Einstein Metrics,

Four-Manifolds, &

Gravitational Instantons

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University of California, Irvine,
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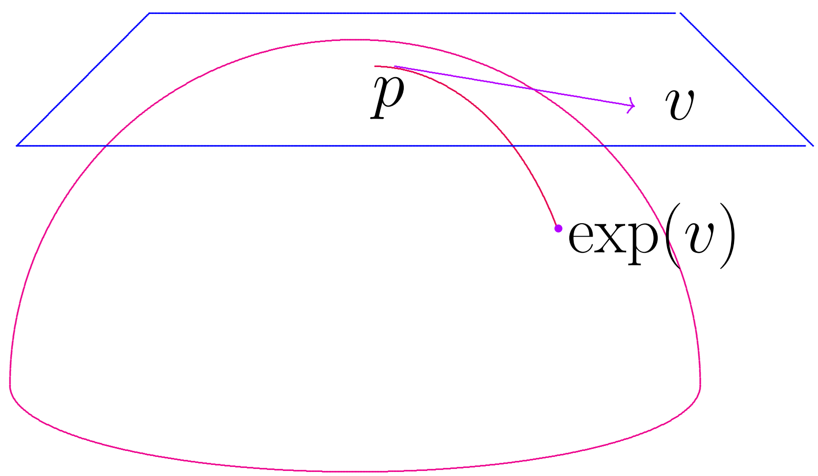

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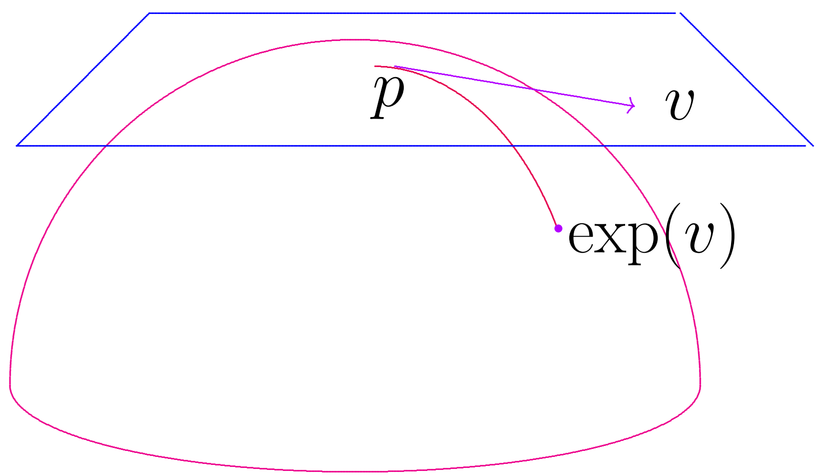

Now choosing $T_pM \xrightarrow{\cong} \mathbb{R}^n$ via some orthonormal basis gives us special coordinates on M.

