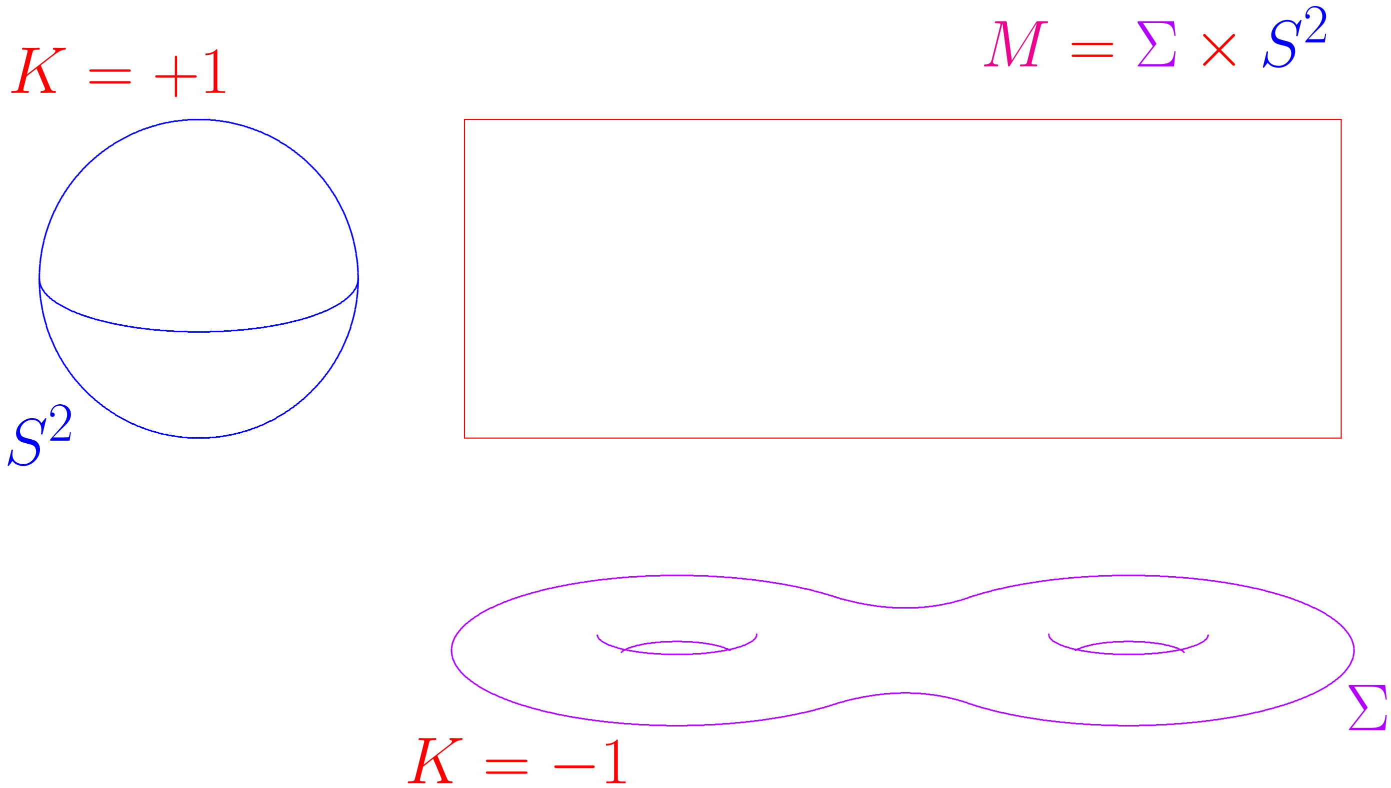

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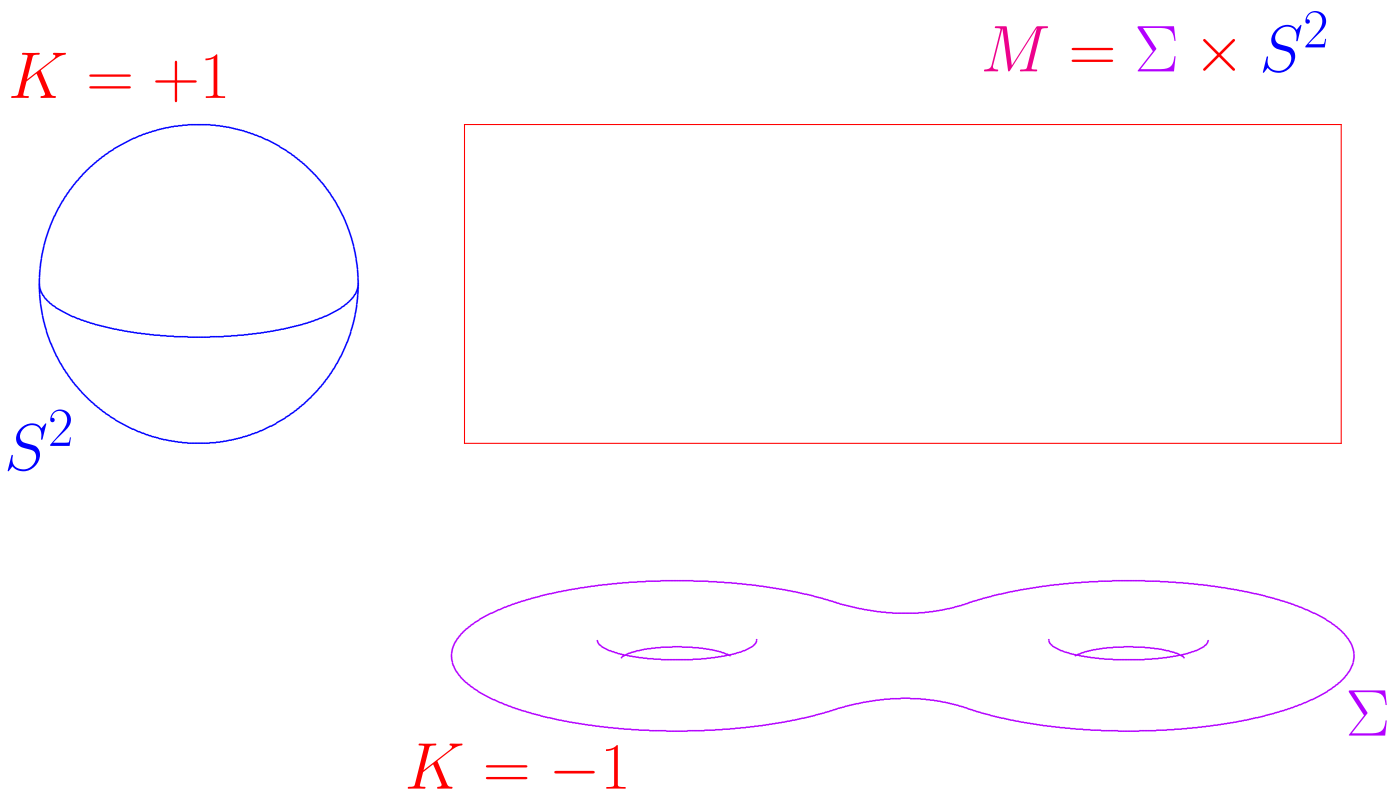

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Same method simultaneously proves. . .

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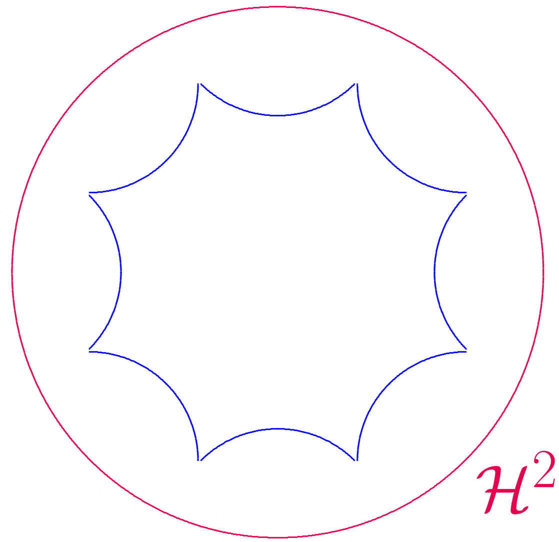
We begin by revisiting hyperbolic metrics on Σ.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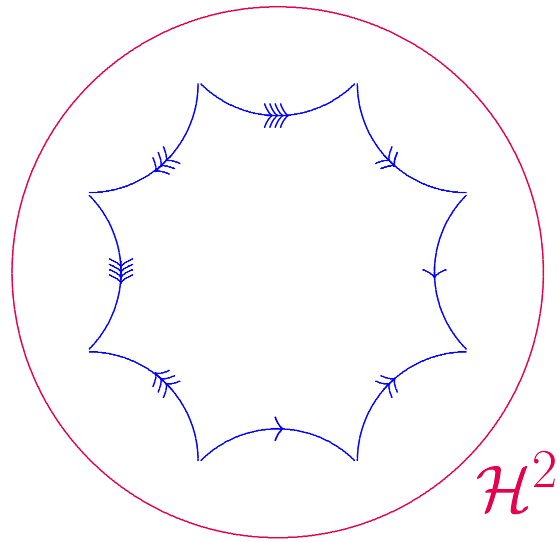
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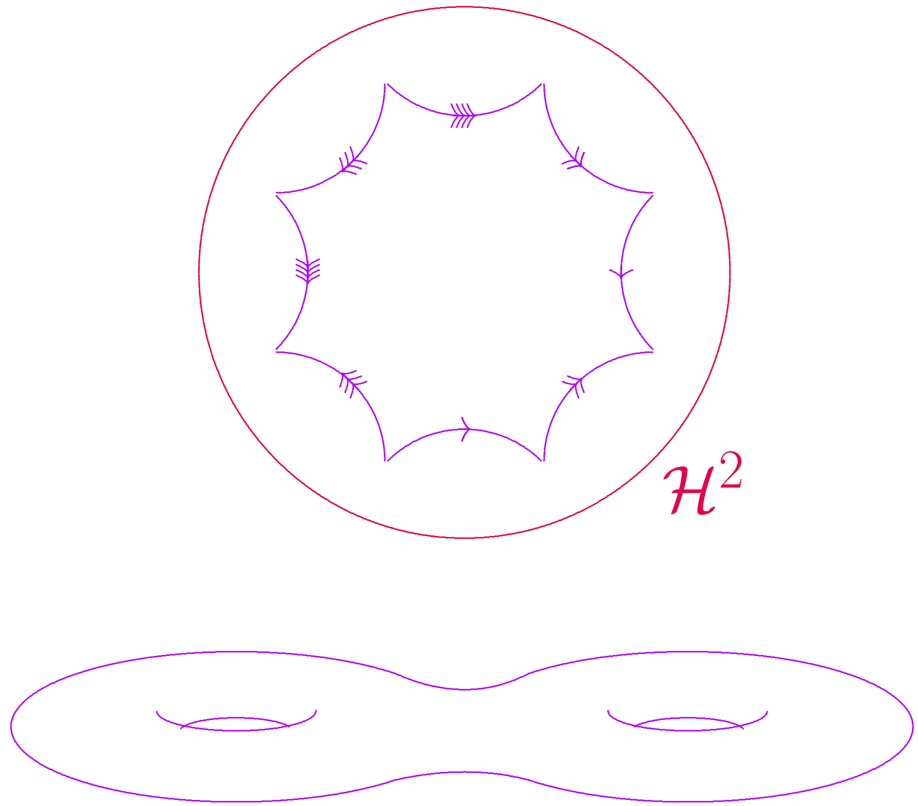
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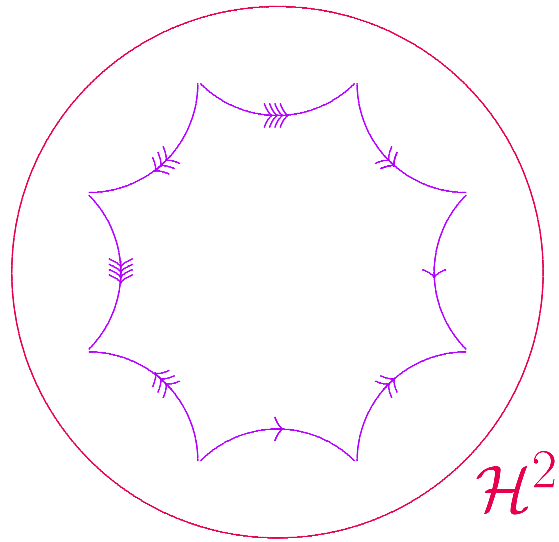

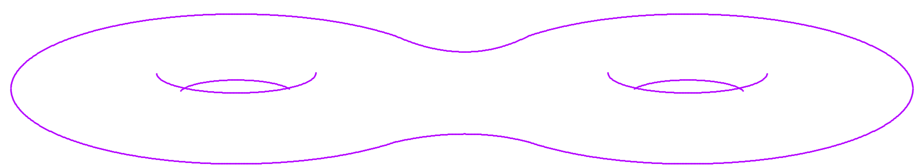

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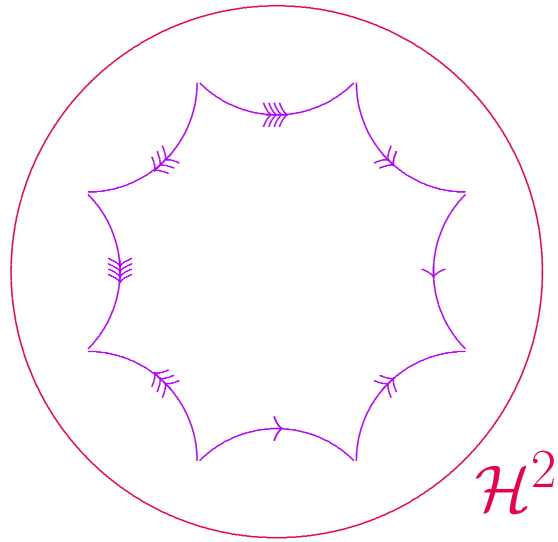

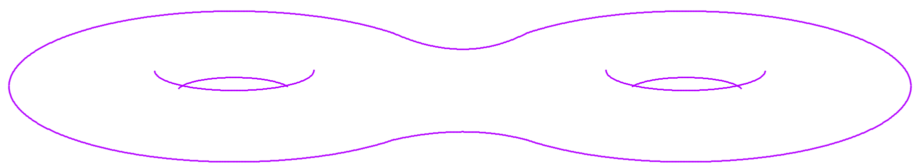

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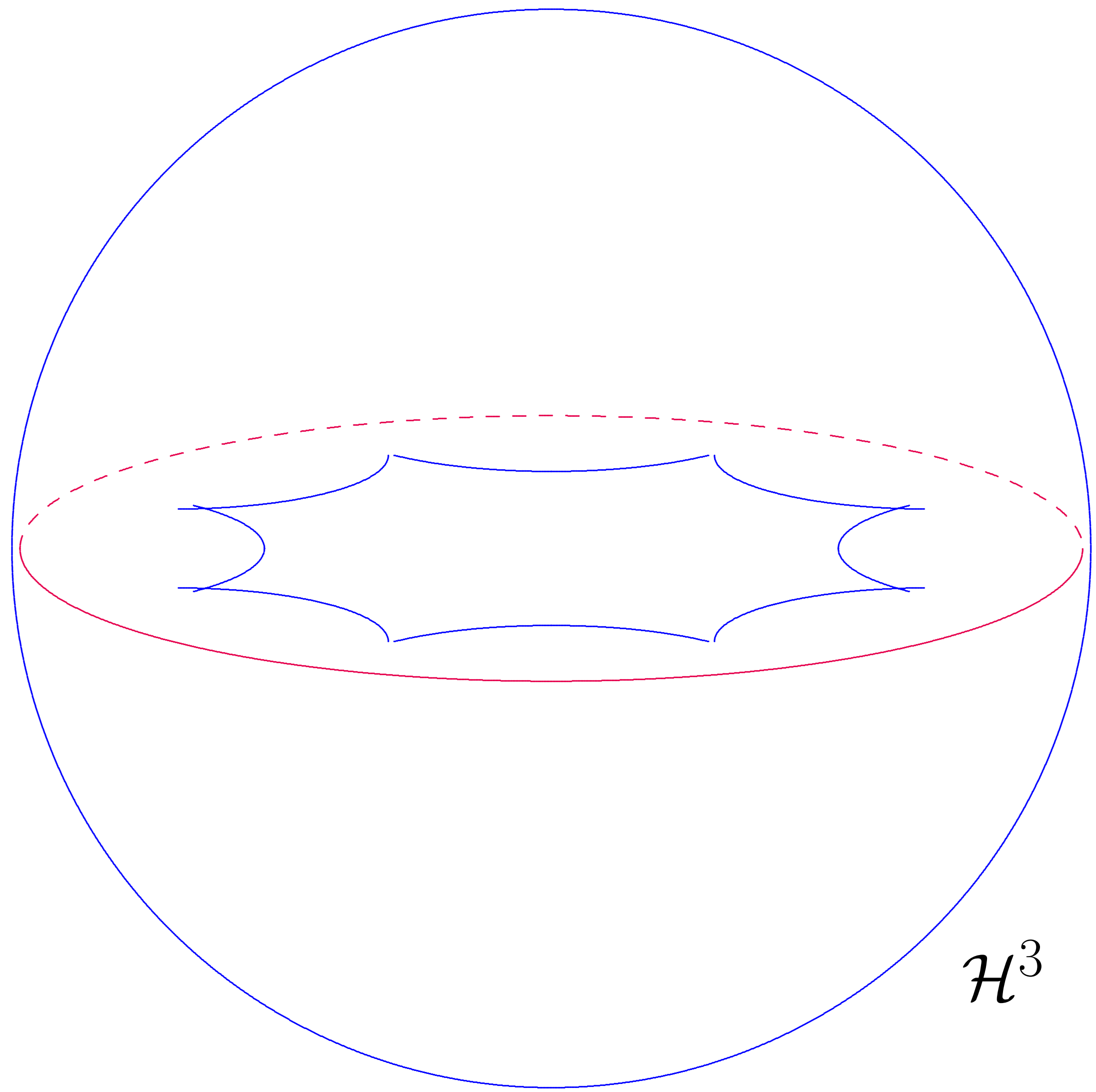
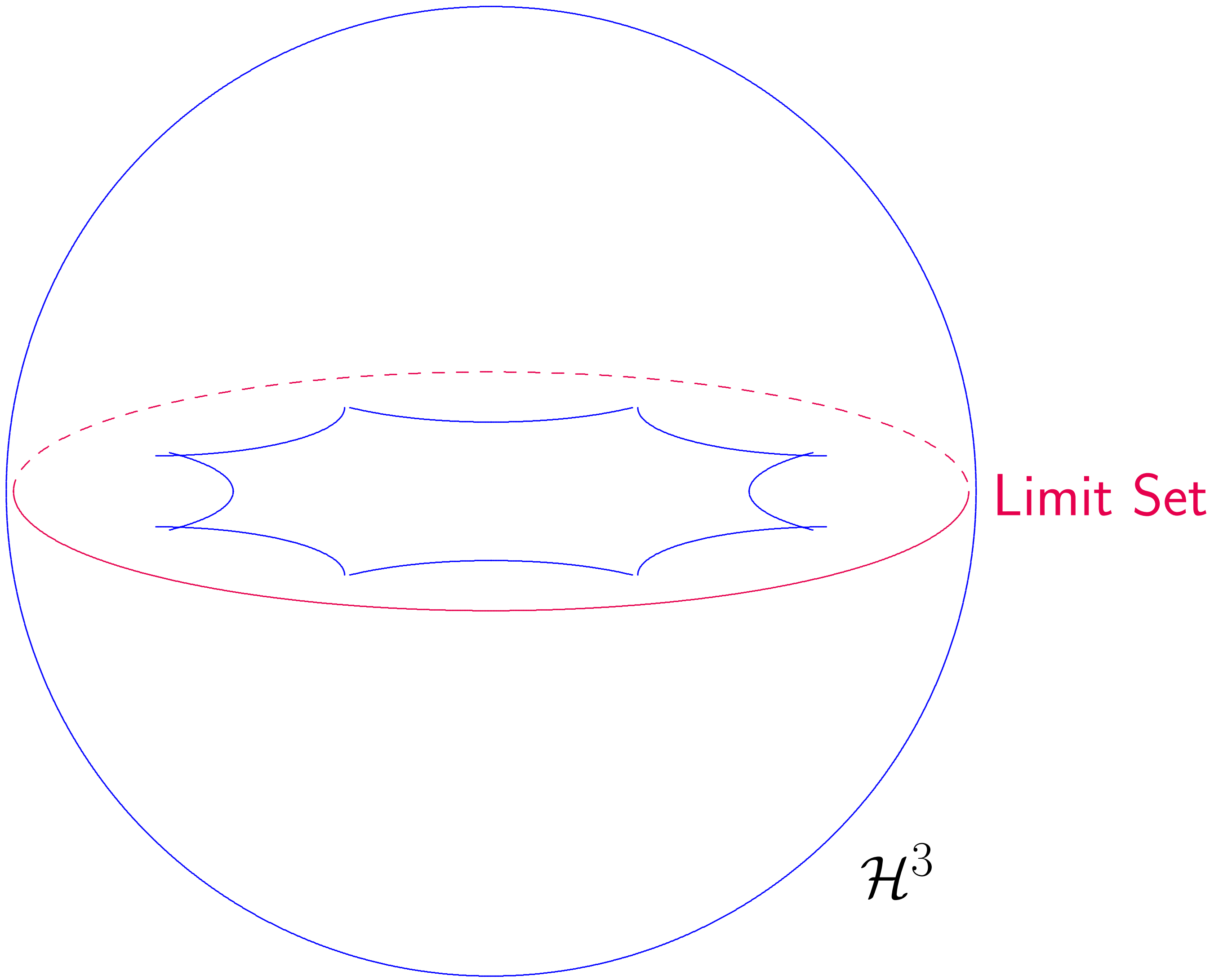
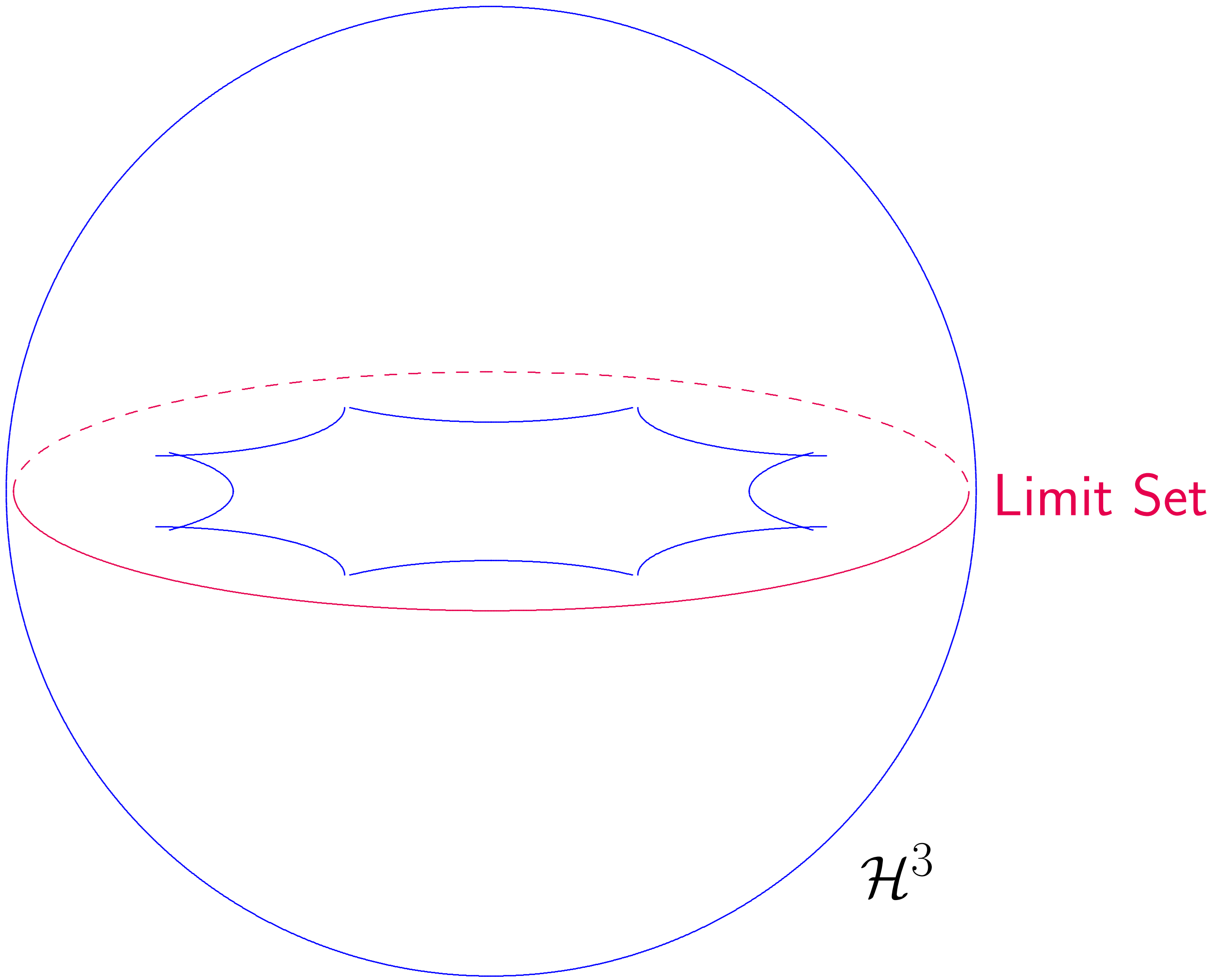
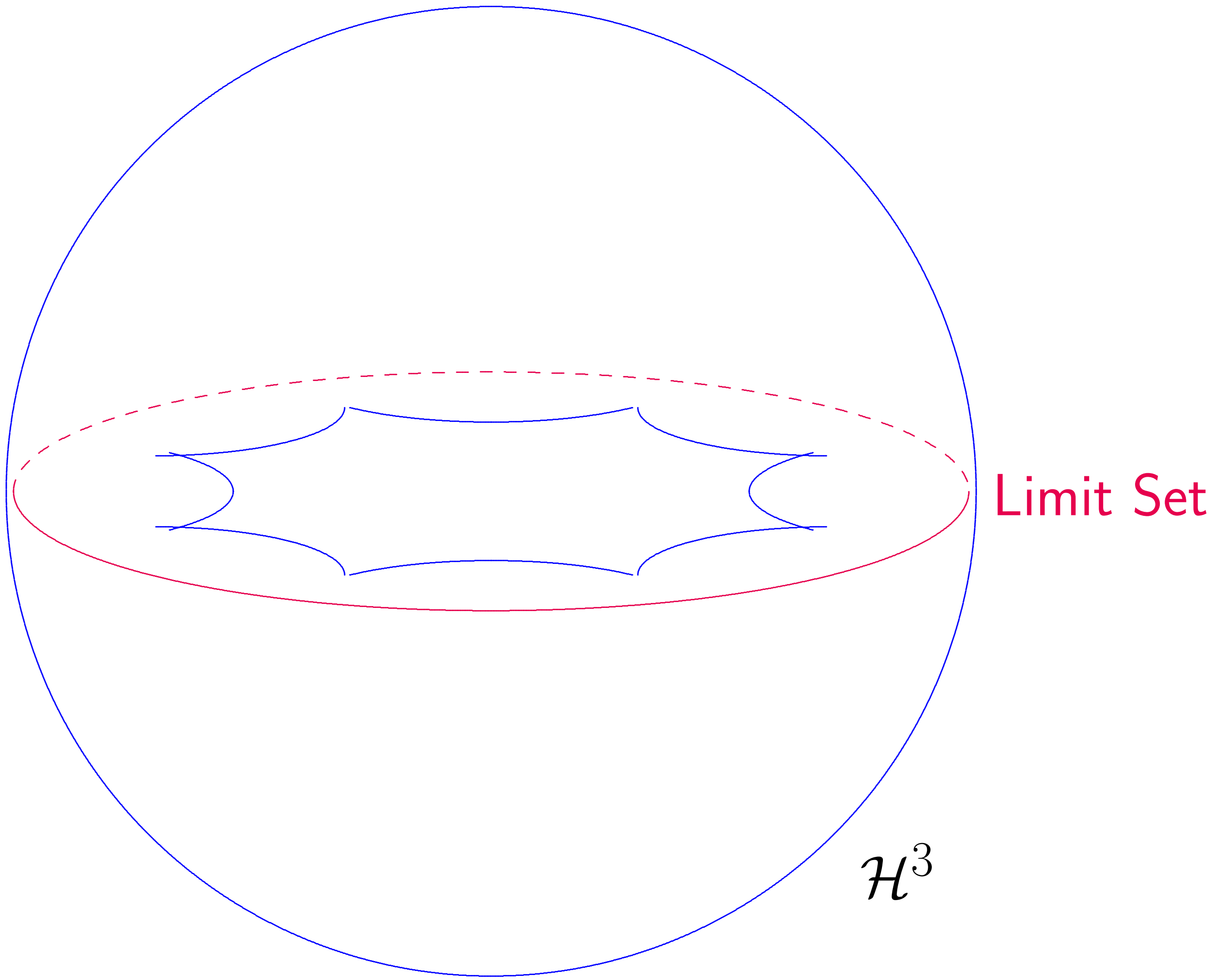