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"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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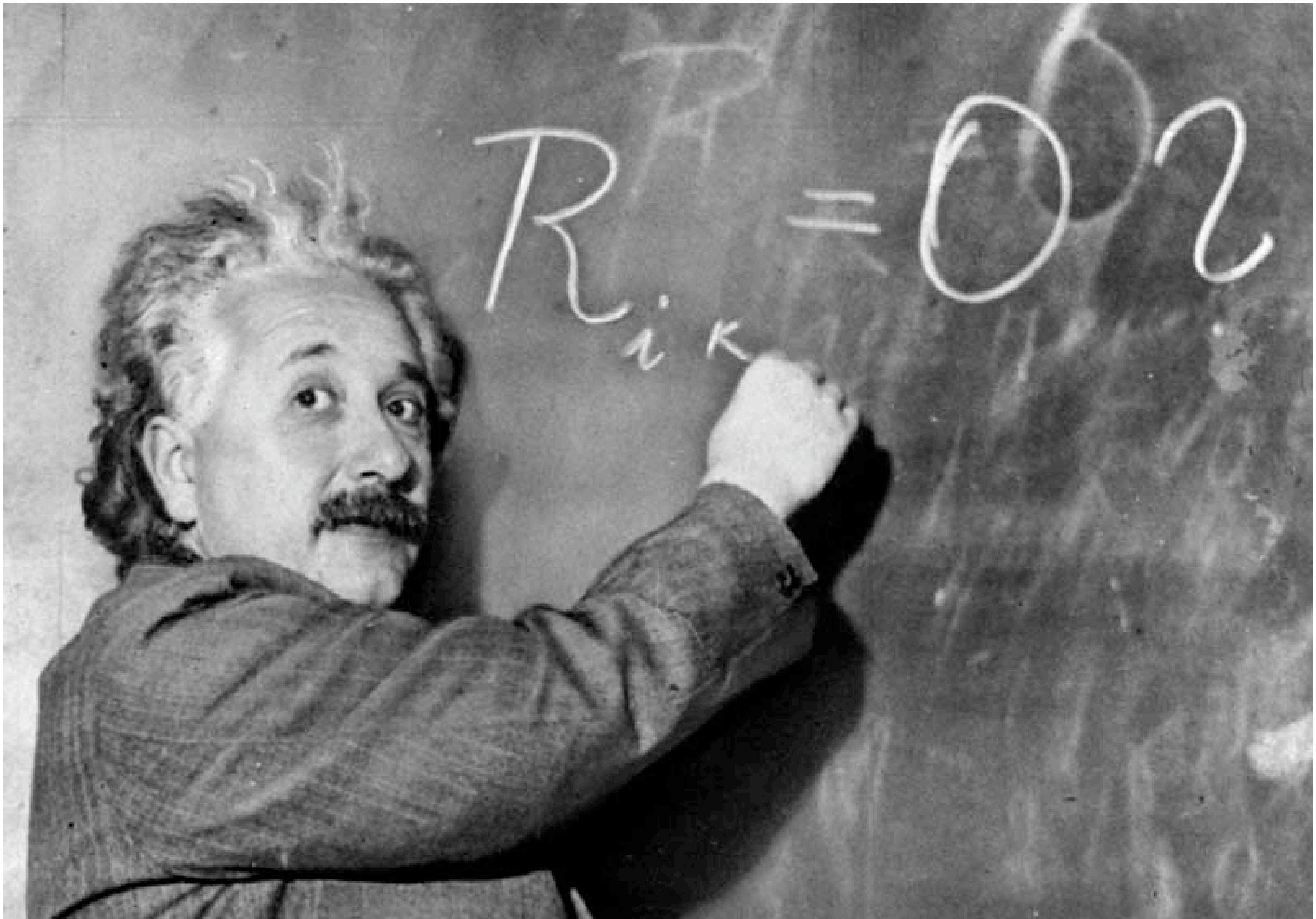

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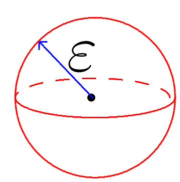

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$$\Delta x^j = 0 \Longrightarrow r_{jk} = \frac{1}{2}\Delta g_{jk} + \ell ots.$$

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- When $n \geq 6$, wide open. Maybe???

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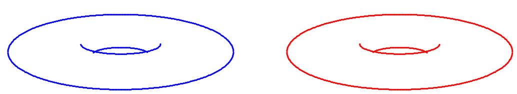

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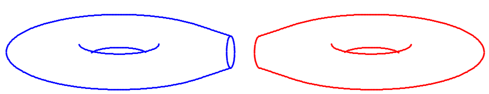

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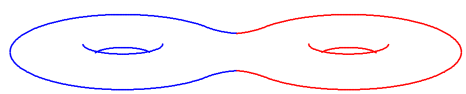

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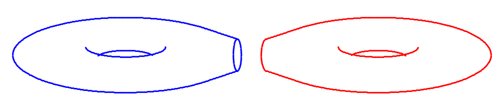

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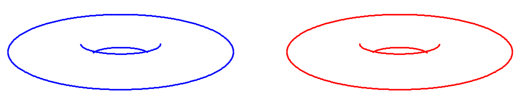

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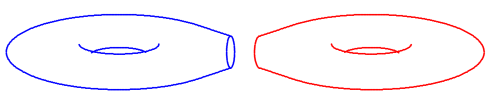

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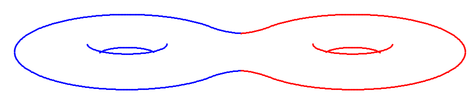

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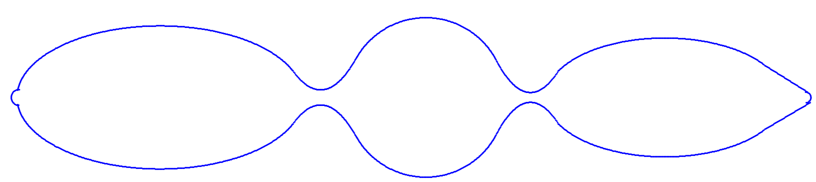

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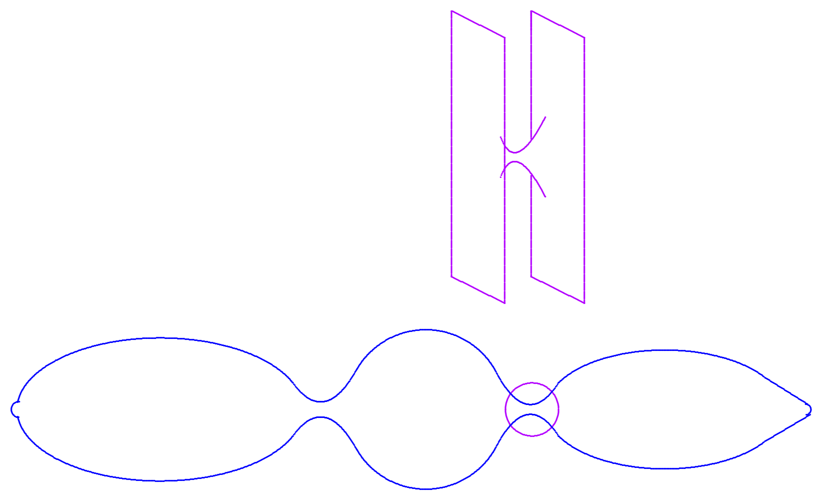

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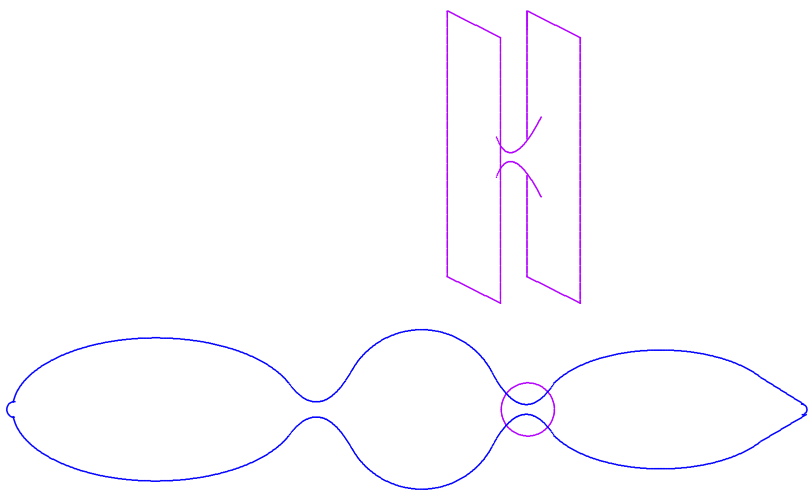

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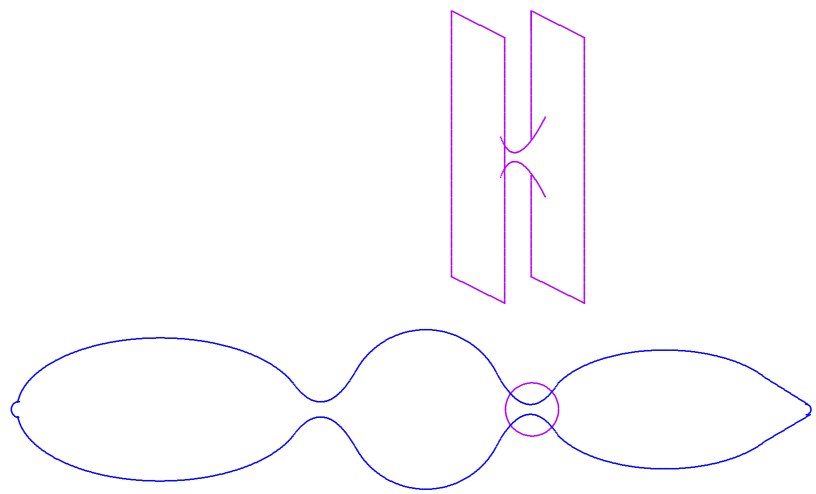