$\mathcal{H}^3$

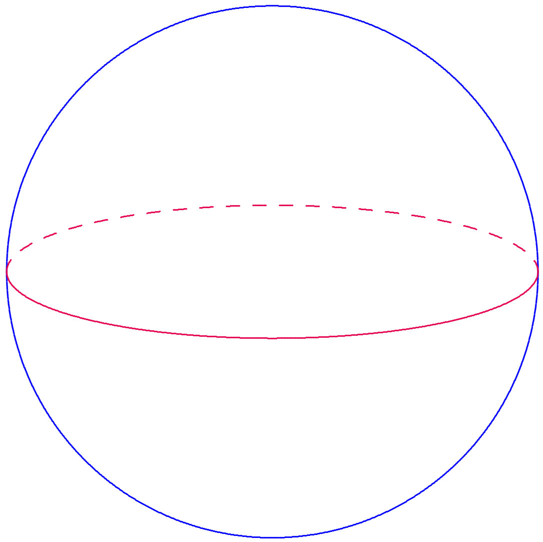

Limit Set
$\mathcal{H}^3$

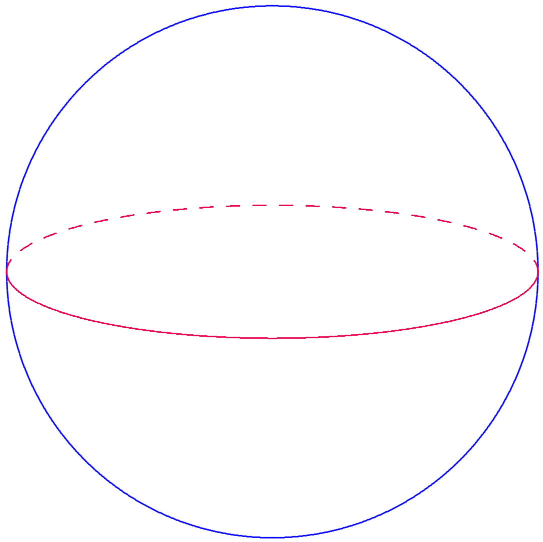


$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2,\mathbb{R})$ Fuchsian group

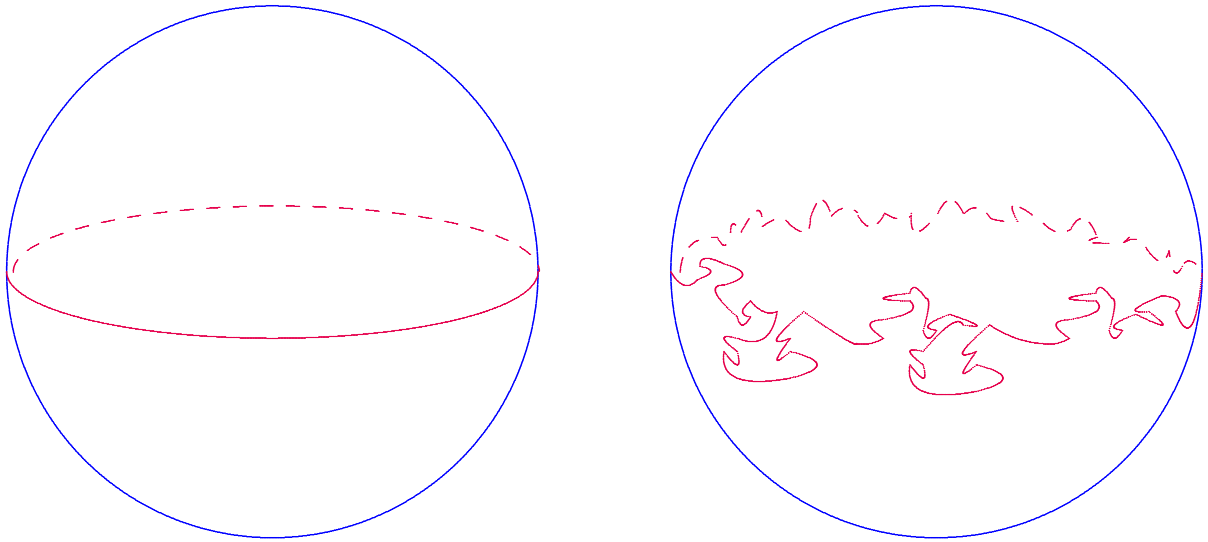


$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2, \mathbb{C})$ Fuchsian group

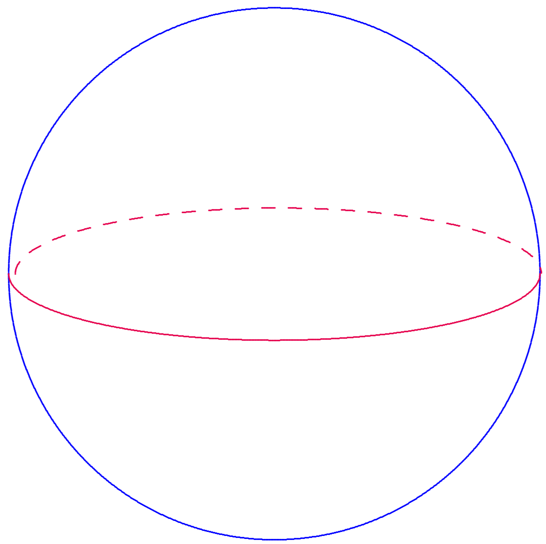




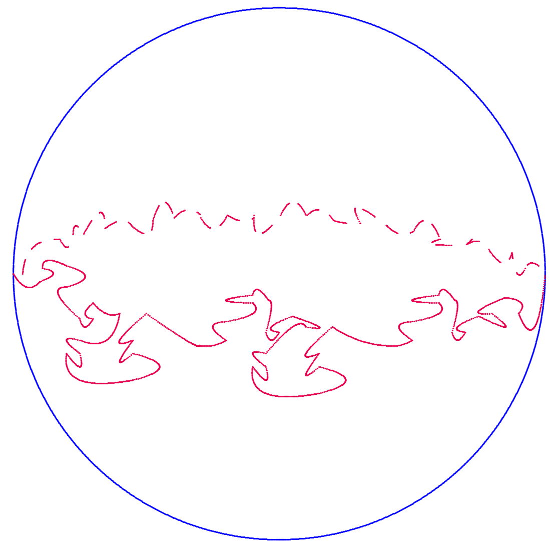
Fuchsian



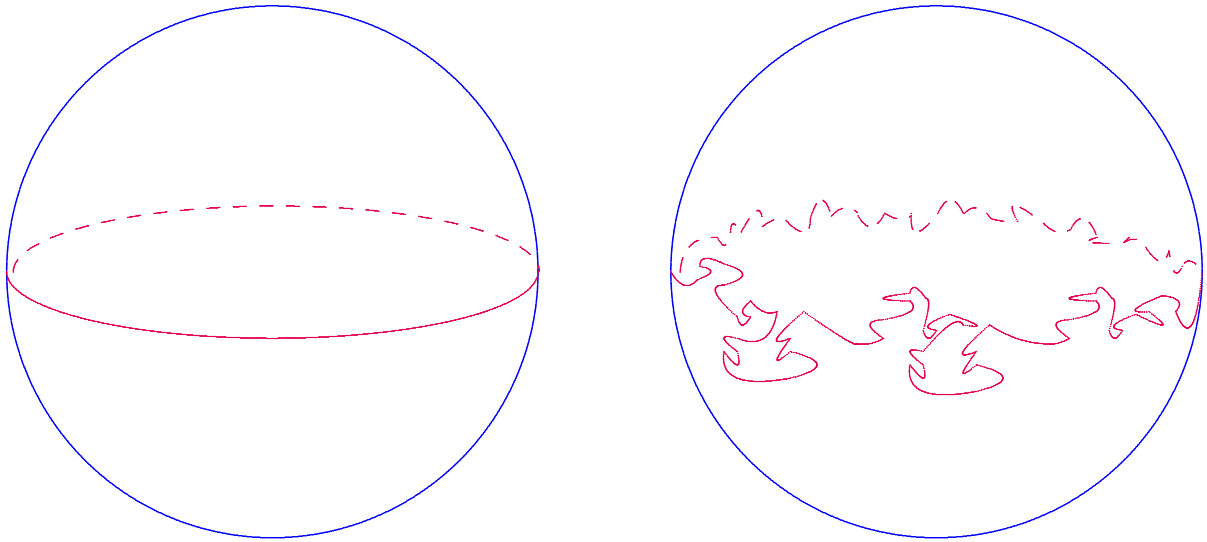
Fuchsian



Fuchsian

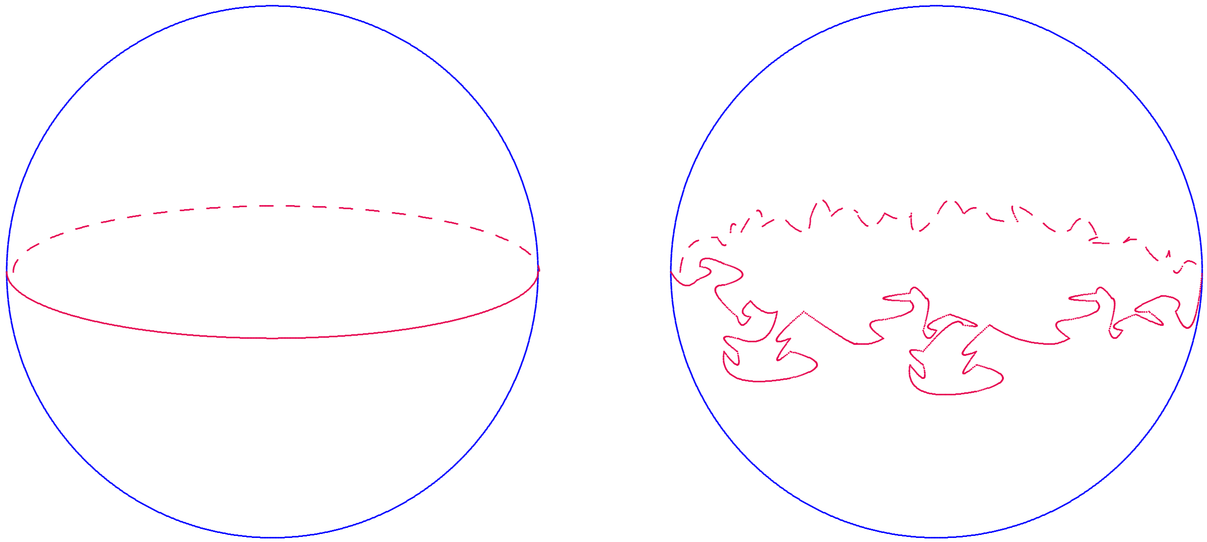


quasi-Fuchsian



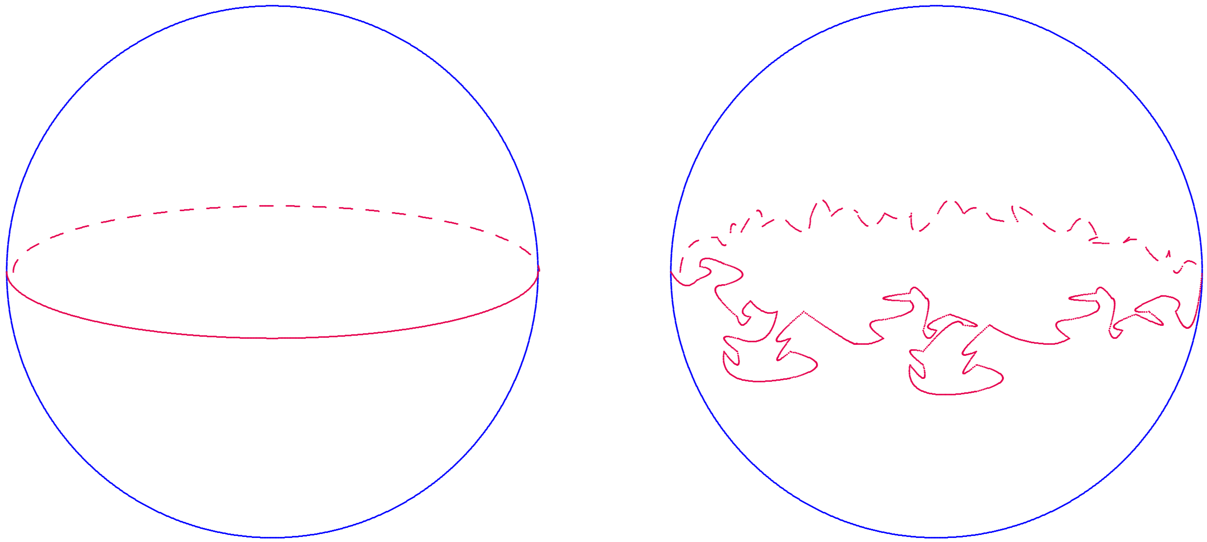
Fuchsian quasi-Fuchsian

$$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2,\mathbb{C})$$ quasi-Fuchsian group



Fuchsian quasi-Fuchsian

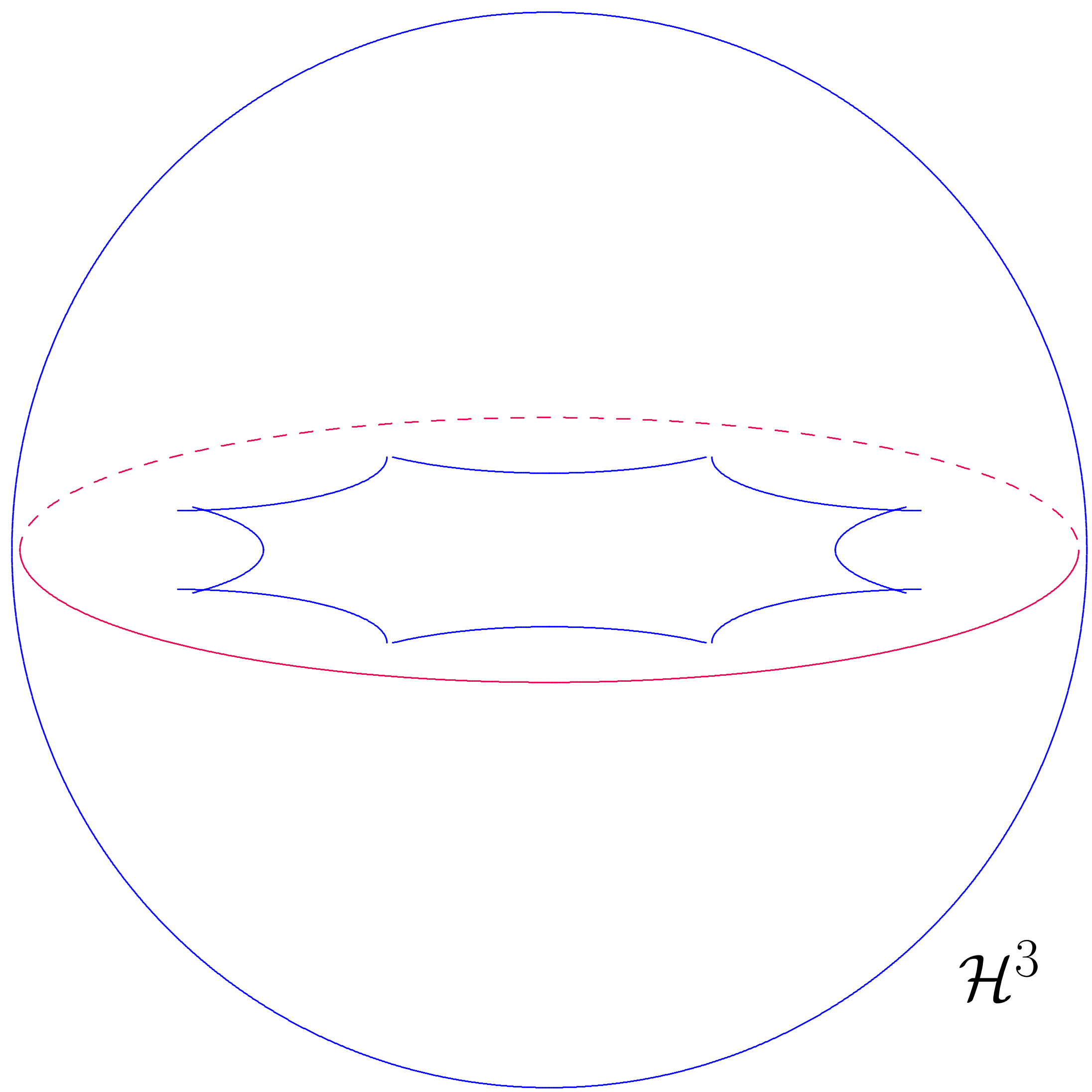
$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2, \mathbb{C})$ quasi-Fuchsian group of Bers type

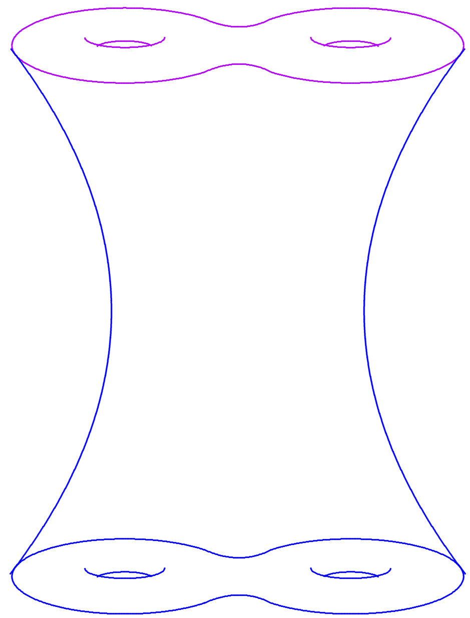


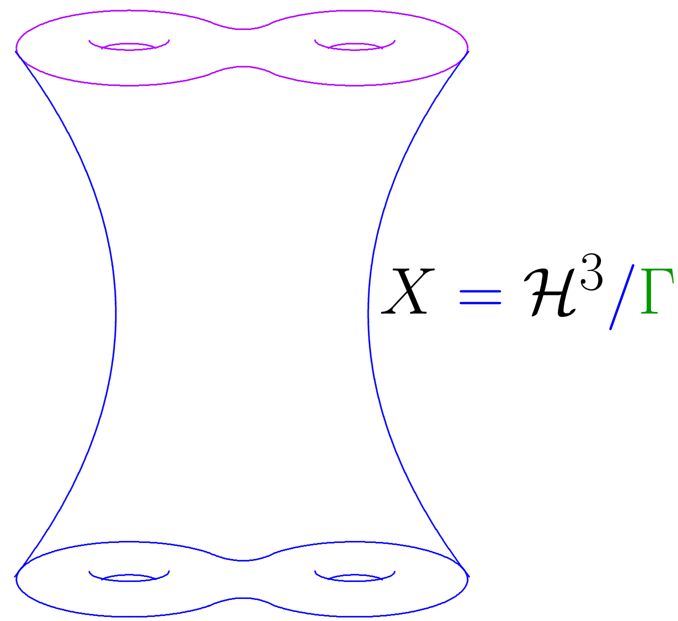
Fuchsian quasi-Fuchsian

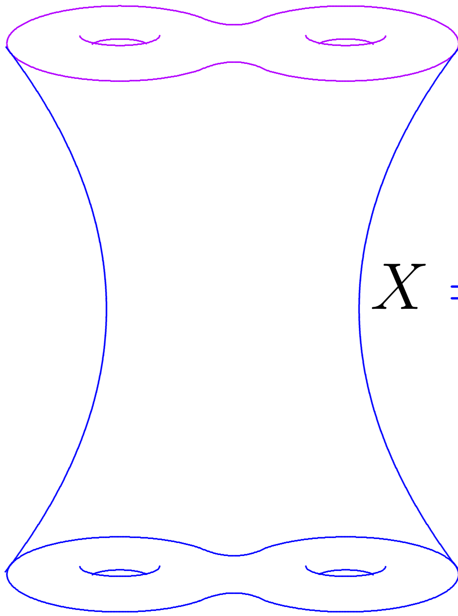
$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2, \mathbb{C})$ quasi-Fuchsian group of Bers type

Quasi-conformally conjugate to Fuchsian.