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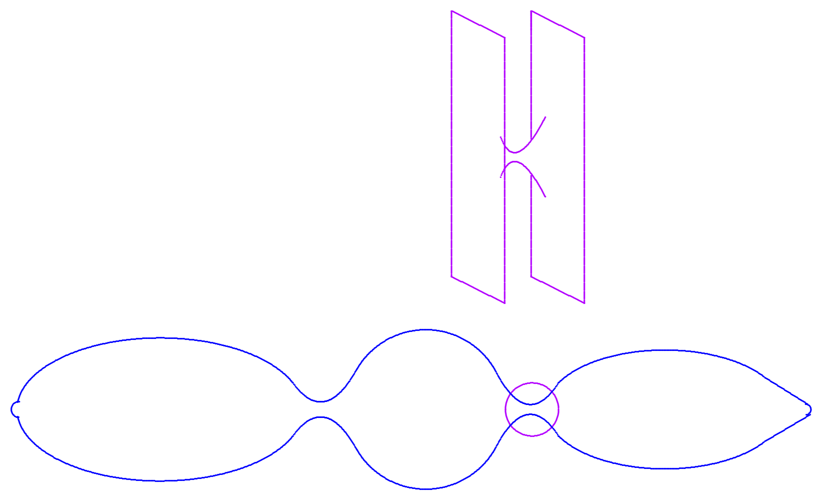

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Similar results for most simply connected spin 5-manifolds. (Boyer-Galicki-Kollár, et al.)

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A. Weil: as hard to climb as $K2$!

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Theorem (L). *There is only one Einstein metric on compact complex-hyperbolic 4-manifold $\mathbb{C}\mathcal{H}_2/\Gamma$, up to scale and diffeos.*

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“Goldilocks Zone” for Einstein metrics:

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Enough rigidity apparently still holds in dimension four to plausibly call this a geometrization.

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Λ^+ self-dual 2-forms.
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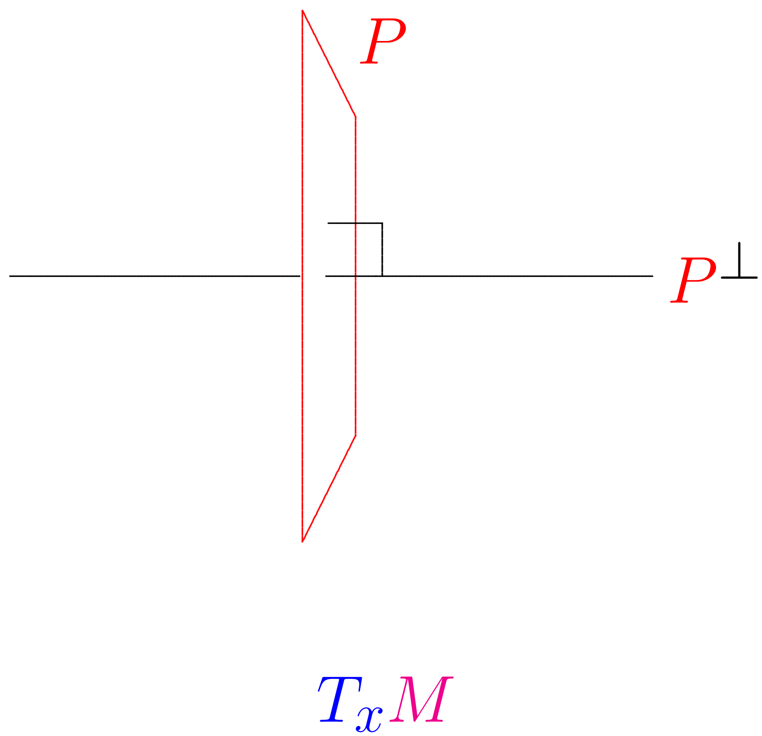

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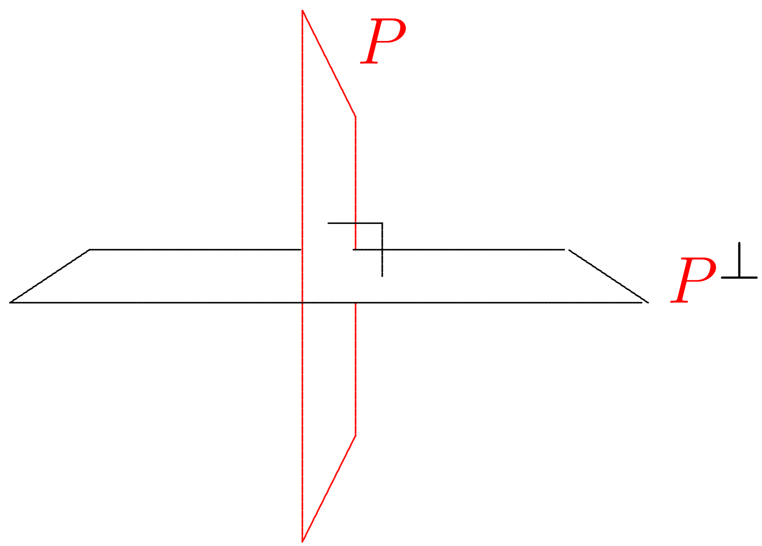