$\mathcal{H}^3$

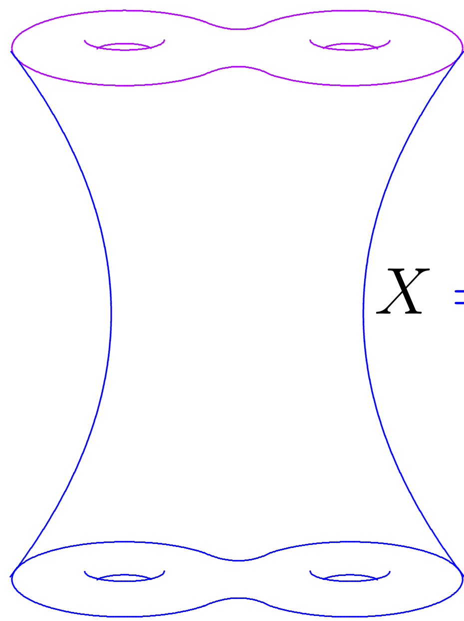



$X = \mathcal{H}^3/\Gamma$



$X = \mathcal{H}^3/\Gamma$

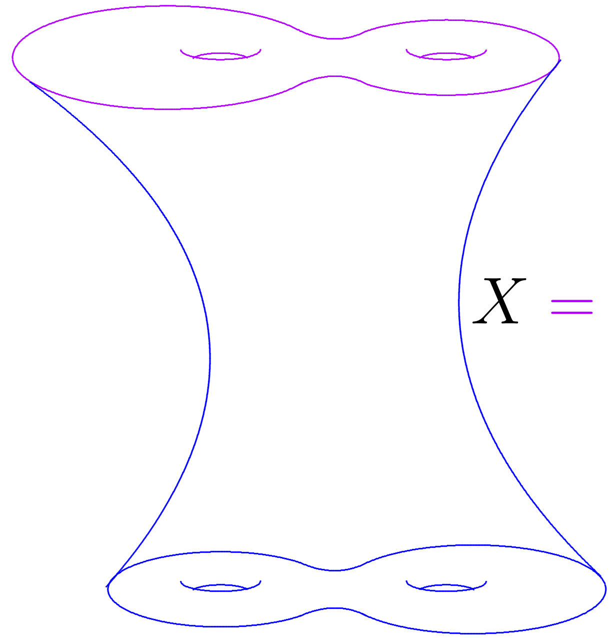
Γ Fuchsian



$X = \mathcal{H}^3/\Gamma$

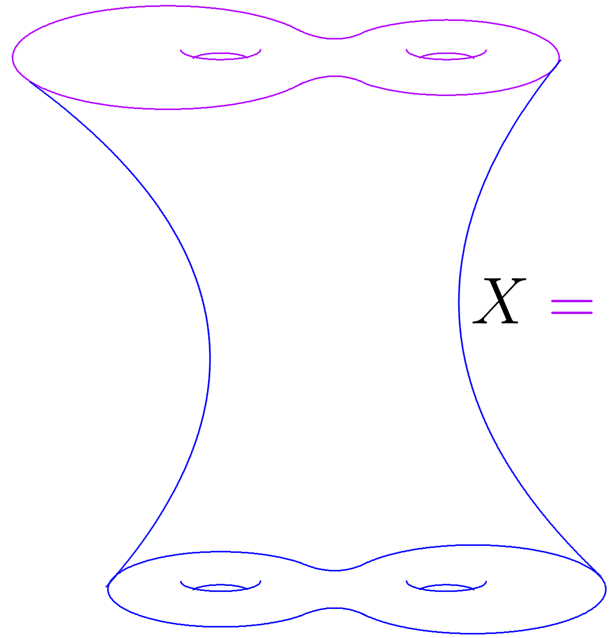
Γ Fuchsian

$X \approx \Sigma \times \mathbb{R}$



Γ quasi-Fuchsian

$X \approx \Sigma \times \mathbb{R}$

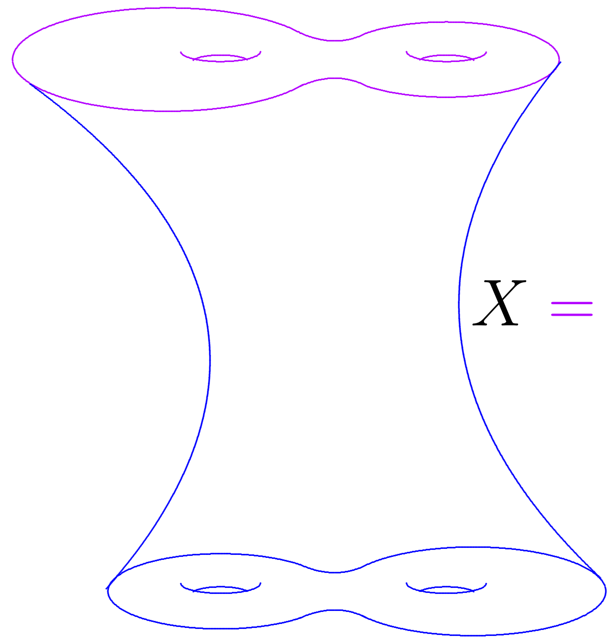


$X = \mathcal{H}^3/\Gamma$

Γ quasi-Fuchsian

$X \approx \Sigma \times \mathbb{R}$

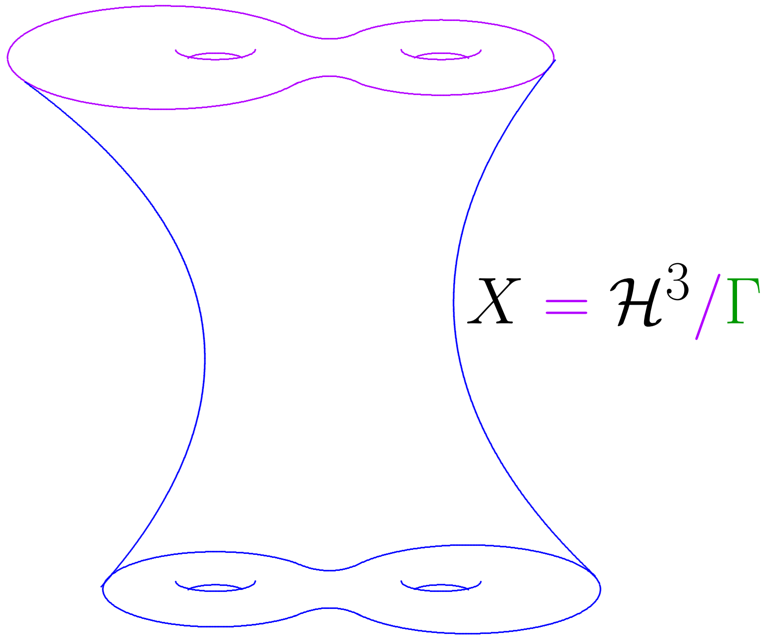
Freedom: two points in Teichmüller space.



$X = \mathcal{H}^3/\Gamma$

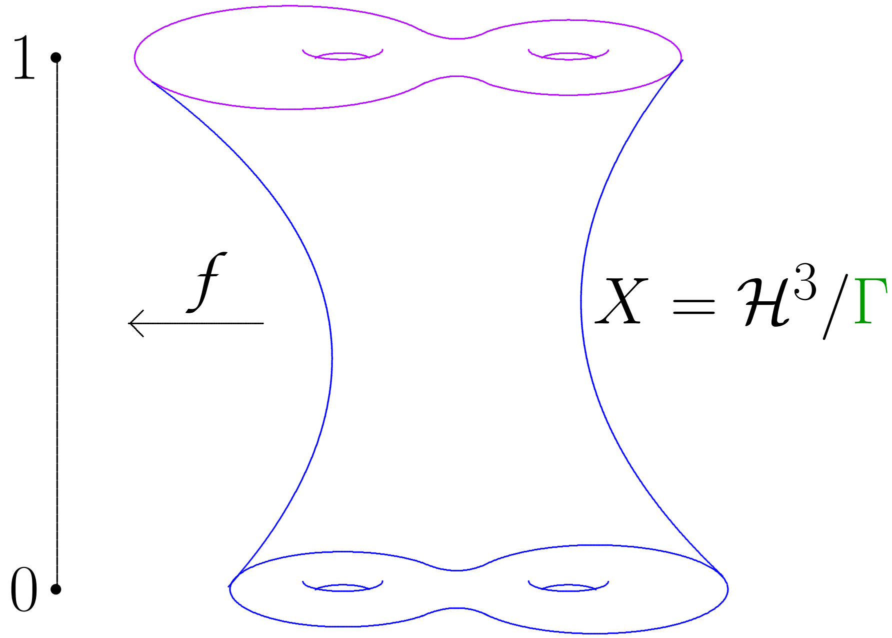
Γ quasi-Fuchsian

$X \approx \Sigma \times \mathbb{R}$



Γ quasi-Fuchsian

$$\overline{X} \approx \Sigma \times [0,1]$$

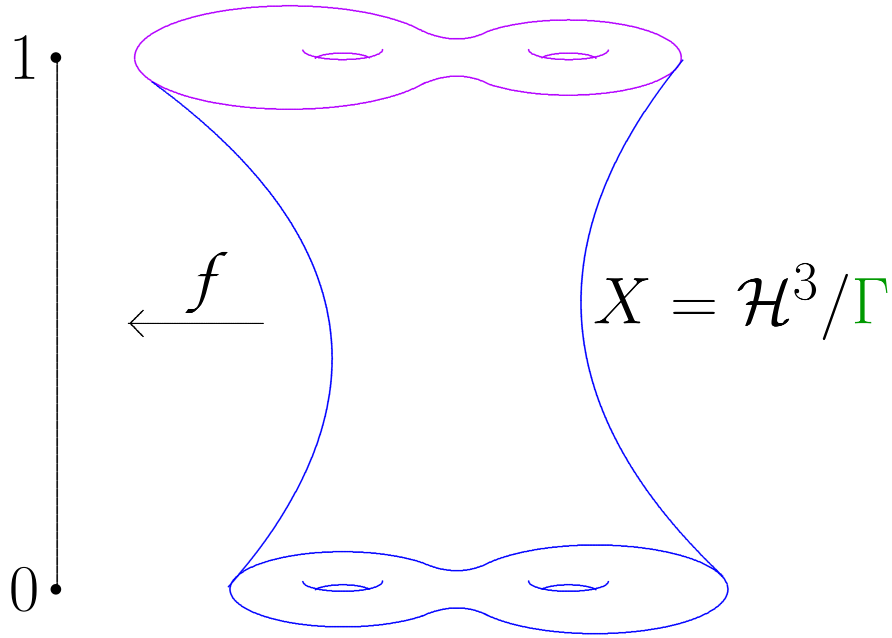


Γ quasi-Fuchsian

$$\overline{X} \approx \Sigma \times [0,1]$$

Tunnel-Vision function:

$$f : \overline{X} \to [0,1]$$



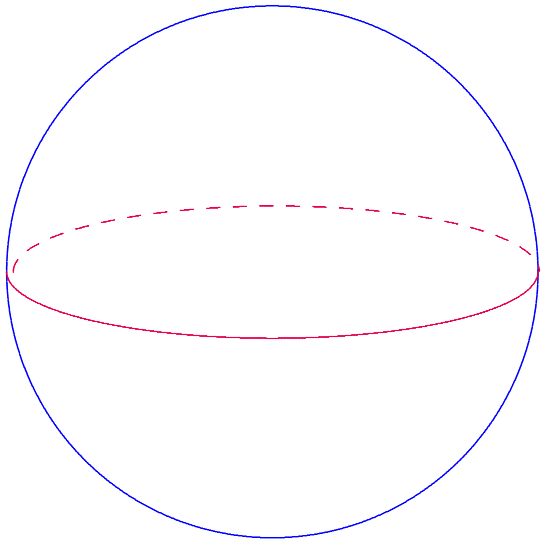
Γ quasi-Fuchsian

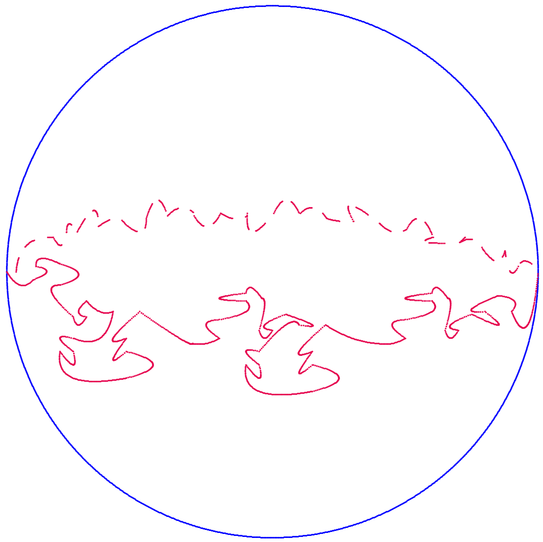
$$\overline{X} \approx \Sigma \times [0,1]$$

Tunnel-Vision function:

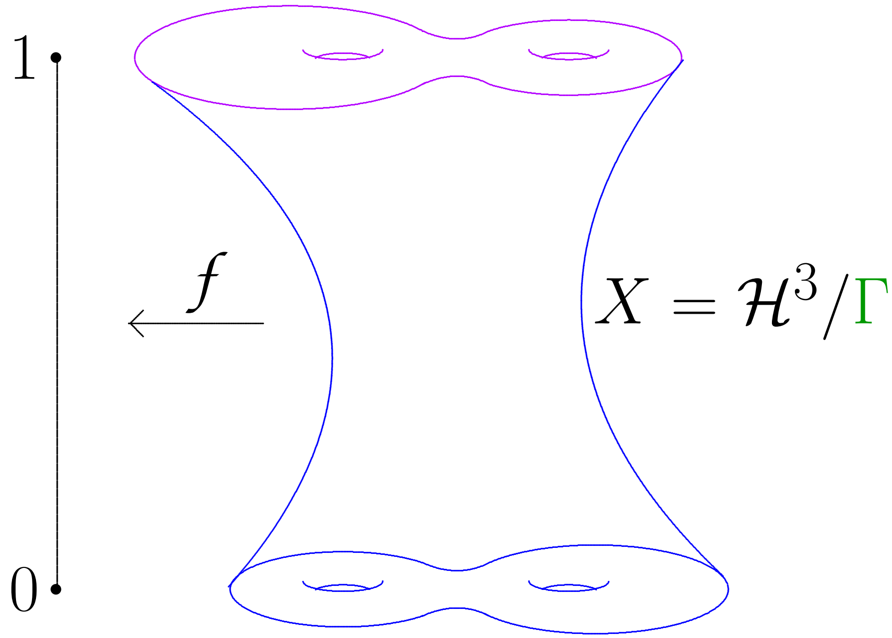
$$f : \overline{X} \to [0,1]$$

$$\Delta f = 0$$





Fuchsian quasi-Fuchsian



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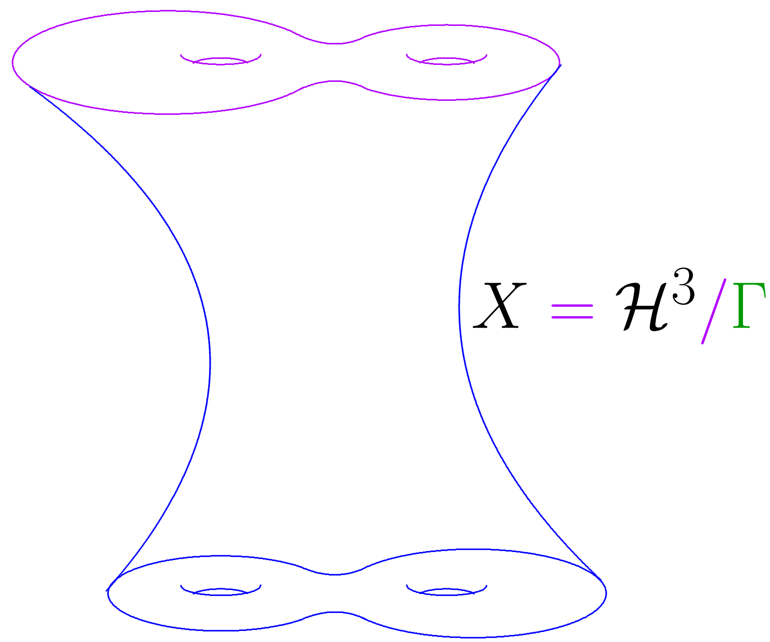
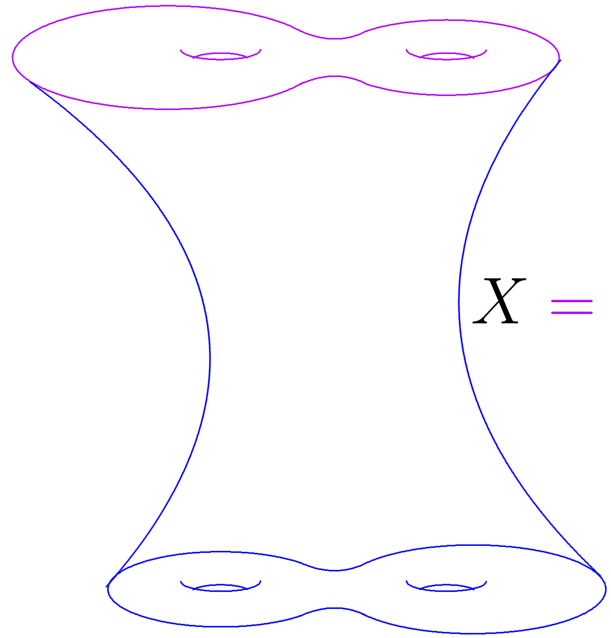
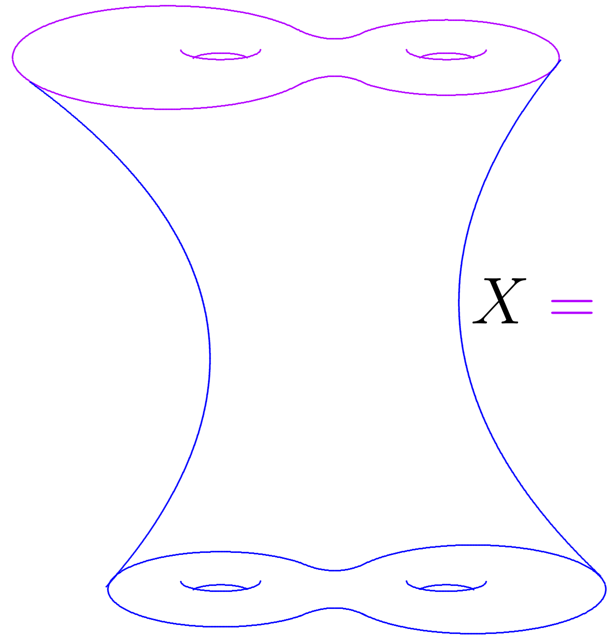

$X = \mathcal{H}^3/\Gamma$



$X = \mathcal{H}^3/\Gamma$

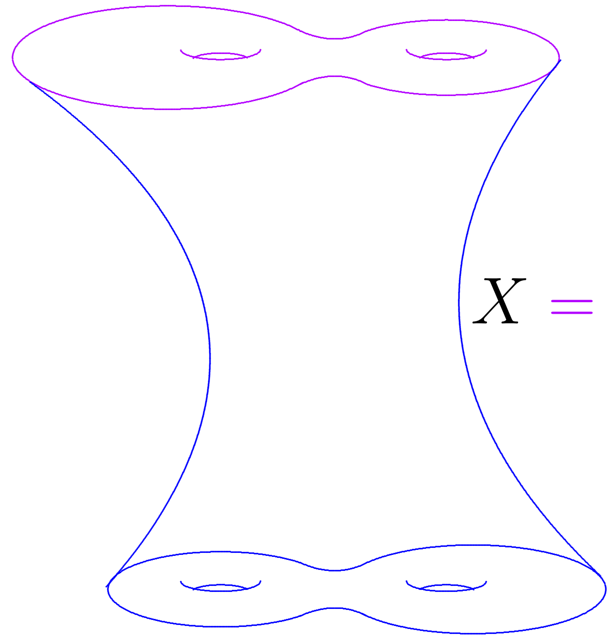
Construction of conformally flat 4-manifolds:



$$X = \mathcal{H}^3/\Gamma$$

Construction of conformally flat 4-manifolds:

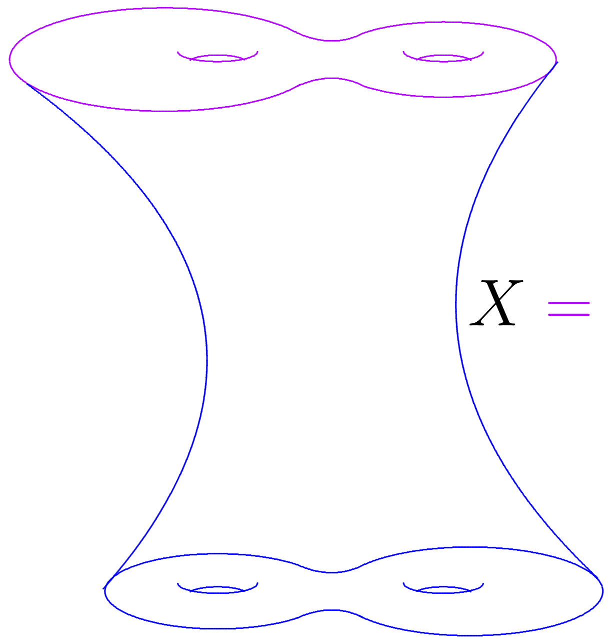
$$M = [\overline{X} \times S^1]/\sim$$



Construction of conformally flat 4-manifolds:

$$M = [\overline{X} \times S^1]/\sim$$

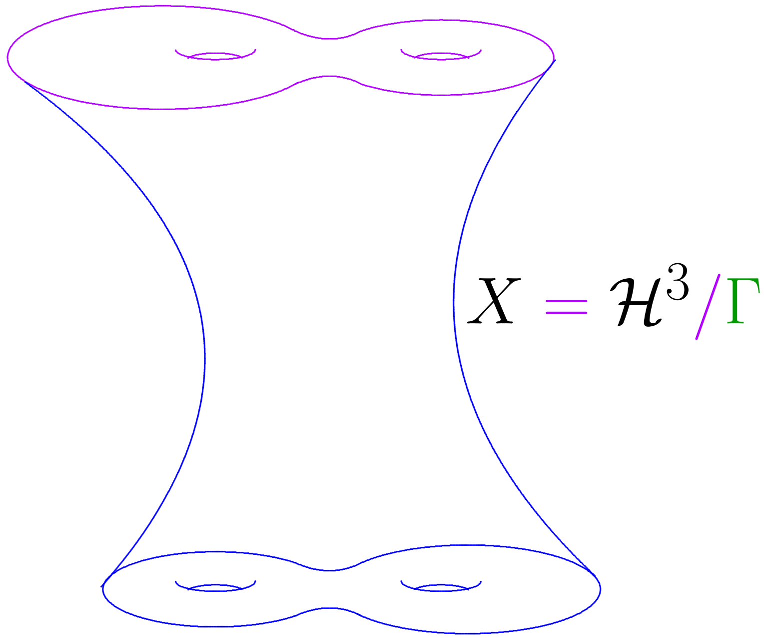
$\sim$: crush $\partial\overline{X} \times S^1$ to $\partial\overline{X}$.