T_xM

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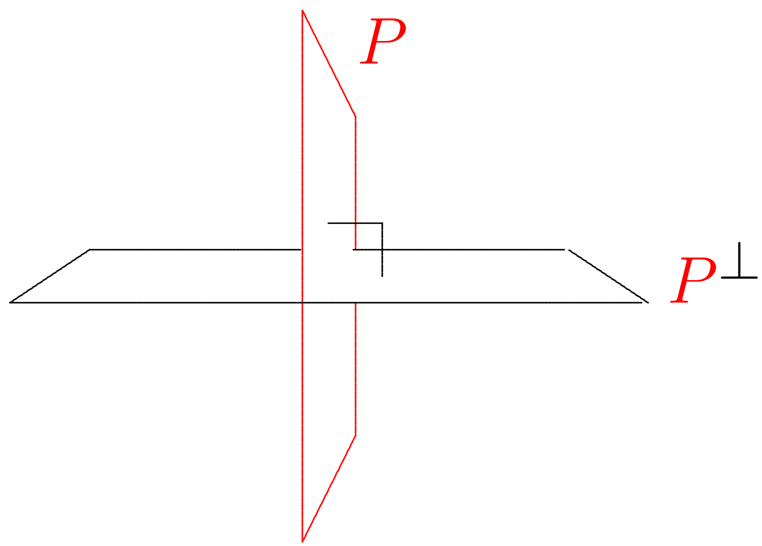

$$K(P) = K(P^{\perp})$$

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Hyper-Kähler?

Hyper-Kähler? Kähler?

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(M^n, g): holonomy

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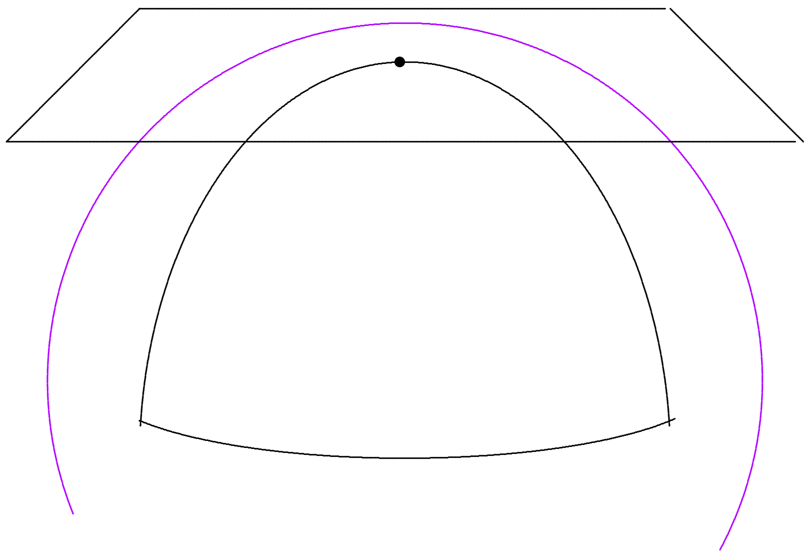

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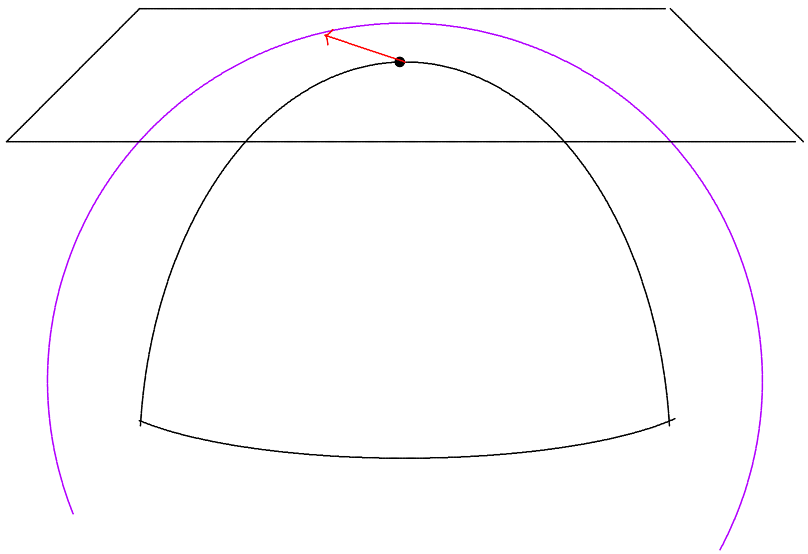