$X = \mathcal{H}^3/\Gamma$

Construction of conformally flat 4-manifolds:

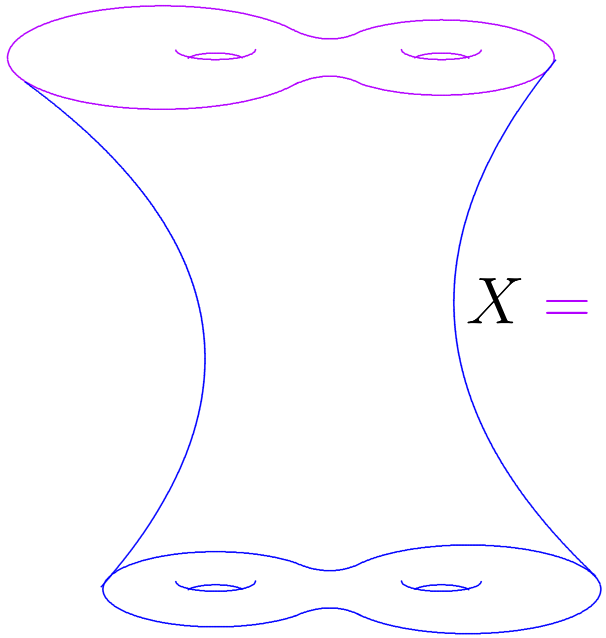
$$M = [\overline{X} \times S^1]/\sim$$



Construction of conformally flat 4-manifolds:

$$M = [\overline{X} \times S^1]/\sim$$

$$g = \qquad\qquad h + dt^2$$

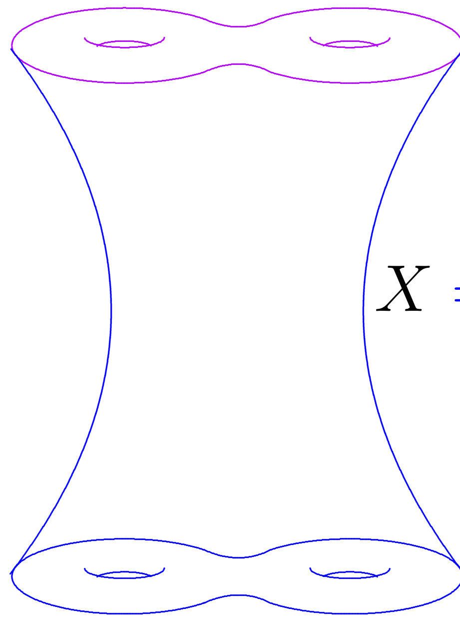


$X = \mathcal{H}^3/\Gamma$

Construction of conformally flat 4-manifolds:

$$M = [\overline{X} \times S^1]/\sim$$

$$g = f(1-f)[h + dt^2]$$



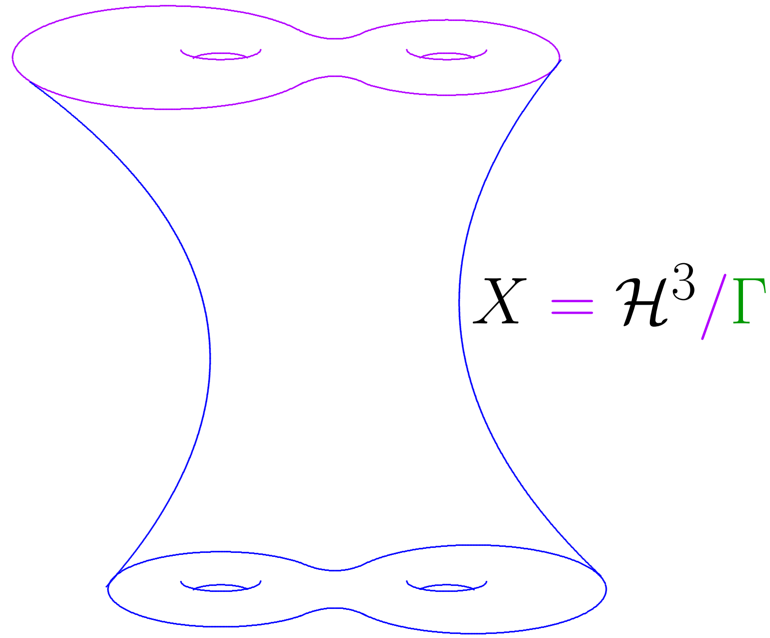
$X = \mathcal{H}^3/\Gamma$

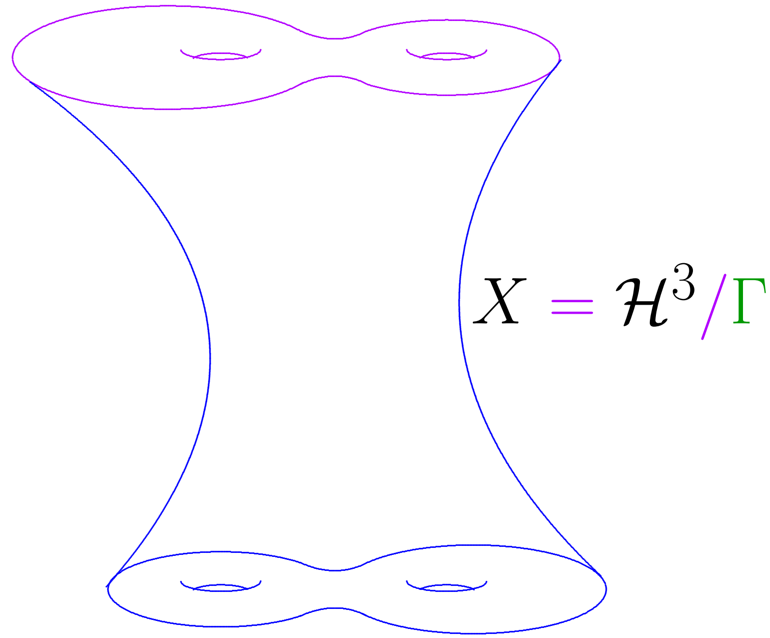
Construction of conformally flat 4-manifolds:

$$M = [\overline{X} \times S^1]/\sim$$

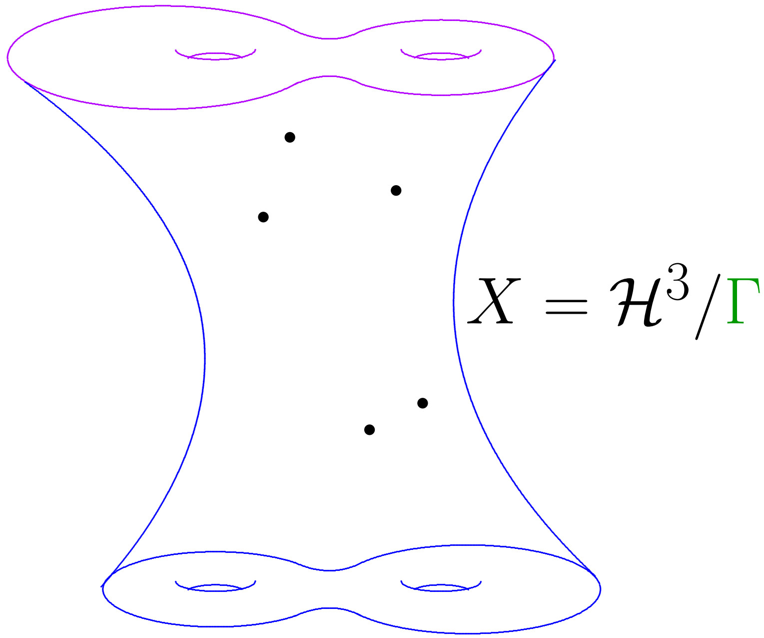
$$g = f(1-f)[h + dt^2]$$

Fuchsian case: $\Sigma \times S^2$ scalar-flat Kähler.


$X = \mathcal{H}^3/\Gamma$

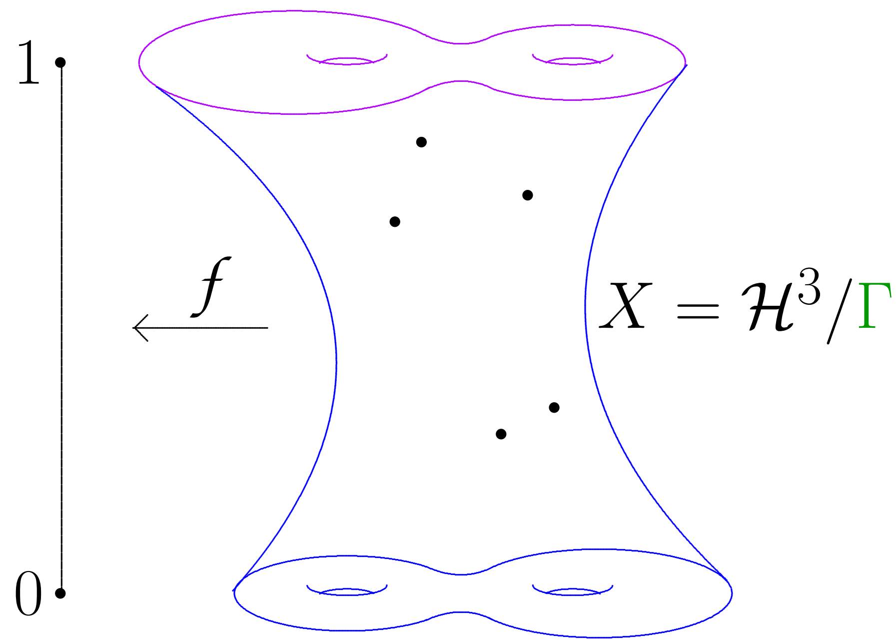


Construction of ASD 4-manifolds:



Construction of ASD 4-manifolds:

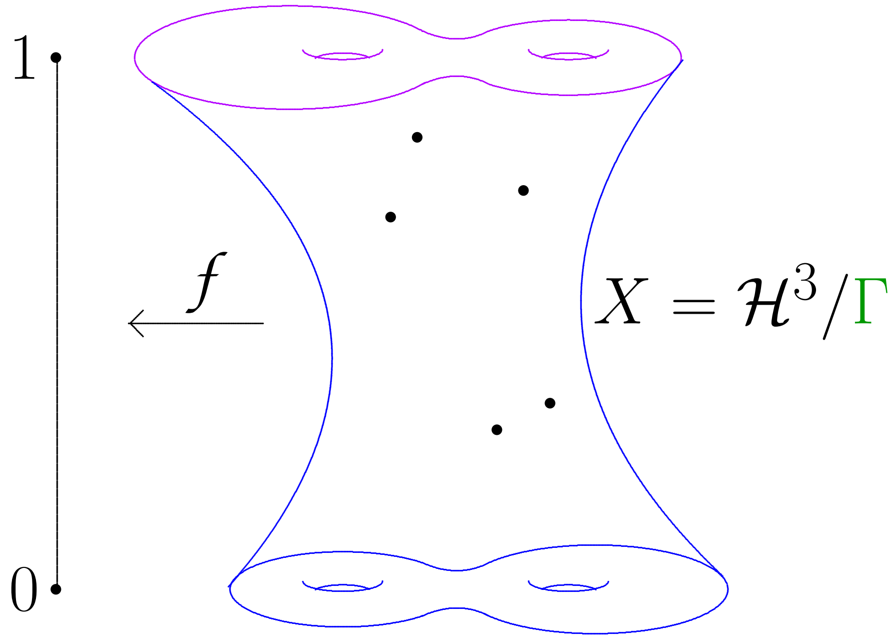
Choose k points $p_1, \ldots, p_k \in X$



Construction of ASD 4-manifolds:

Choose k points $p_1, \ldots, p_k \in X$

satisfying $\sum_{j=1}^{k} f(p_j) \in \mathbb{Z}$.

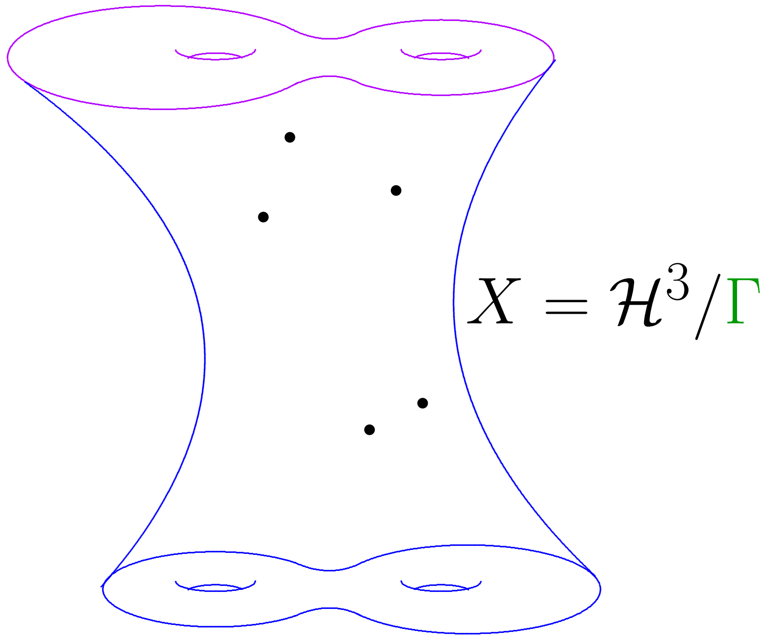


Construction of ASD 4-manifolds:

Choose k points $p_1, \ldots, p_k \in X$

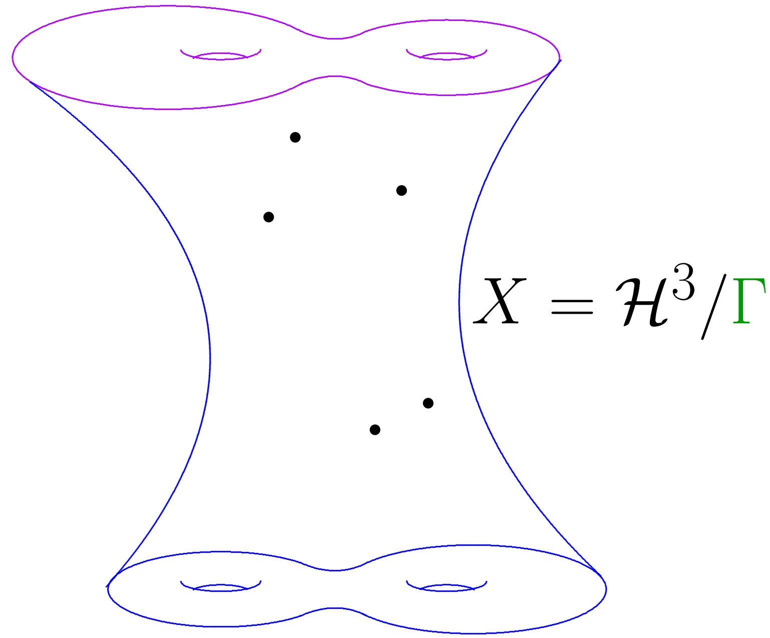
satisfying $\sum_{j=1}^{k} f(p_j) \in \mathbb{Z}$.

Can do if $k \neq 1$.



Construction of ASD 4-manifolds:

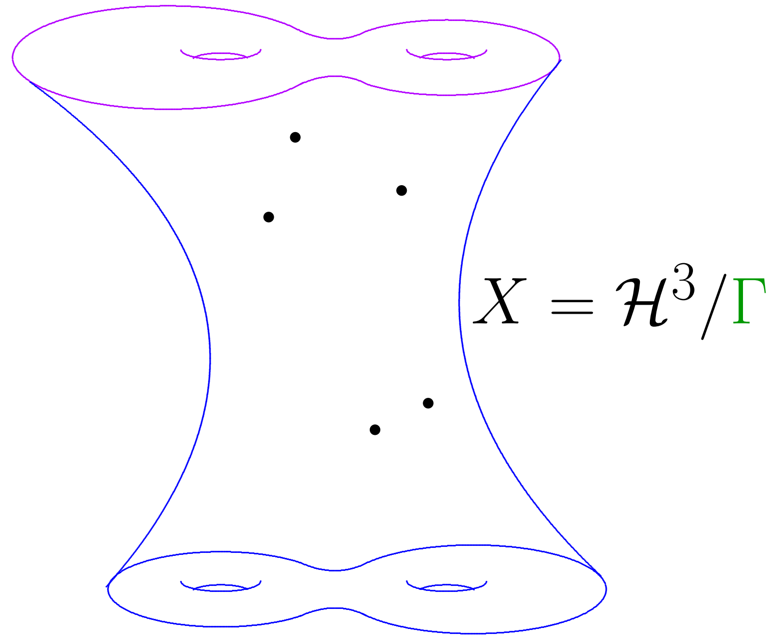
Let G_j be the Green's function of p_j:



Construction of ASD 4-manifolds:

Let G_j be the Green's function of p_j:

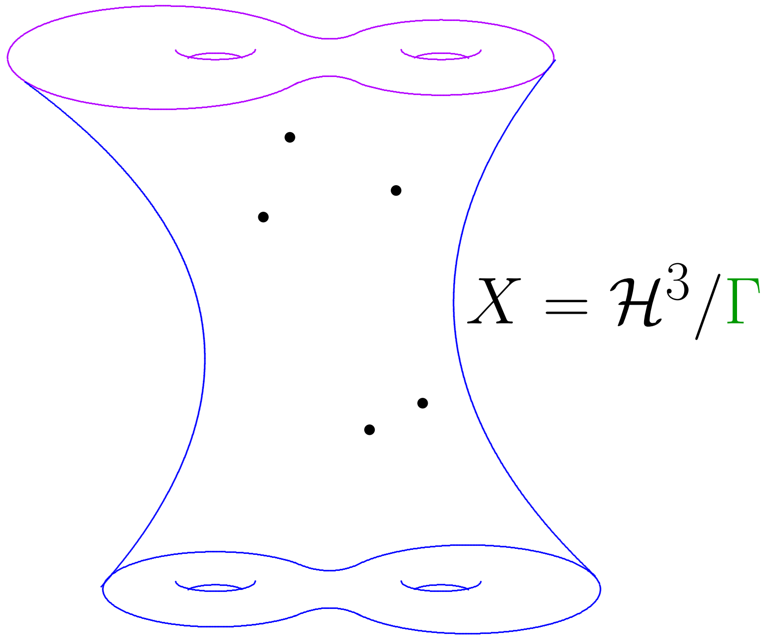
$$\Delta G_j = 2\pi\delta_{p_j}, \qquad G_j \to 0 \text{ at } \partial\overline{X}$$



Construction of ASD 4-manifolds:

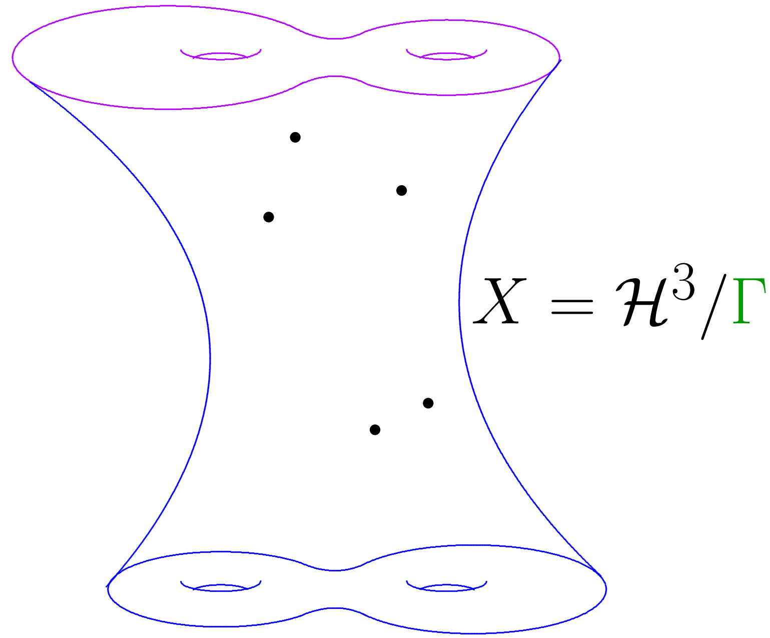
Let G_j be the Green's function of p_j, and set

$$V = 1 + \sum_{j=1}^{k} G_j .$$



Construction of ASD 4-manifolds:

$$V = 1 + \sum_{j=1}^{k} G_j.$$

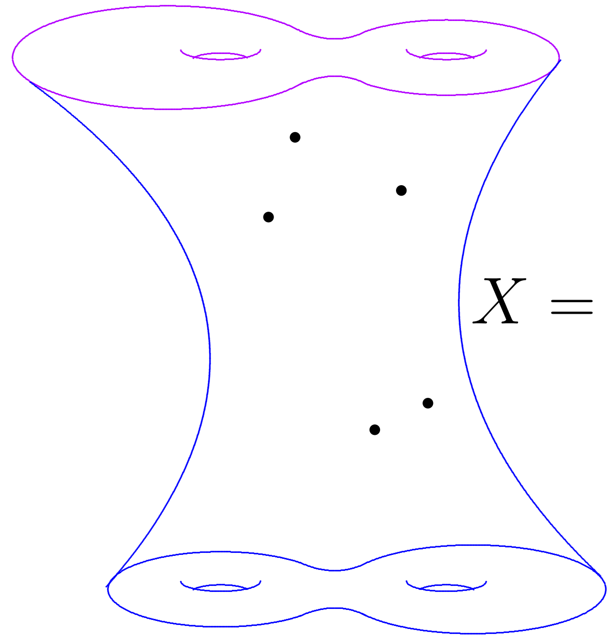


Construction of ASD 4-manifolds:

$$V = 1 + \sum_{j=1}^{k} G_j .$$

Choose $P \to (X - \{p_1, \ldots, p_k\})$ circle bundle with connection form θ such that

$$d\theta = \star dV .$$

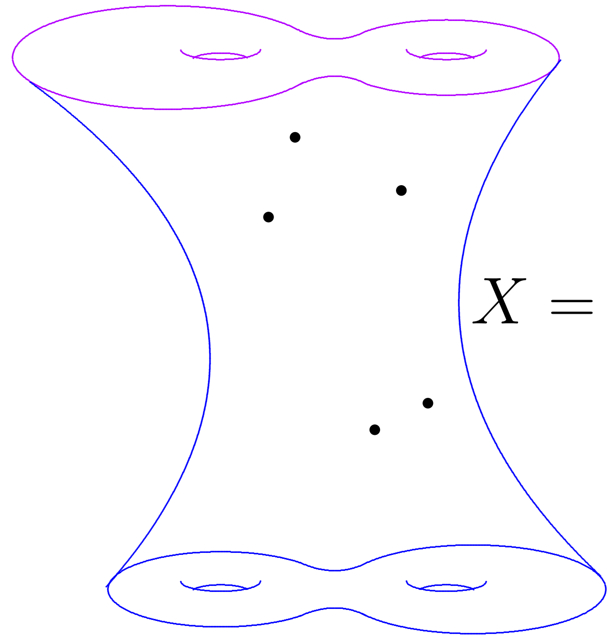


Construction of ASD 4-manifolds:

$$g = \quad V h + V^{-1}\theta^2$$

$$V = 1 + \sum_{j=1}^{k} G_j$$

$$d\theta = \star dV$$

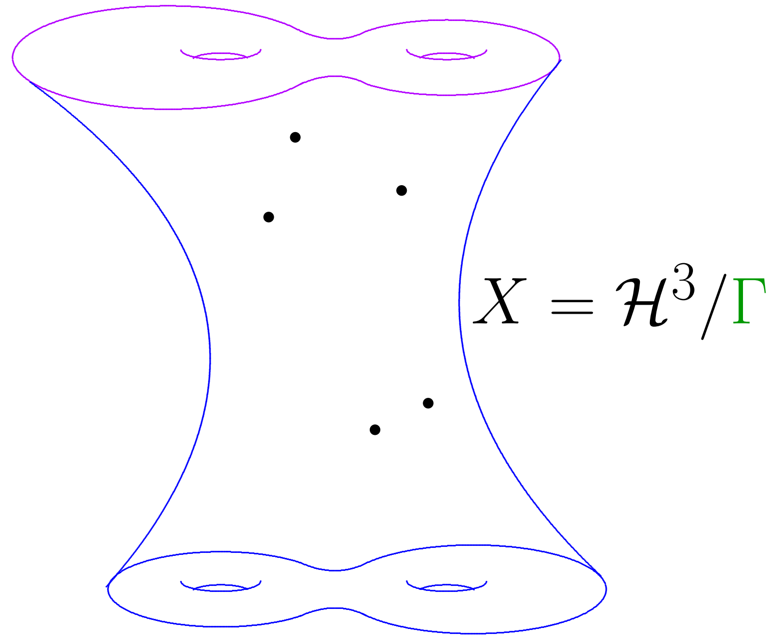


Construction of ASD 4-manifolds:

$$g = f(1-f)[Vh + V^{-1}\theta^2]$$

$$V = 1 + \sum_{j=1}^{k} G_j$$

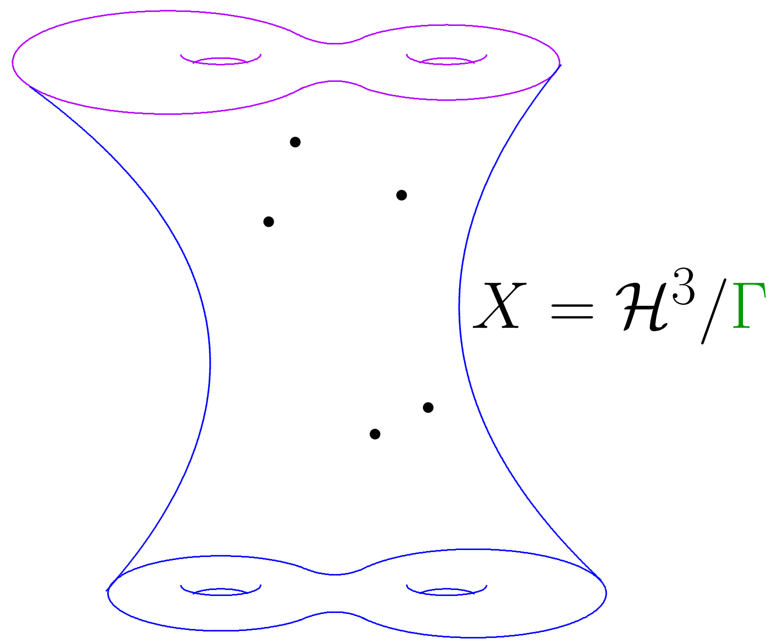
$$d\theta = \star dV$$



Construction of ASD 4-manifolds:

$$g = f(1-f)[Vh + V^{-1}\theta^2]$$

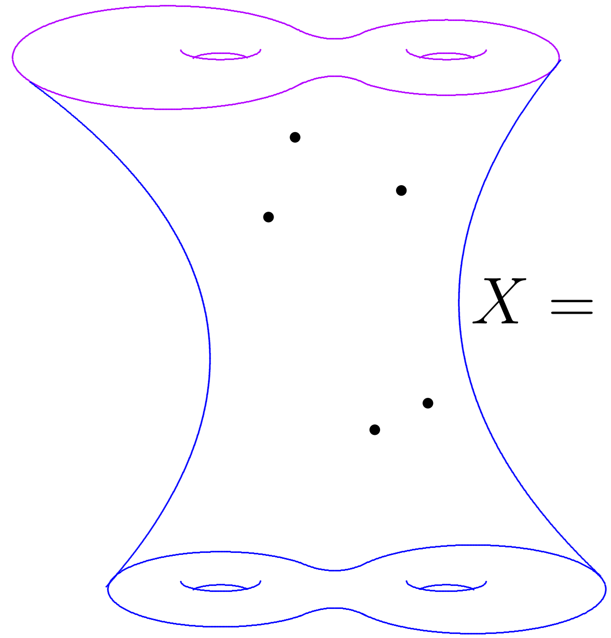
$$M = P \cup \{\hat{p}_1, \ldots, \hat{p}_k\} \cup \partial\overline{X}$$



Construction of ASD 4-manifolds:

$$g = f(1-f)[Vh + V^{-1}\theta^2]$$

$$\begin{array}{ccccccc}
M & = & P & \cup & \{\hat{p}_1, \ldots, \hat{p}_k\} & \cup & \partial\overline{X} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\overline{X} & = & X - \{p_1, \ldots, p_k\} & \cup & \{p_1, \ldots, p_k\} & \cup & \partial\overline{X}
\end{array}$$



$X = \mathcal{H}^3/\Gamma$

Construction of ASD 4-manifolds:

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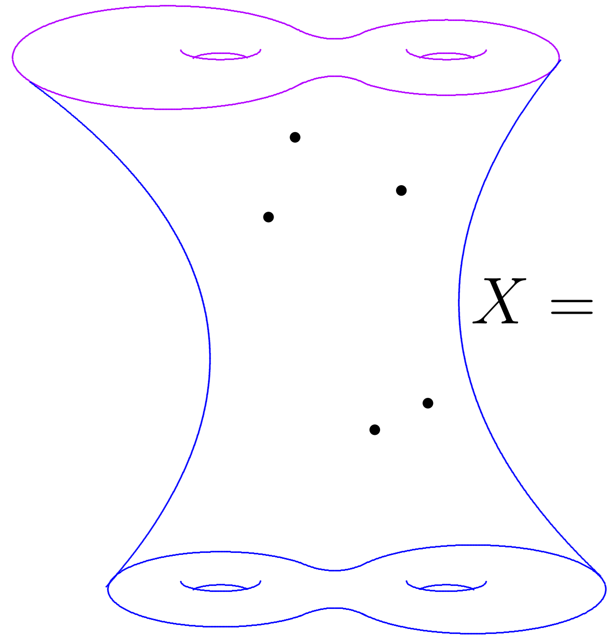

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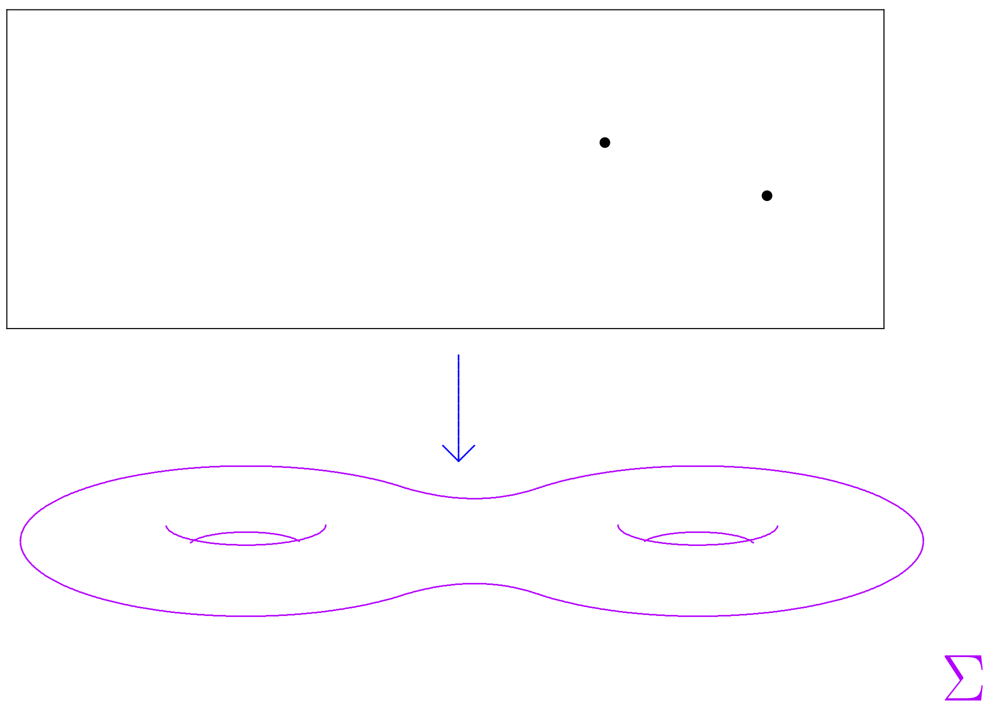
$$\approx (\Sigma \times S^2) \# k \overline{\mathbb{CP}}_2$$

$$\overline{X} = \Sigma \times [0,1]$$



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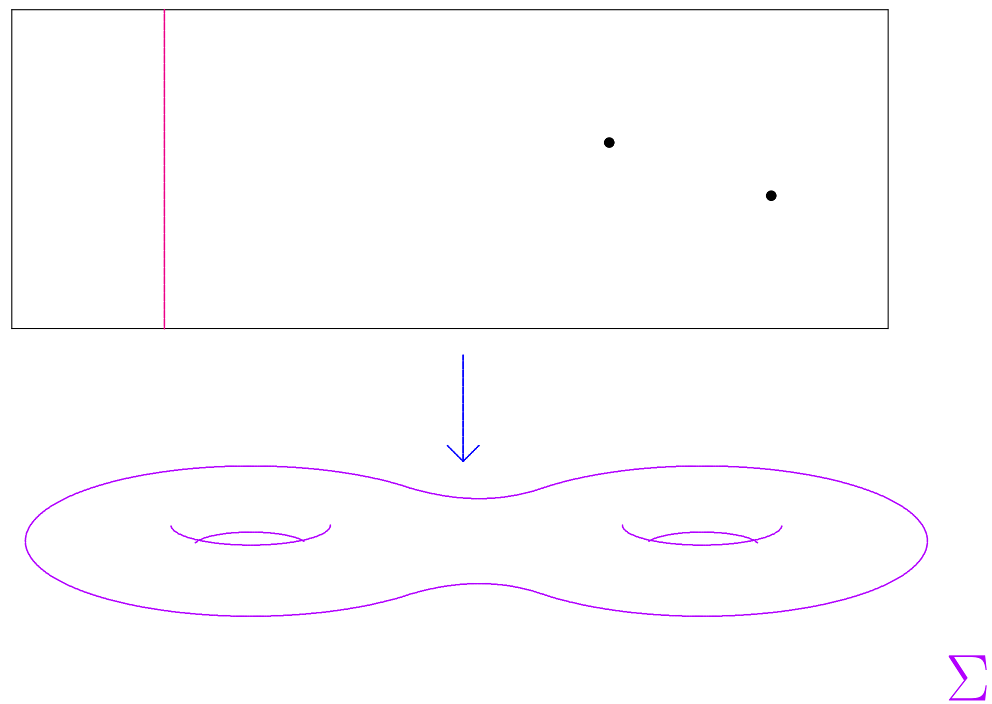
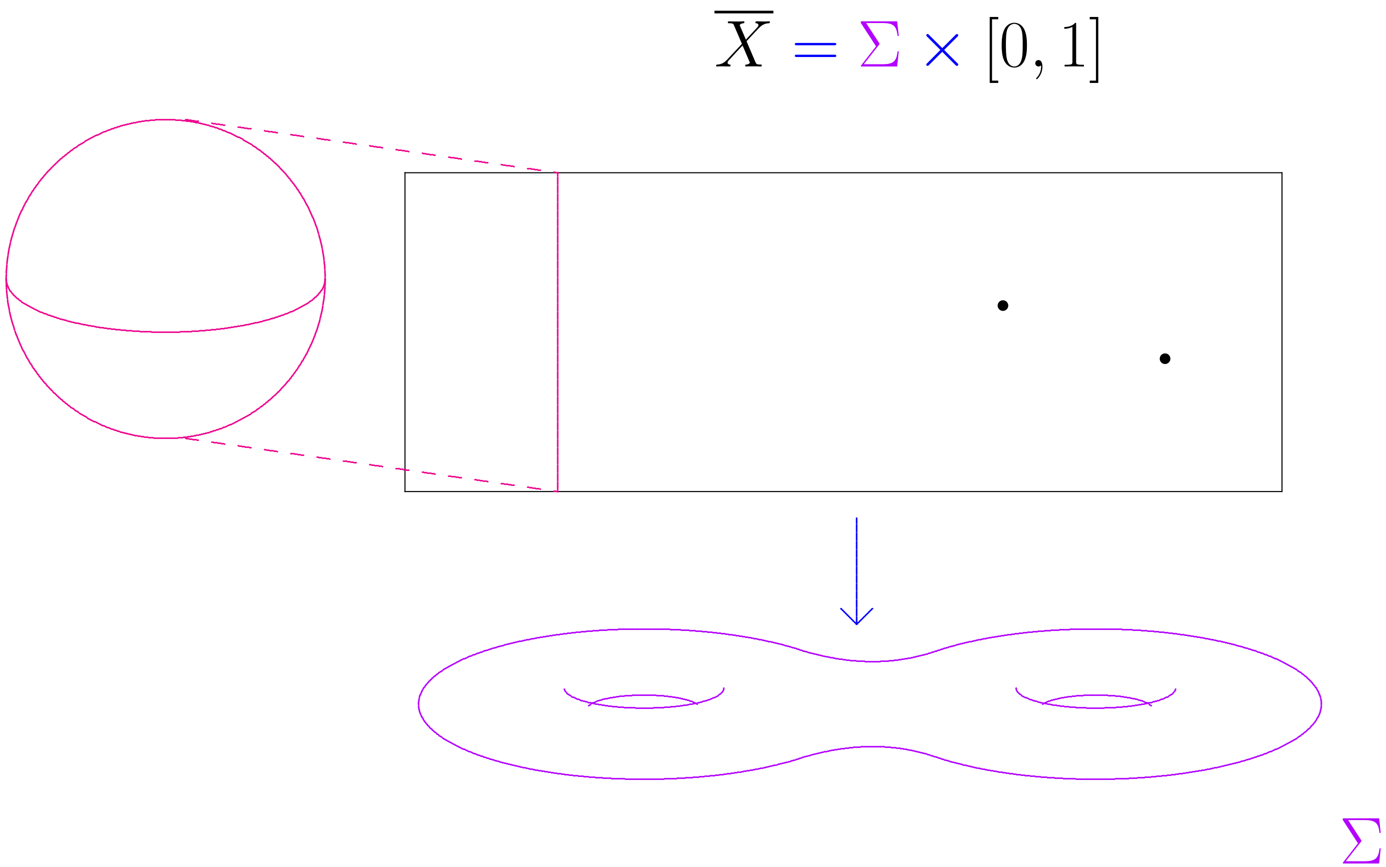
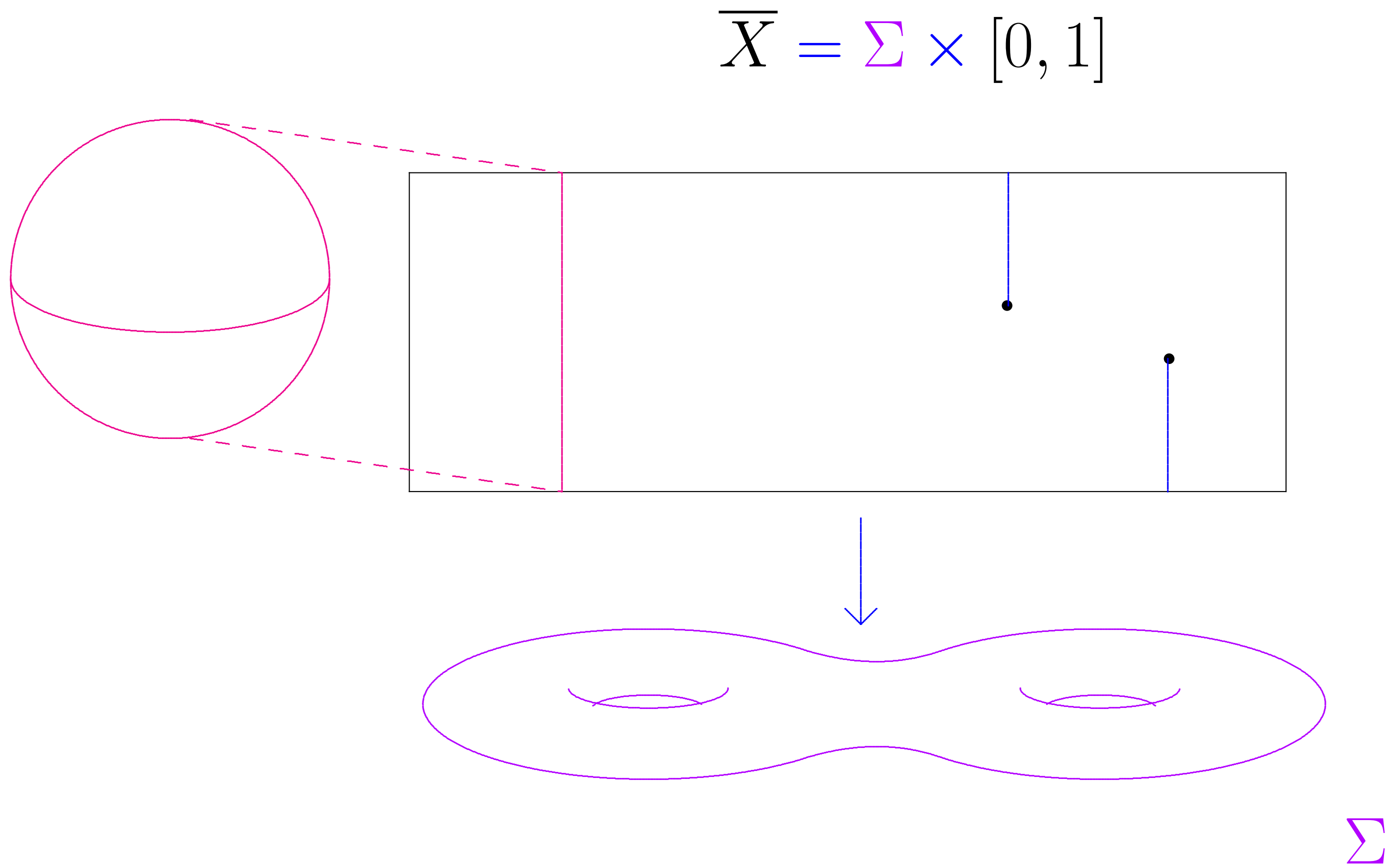