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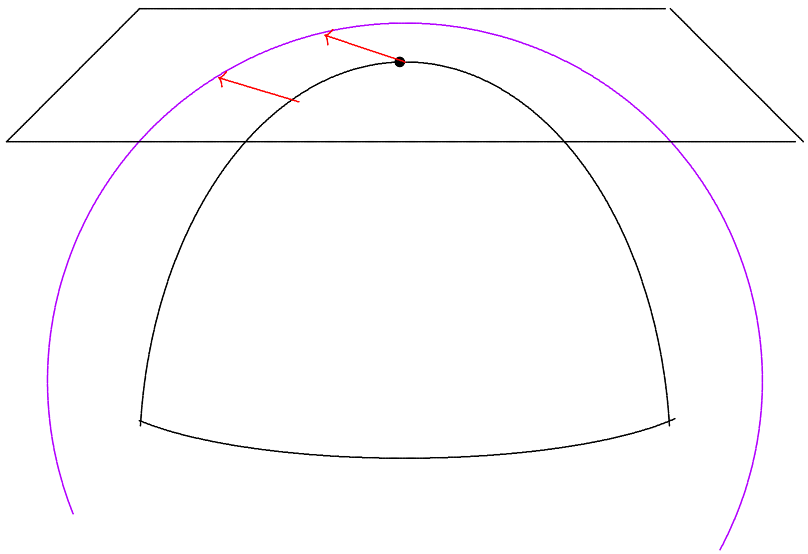

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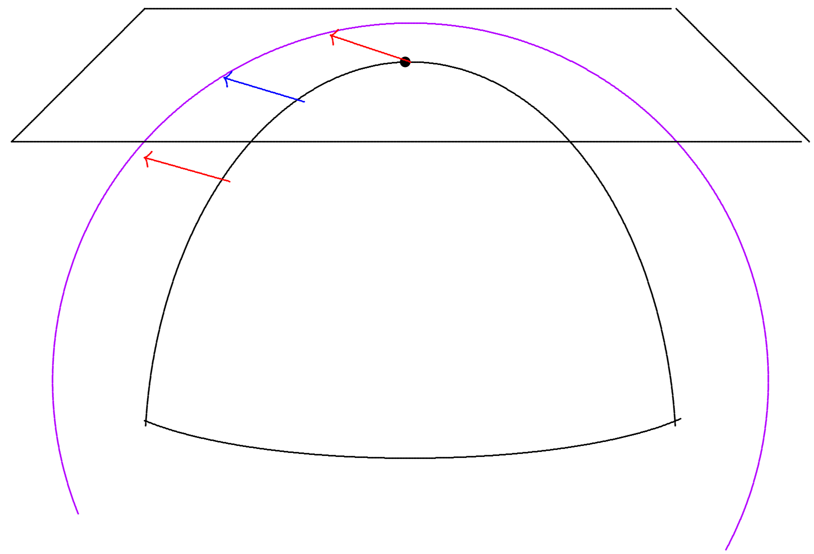

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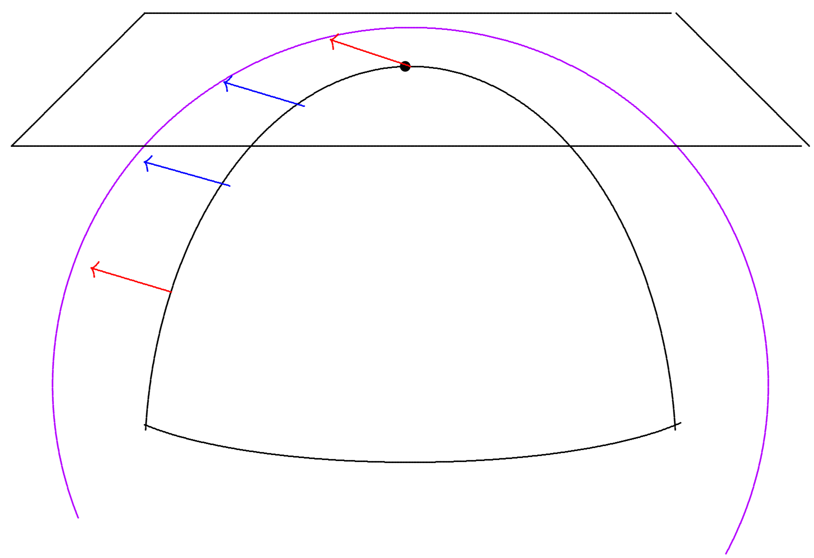