$\overline{X} = \Sigma \times [0,1]$
Σ



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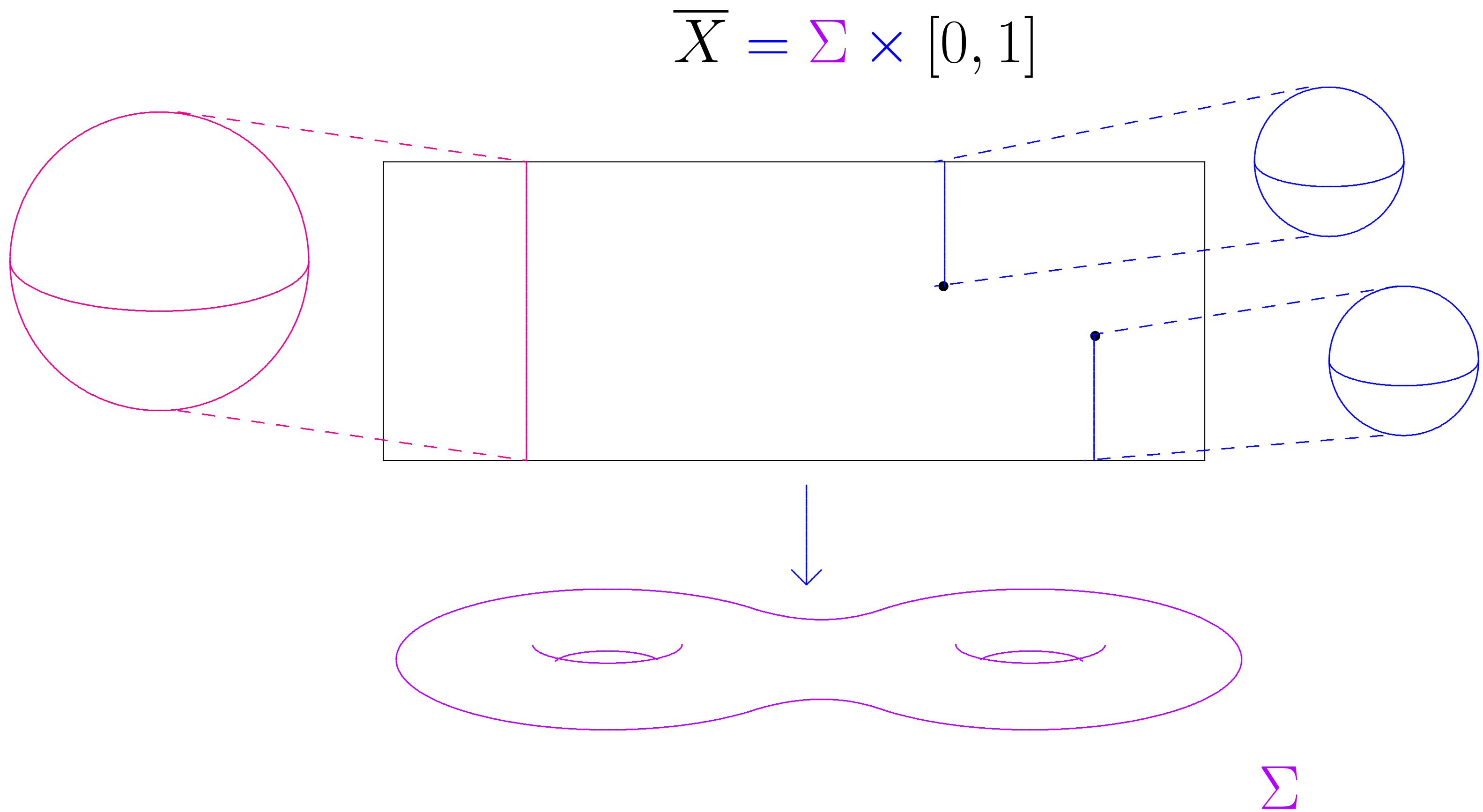
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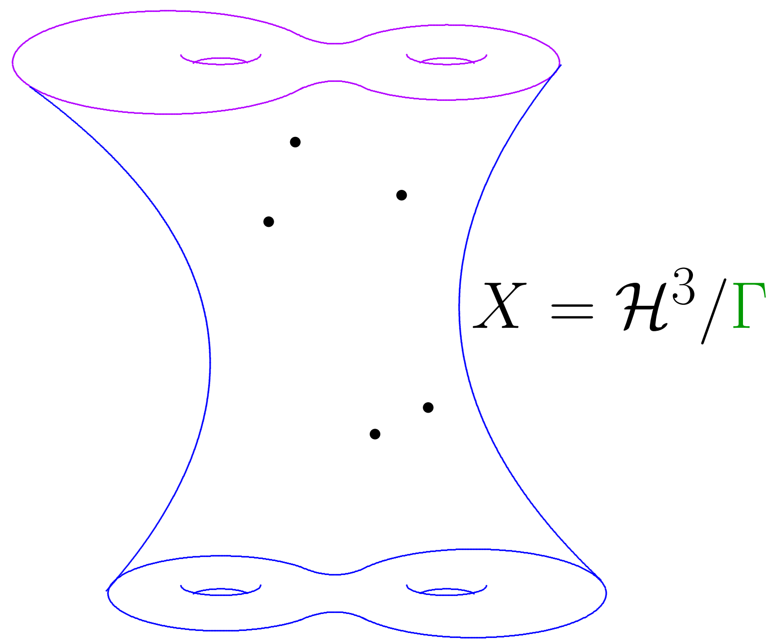
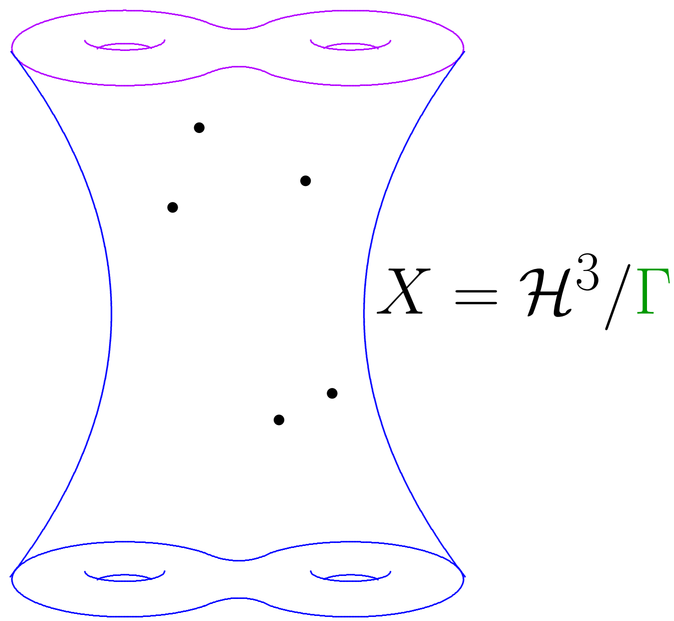


Construction of ASD 4-manifolds:

$$g = f(1-f)[Vh + V^{-1}\theta^2]$$

$$M = P \cup \{\hat{p}_1, \ldots, \hat{p}_k\} \cup \partial\overline{X}$$

$$\approx (\Sigma \times S^2)\# k\overline{\mathbb{CP}}_2$$



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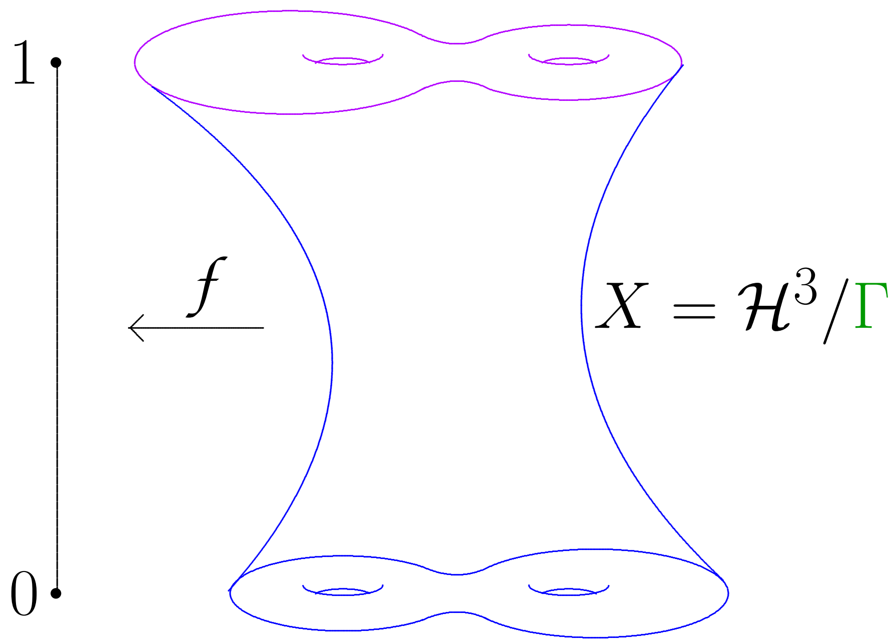
$$g = f(1-f)[Vh + V^{-1}\theta^2]$$

$$M = P \cup \{\hat{p}_1, \ldots, \hat{p}_k\} \cup \partial \overline{X}$$

Fuchsian case: $(\Sigma \times S^2)\# k\overline{\mathbb{CP}}_2$ scalar-flat Kähler



Γ quasi-Fuchsian

$$\overline{X} \approx \Sigma \times [0,1]$$

Tunnel-Vision function:

$$f : \overline{X} \to [0,1]$$

$$\Delta f = 0$$

Theorem.

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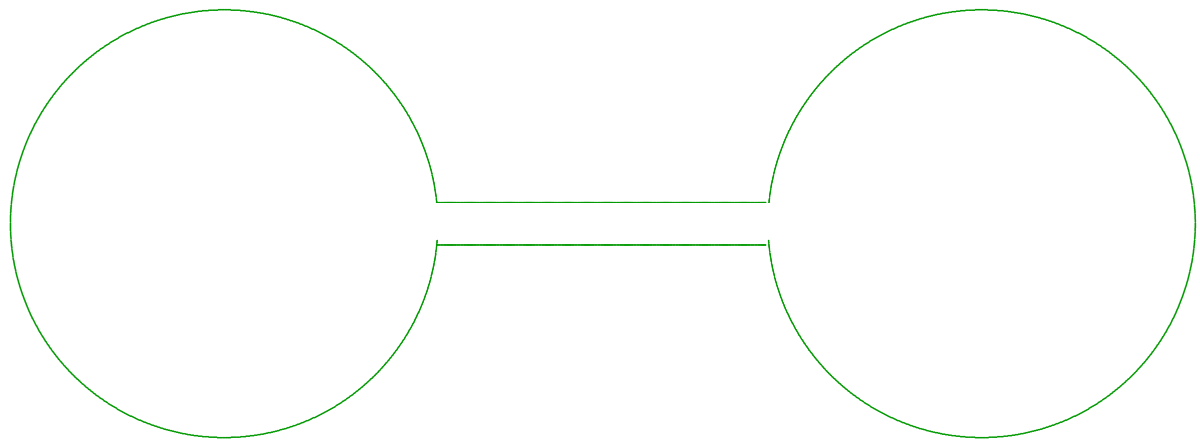
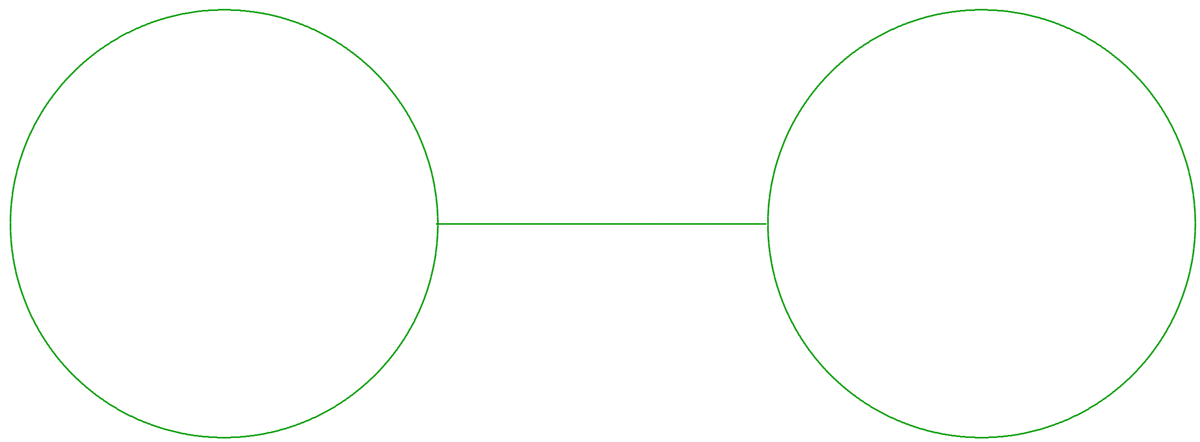
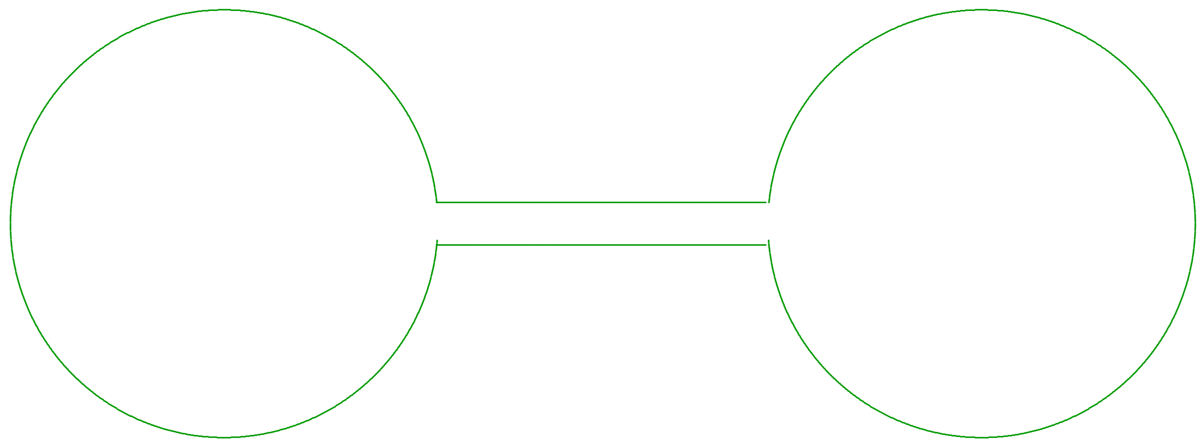
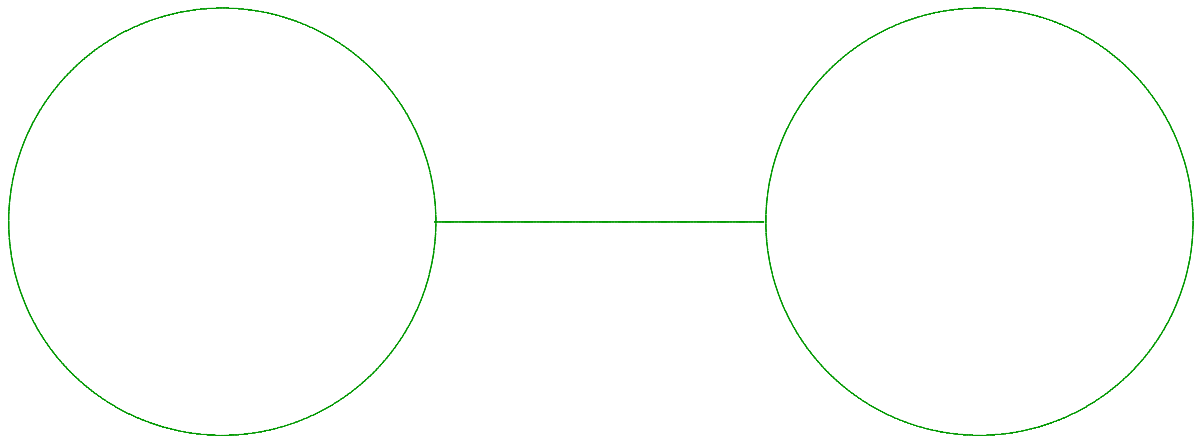
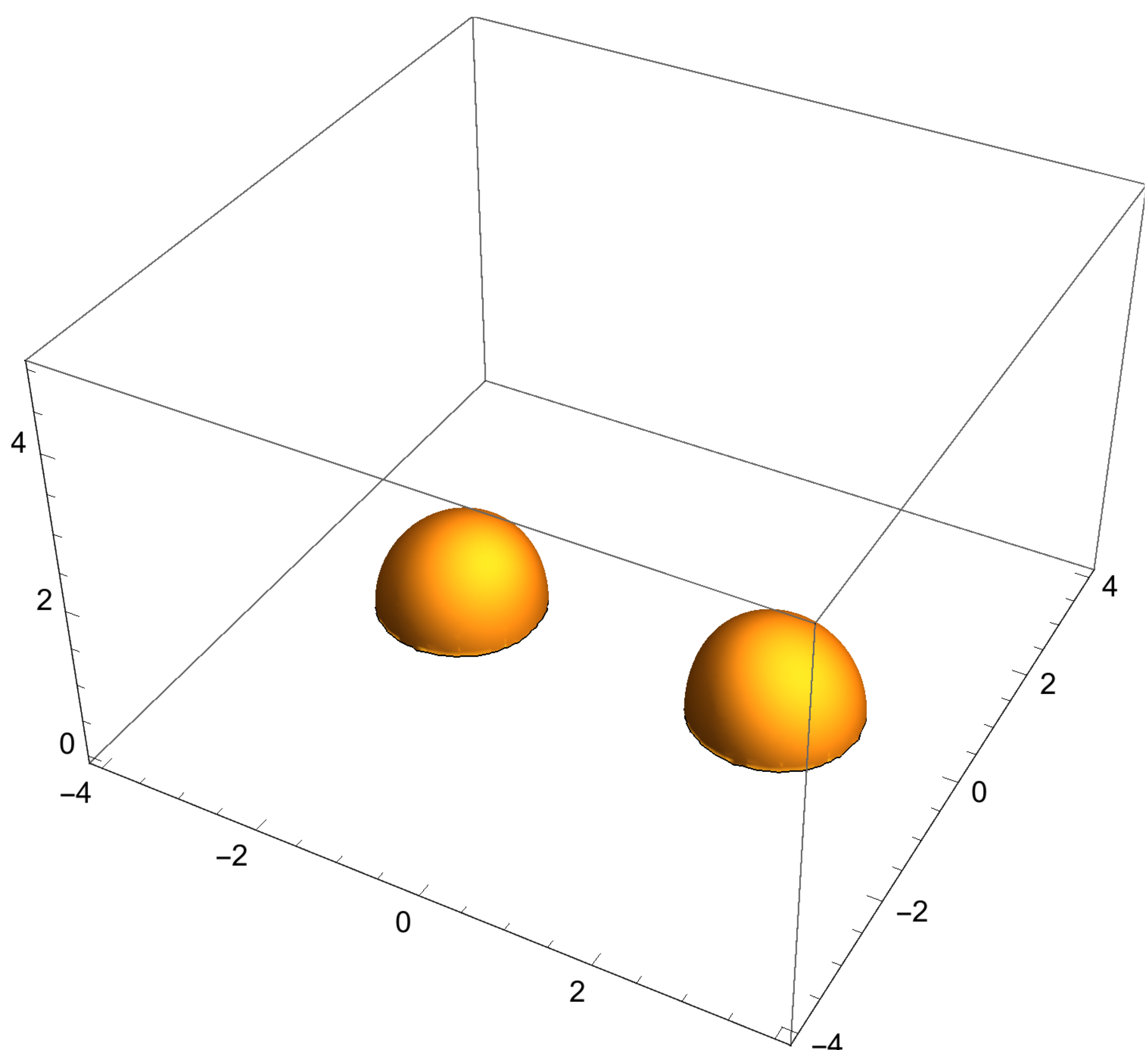
Ahlfors-Bers: Quasi-conformal mappings











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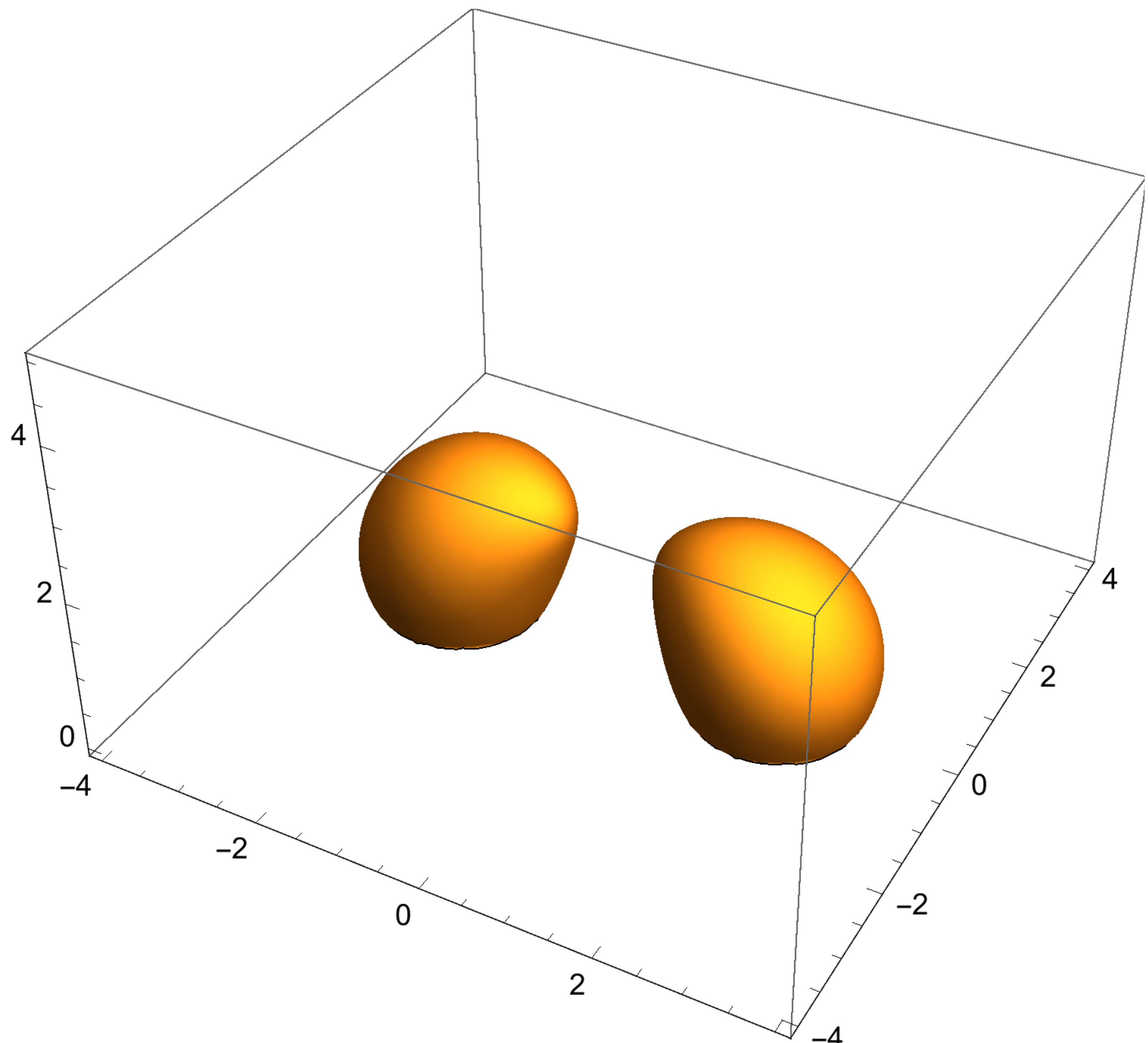
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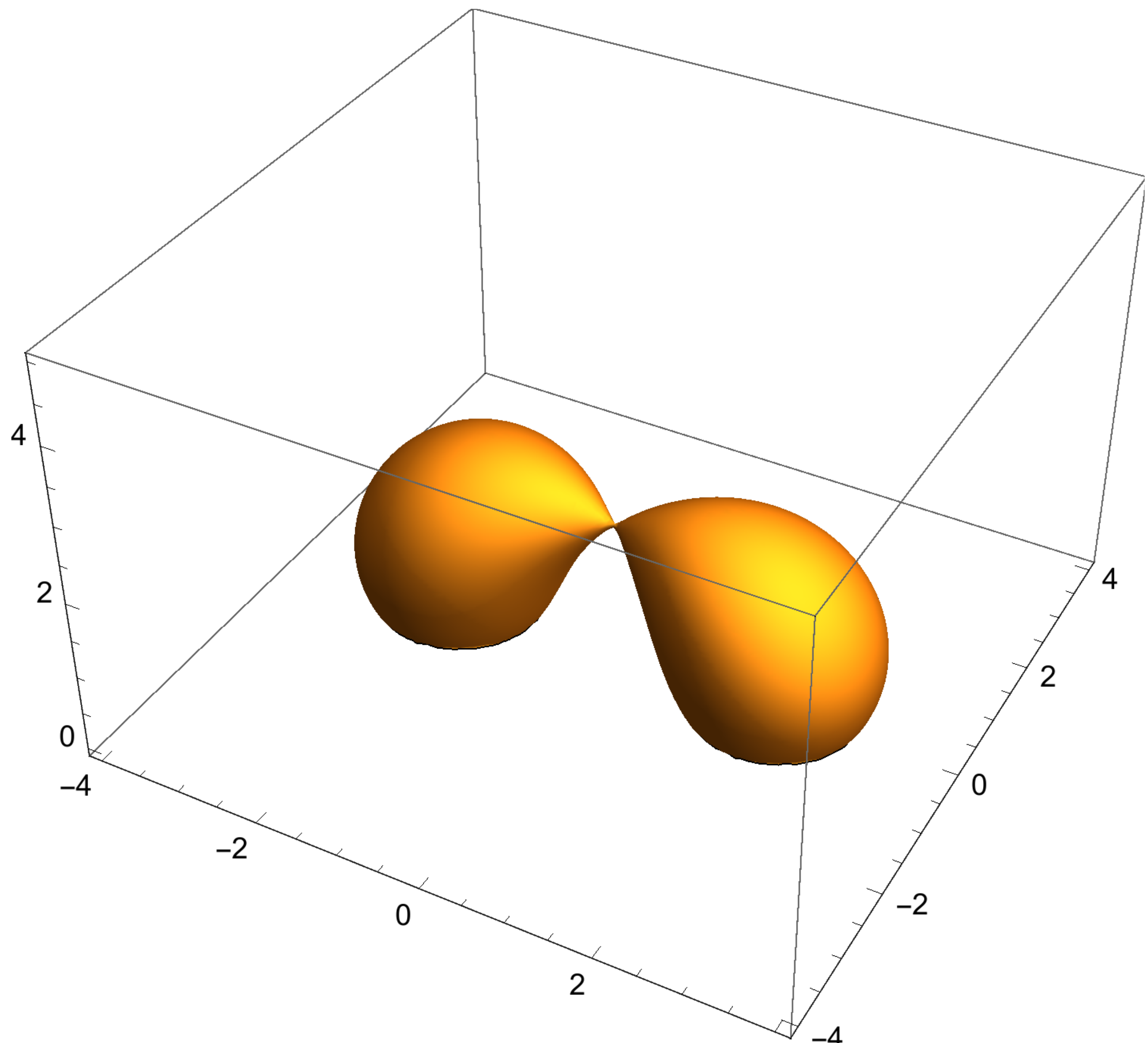
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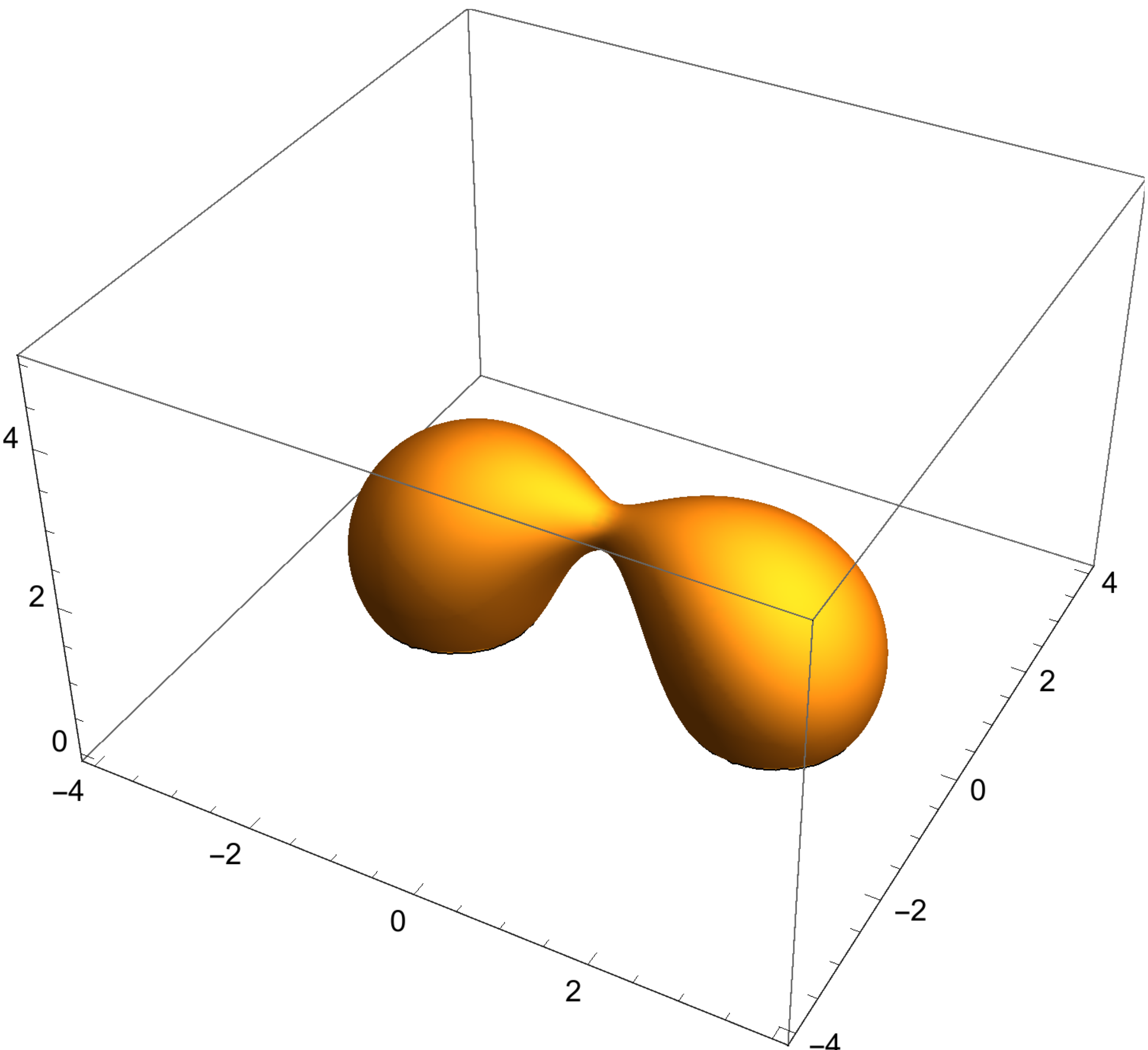