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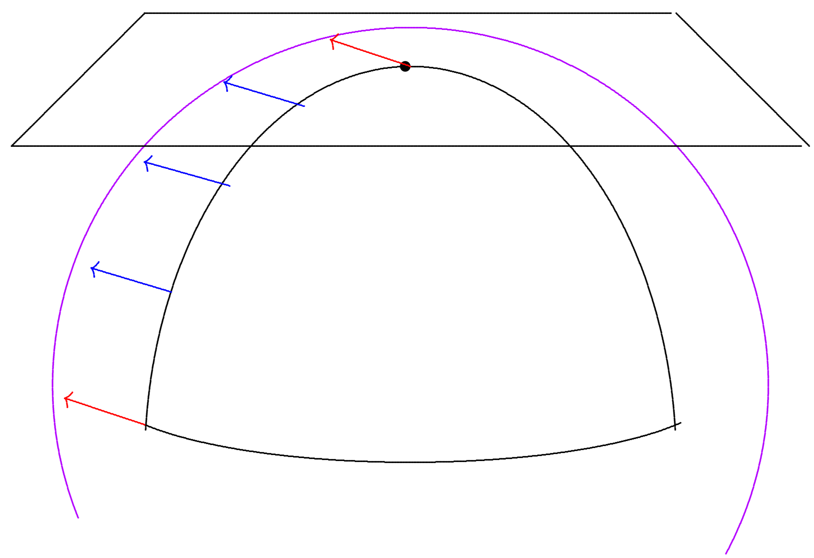

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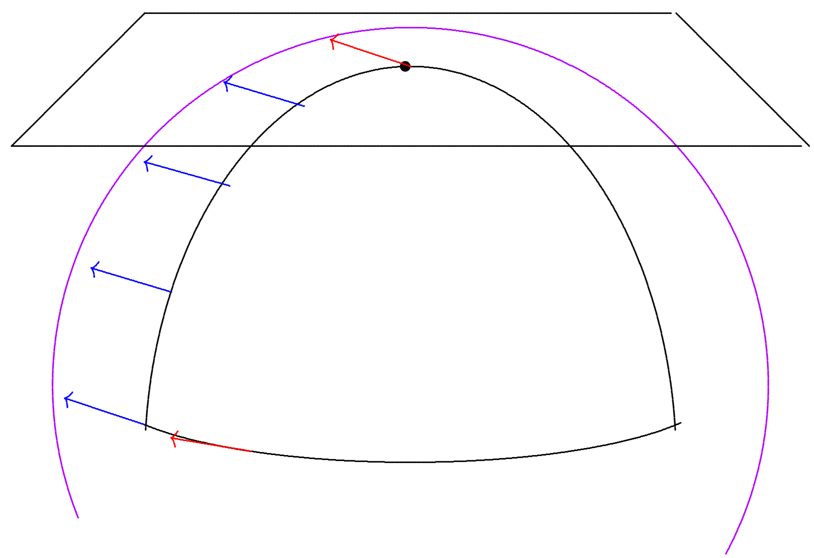

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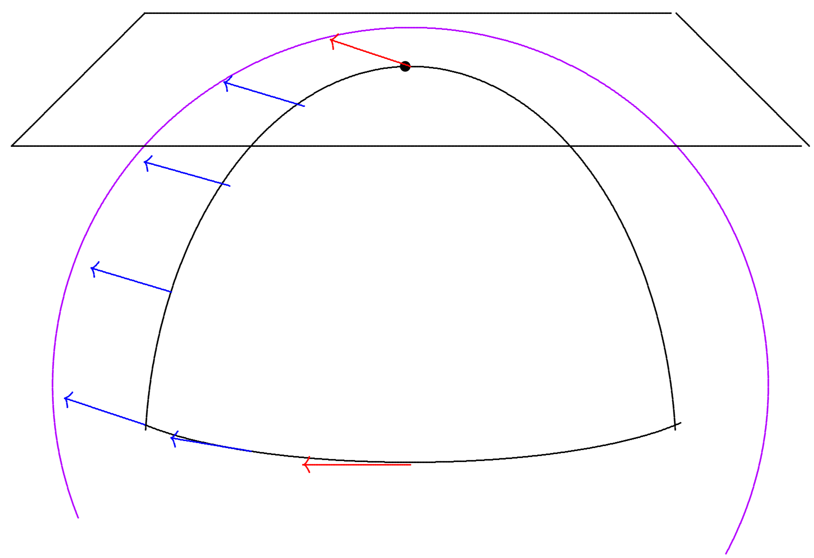