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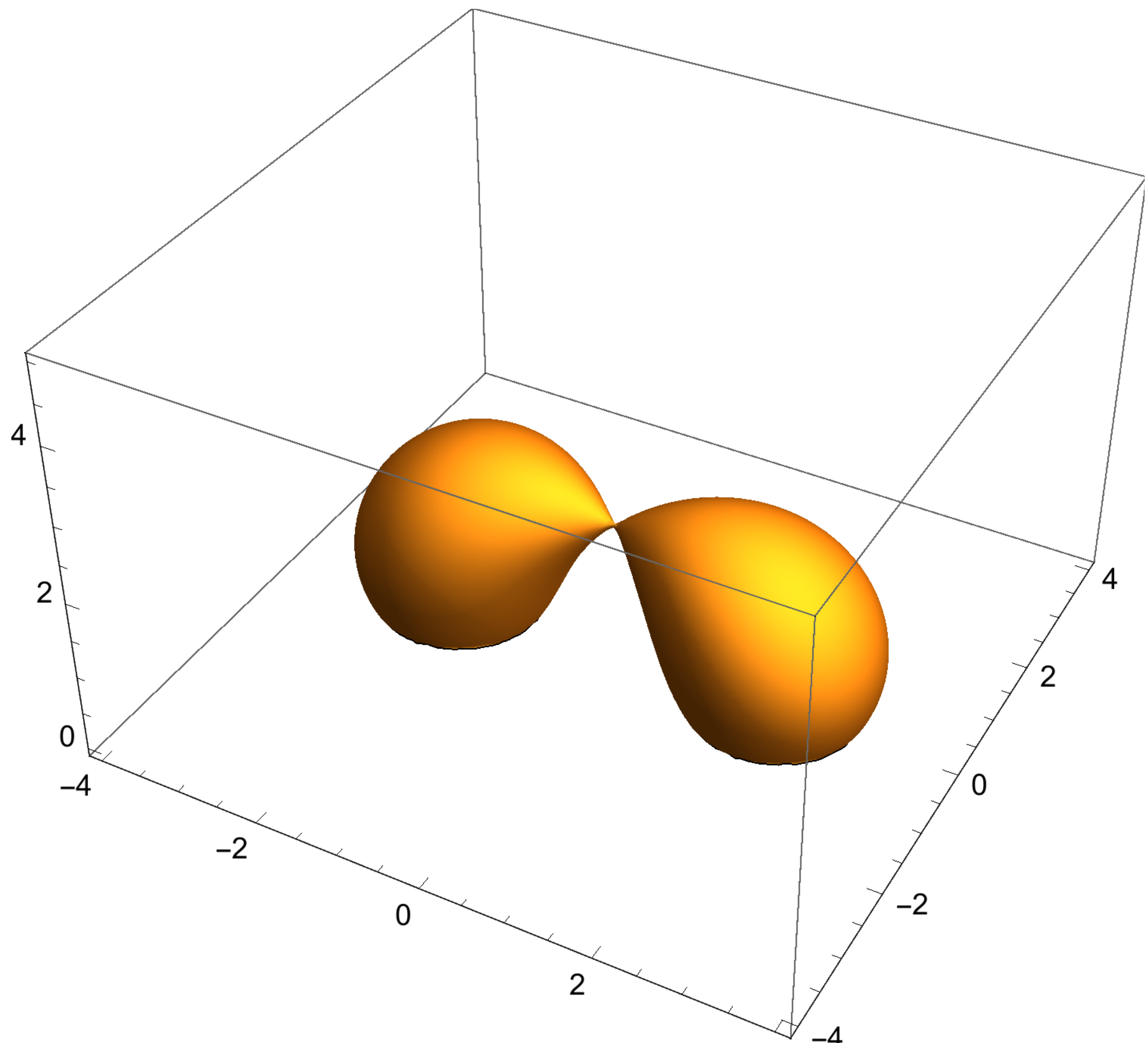
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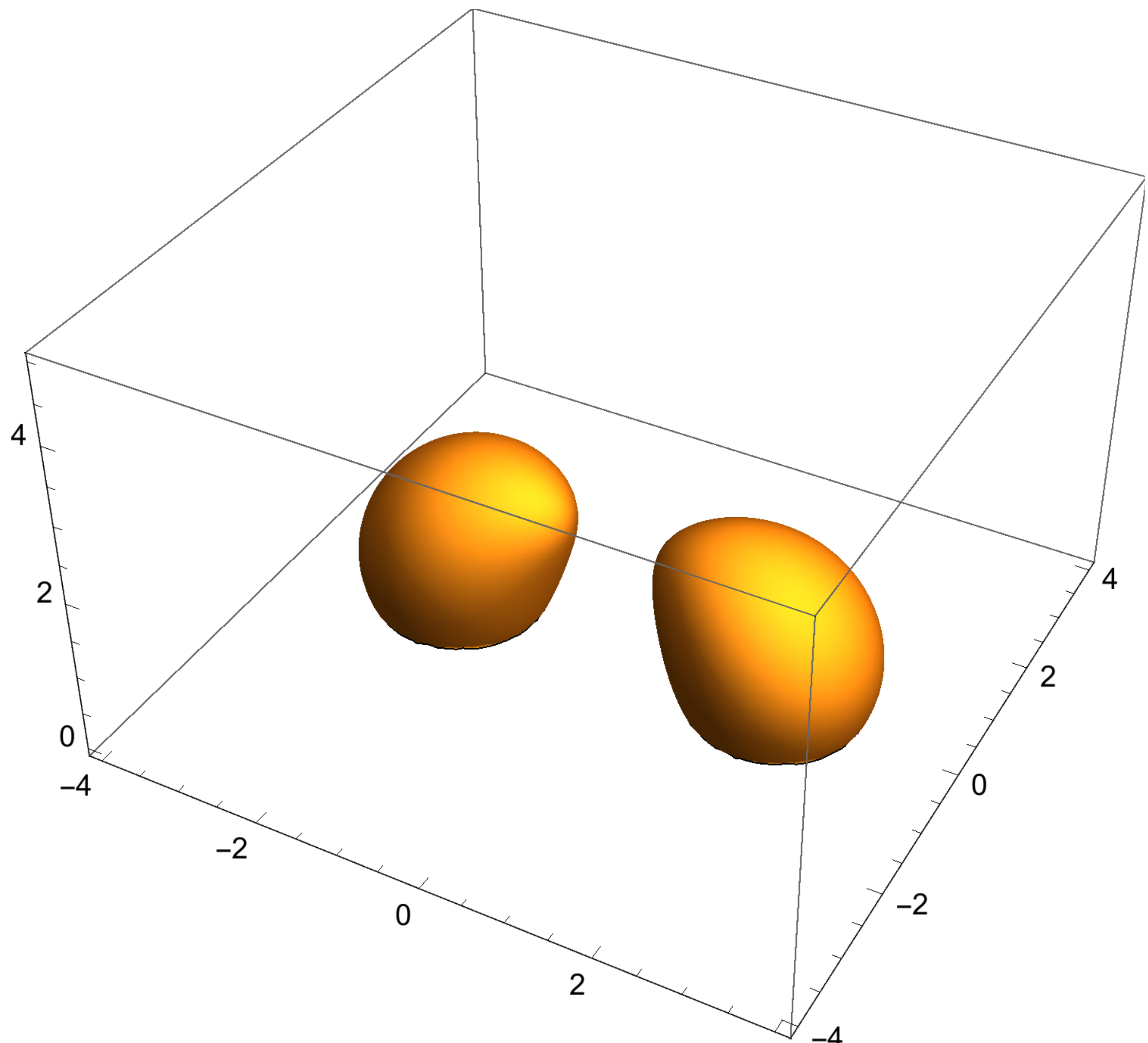
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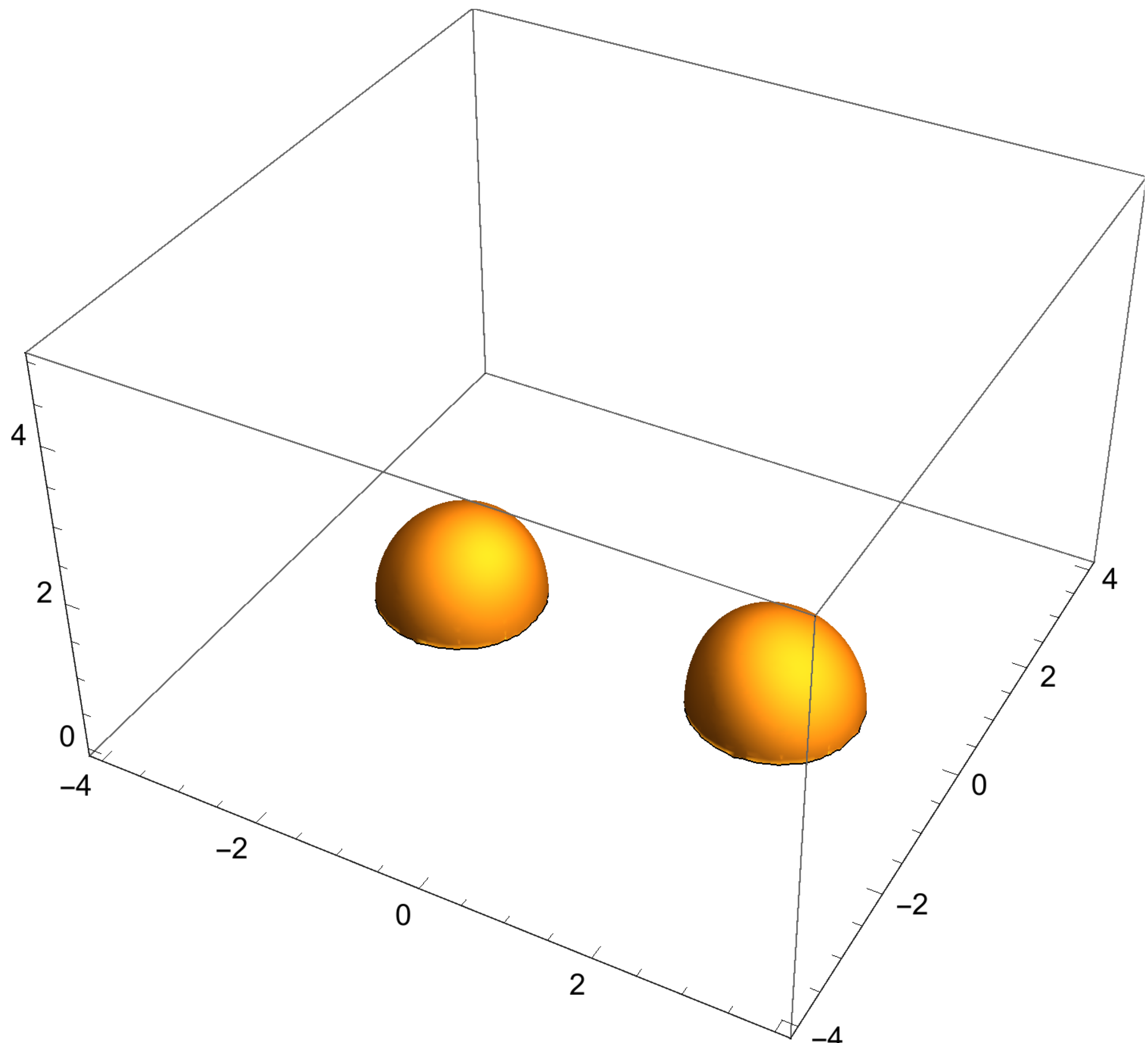

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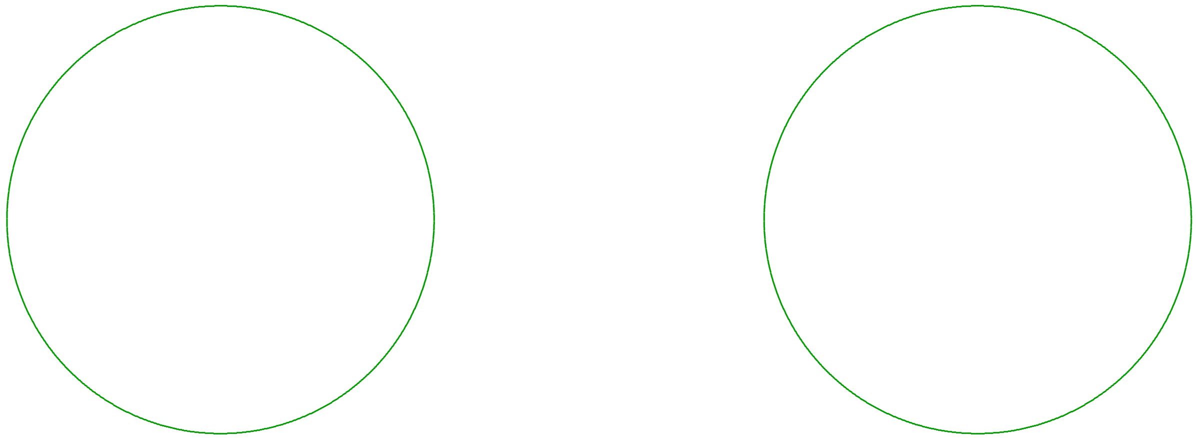
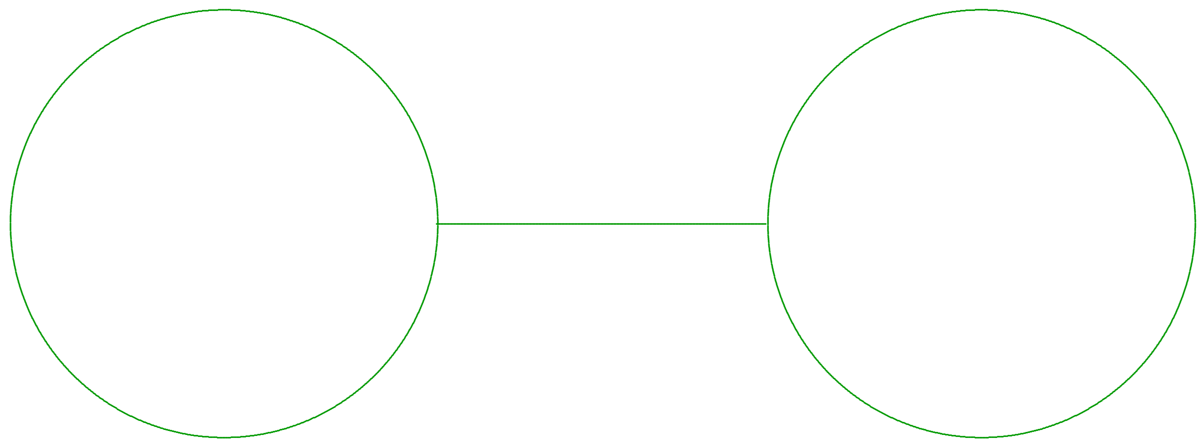
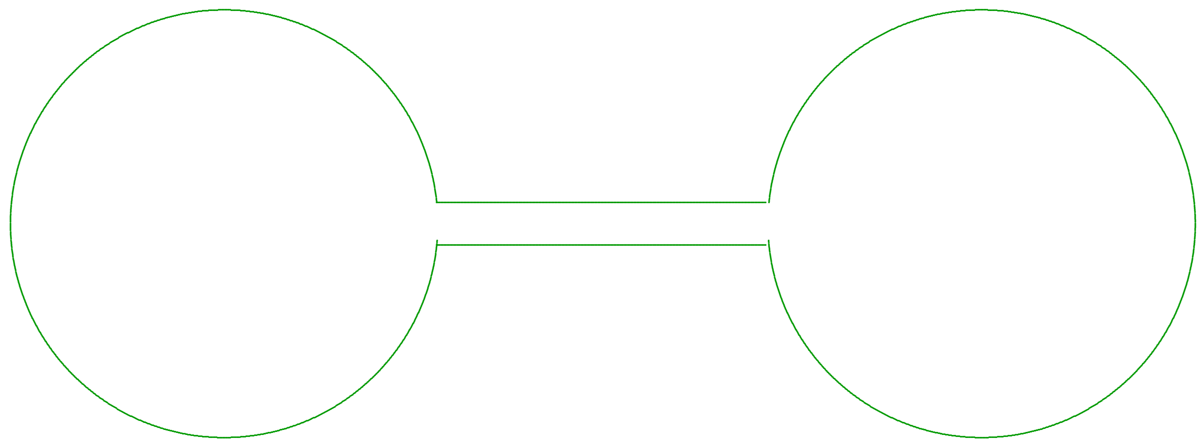
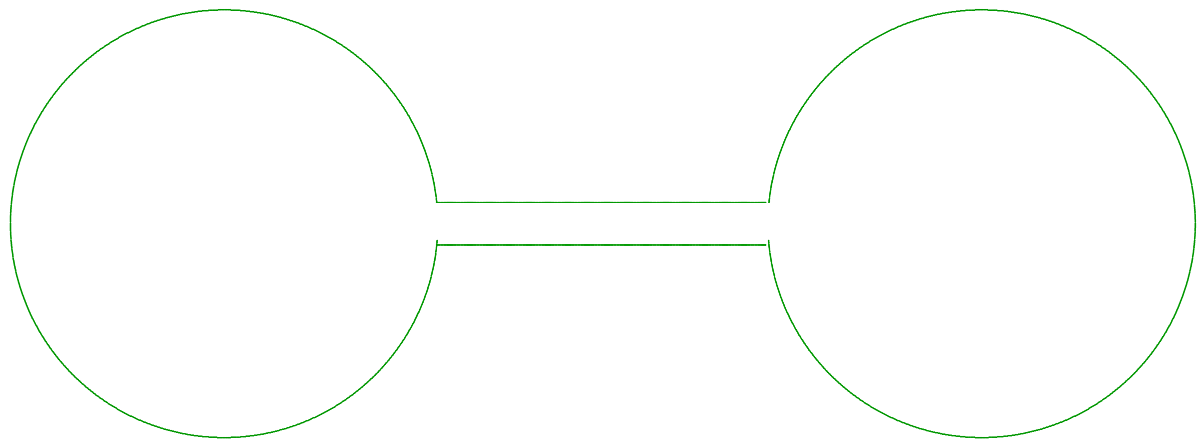
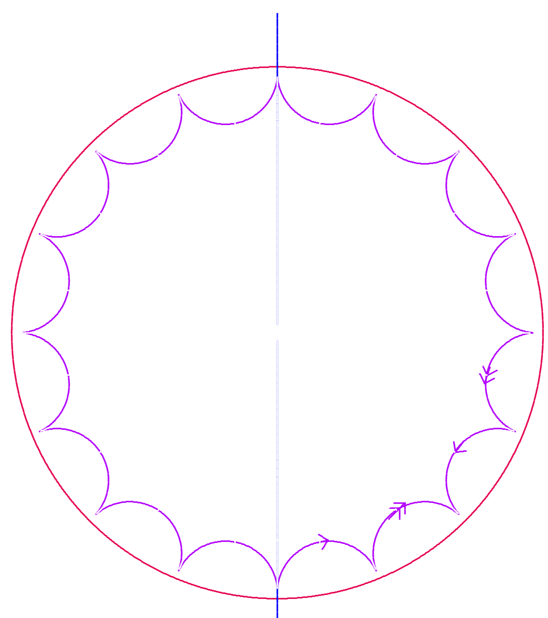
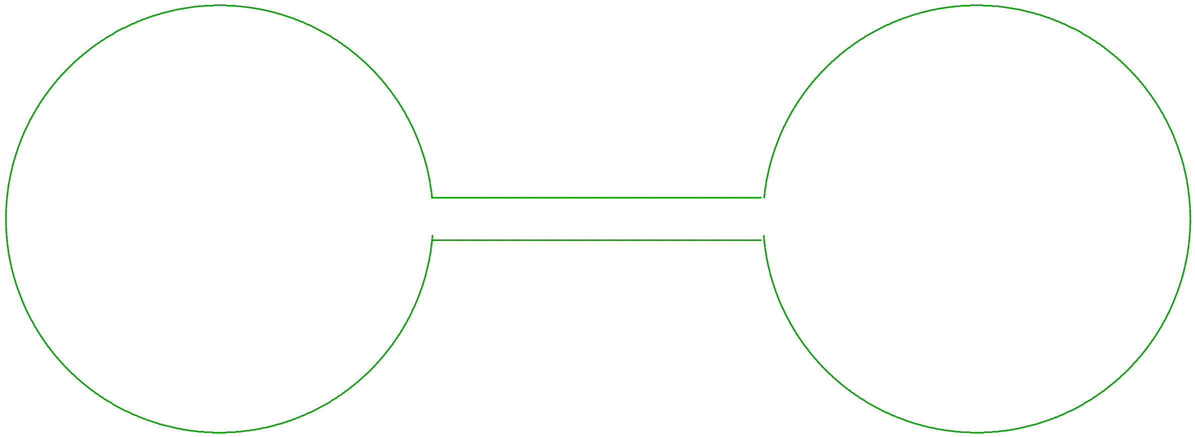


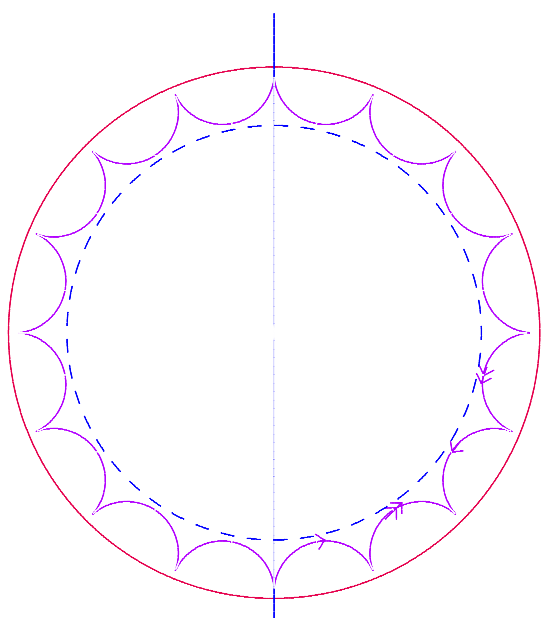


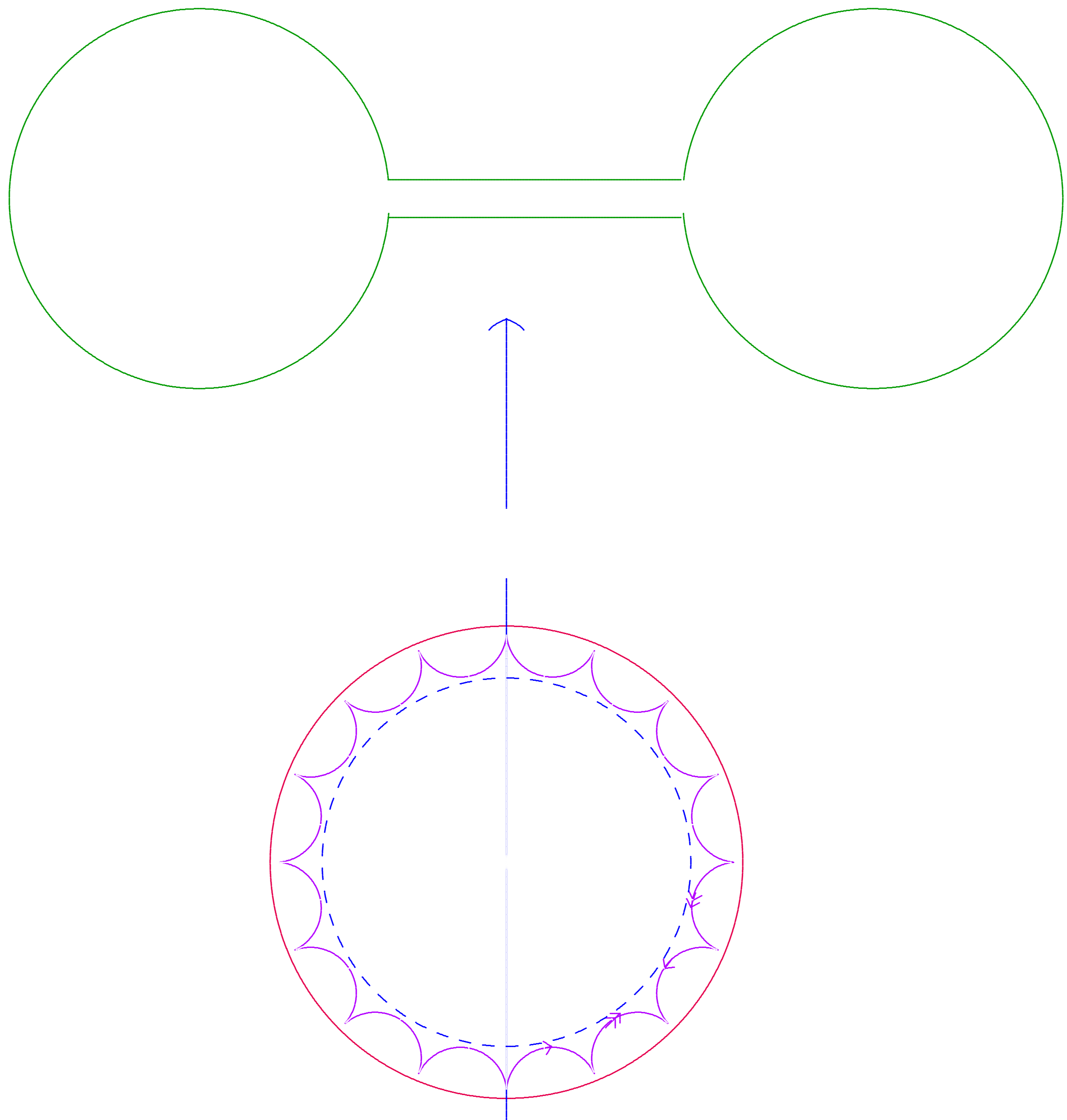


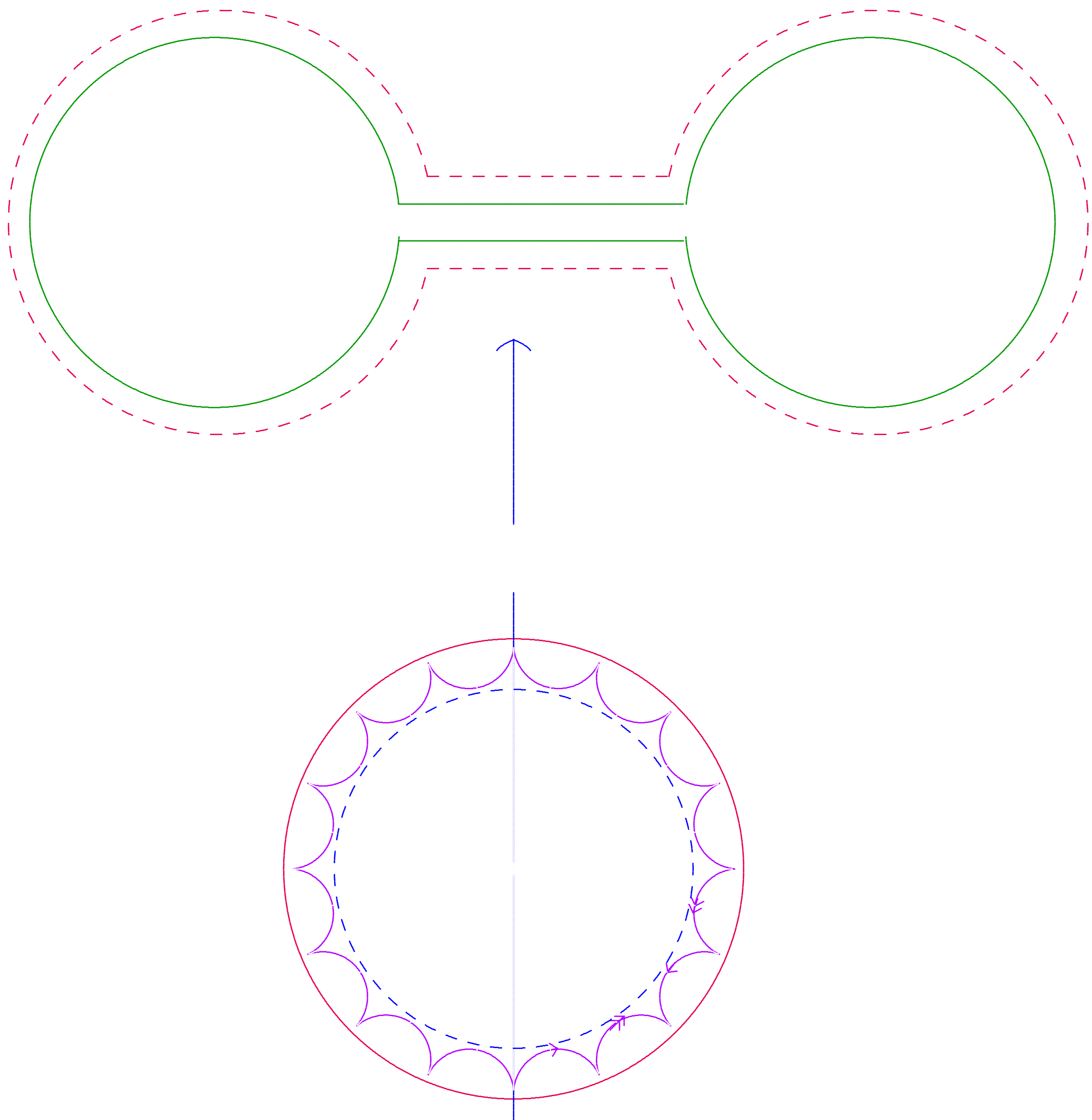


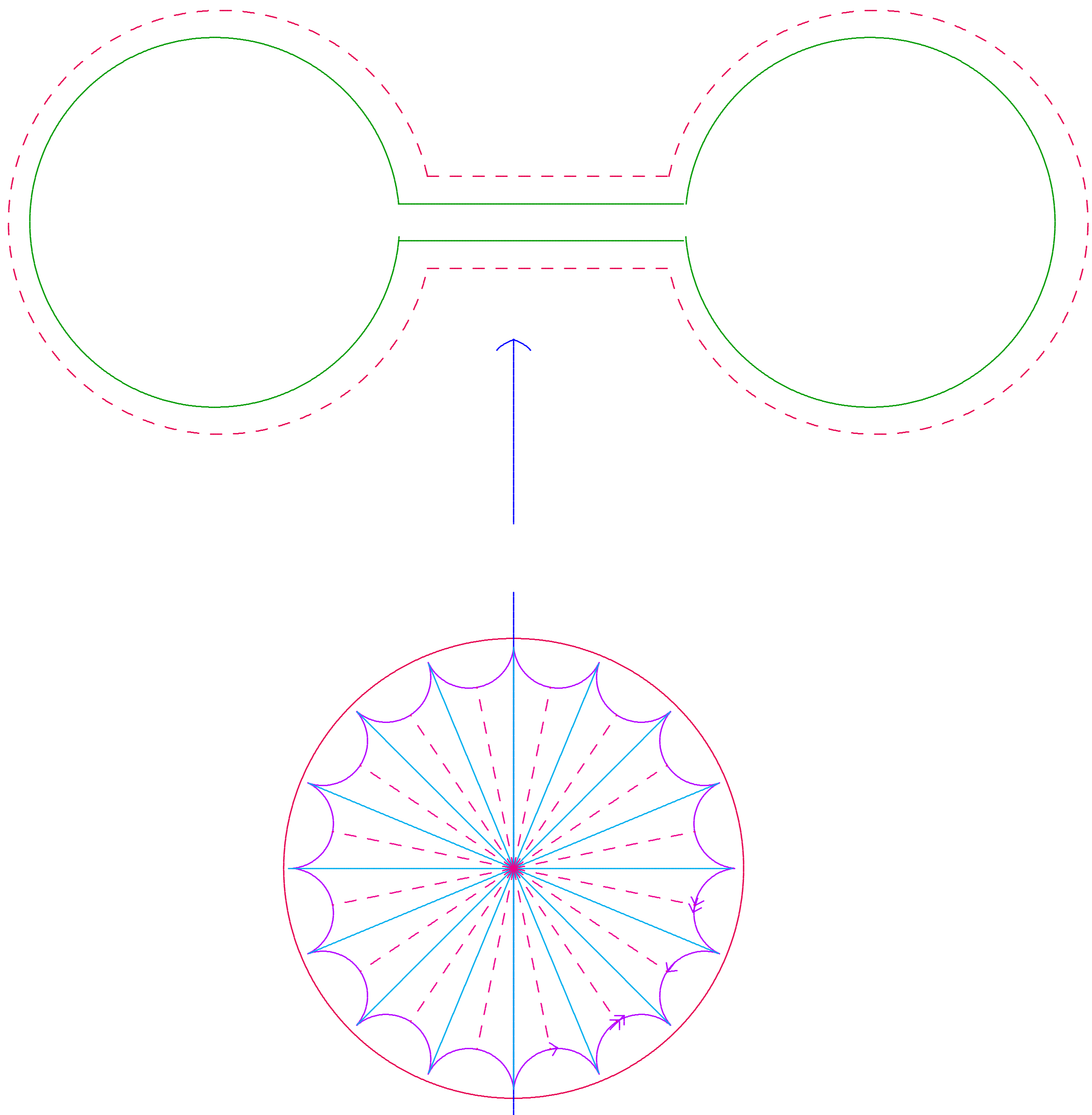


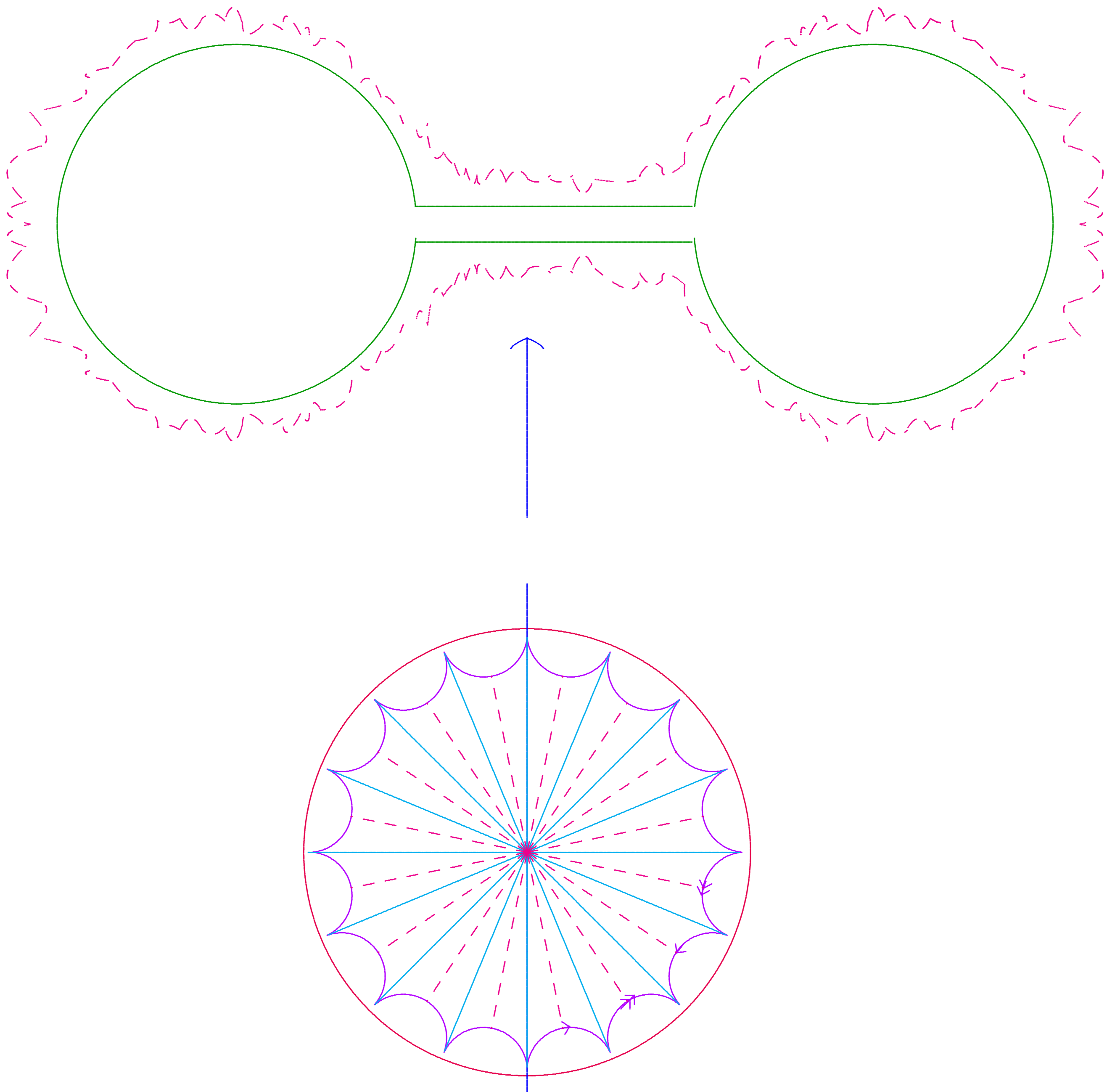








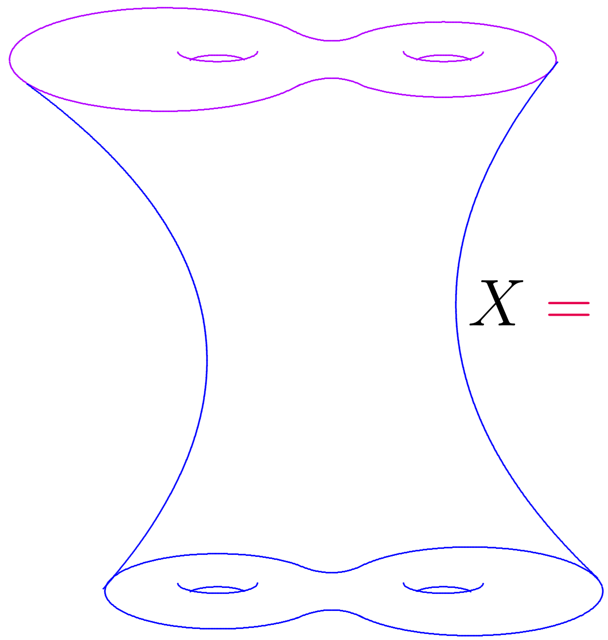




Theorem A. *Consider* 4*-manifolds* $M = \Sigma \times S^2$, *where* Σ *compact Riemann surface of genus* $\mathbf{g}$.

Then $\forall$ *even* $\mathbf{g} \gg 0$, $\exists$ *family* $[g_t]$, $t \in [0, 1]$, *of locally-conformally-flat classes on* M, *such that*

- $\exists$ *scalar-flat Kähler metric* $g_0 \in [g_0]$*; but*
- $\nexists$ *almost-Kähler metric* $g \in [g_1]$.



$$X = \mathcal{H}^3/\Gamma$$

Construction of conformally flat 4-manifolds:

$$M = [\overline{X} \times S^1]/\sim$$

$$g = f(1-f)[h + dt^2]$$

Theorem B. *Fix an integer* $k \geq 2$, *and then consider the 4-manifolds* $M = (\Sigma \times S^2) \# k\overline{\mathbb{CP}}_2$, *where* Σ *compact Riemann surface of genus* g.

Then $\forall$ *even* $\mathsf{g} \gg 0$, $\exists$ *family* $[g_t]$, $t \in [0, 1]$, *of anti-self-dual conformal classes on* M, *such that*

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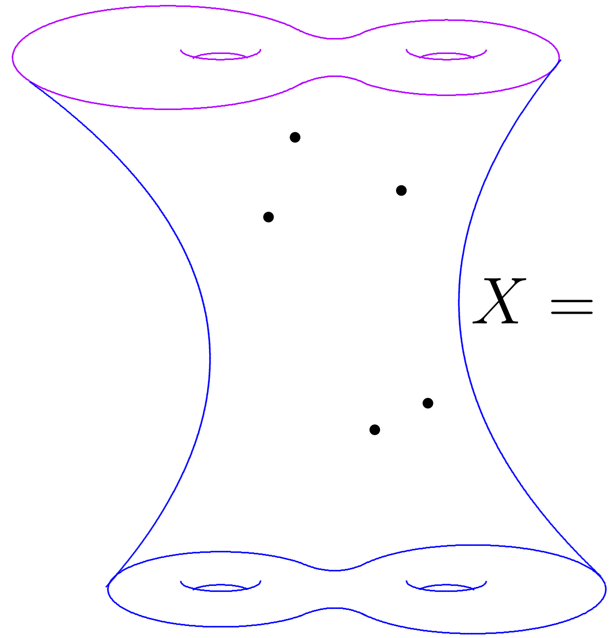

Construction of ASD 4-manifolds:

$$g = f(1-f)[Vh + V^{-1}\theta^2]$$

$$V = 1 + \sum_{j=1}^{k} G_j$$

$$d\theta = \star dV$$

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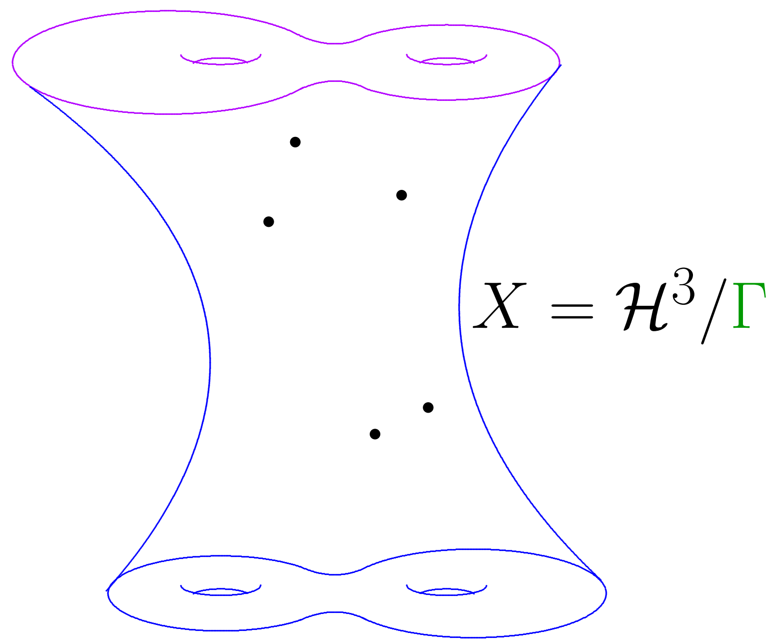

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¡Muchas Gracias por la Invitación!