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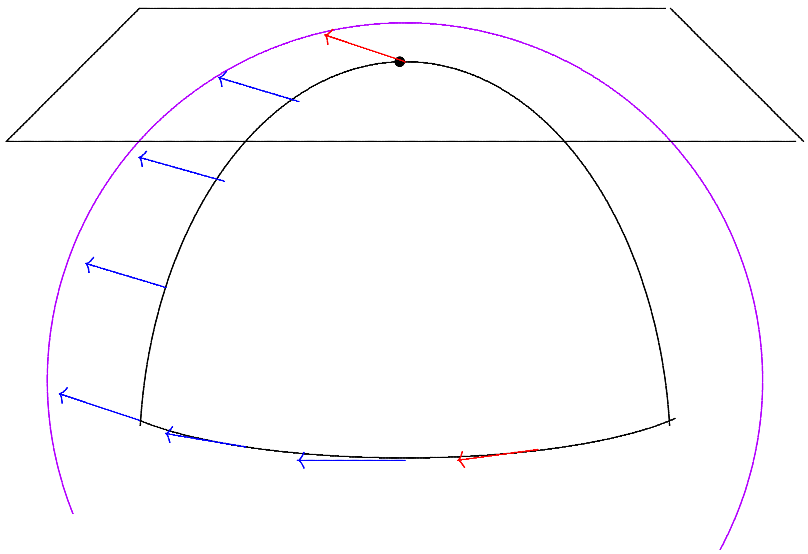

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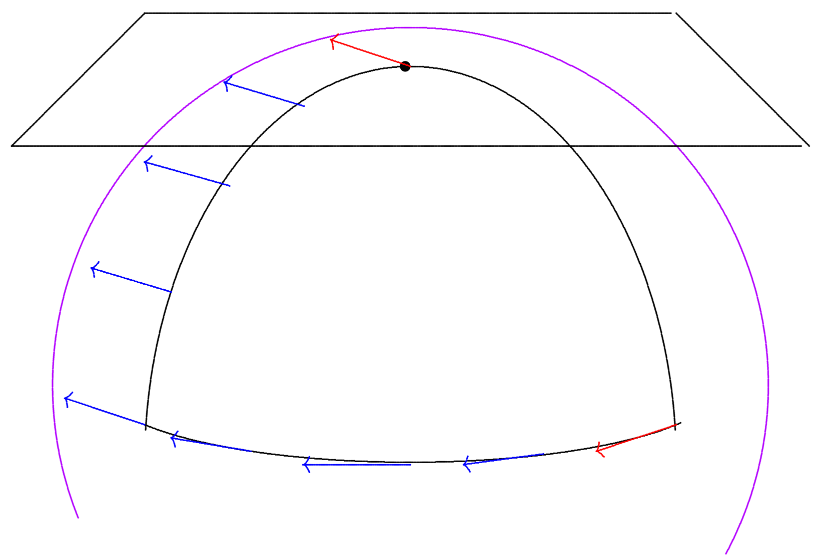

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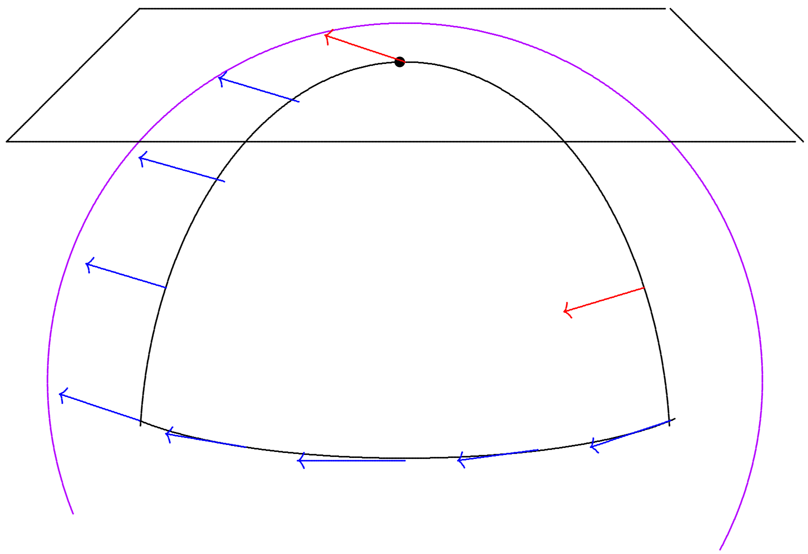

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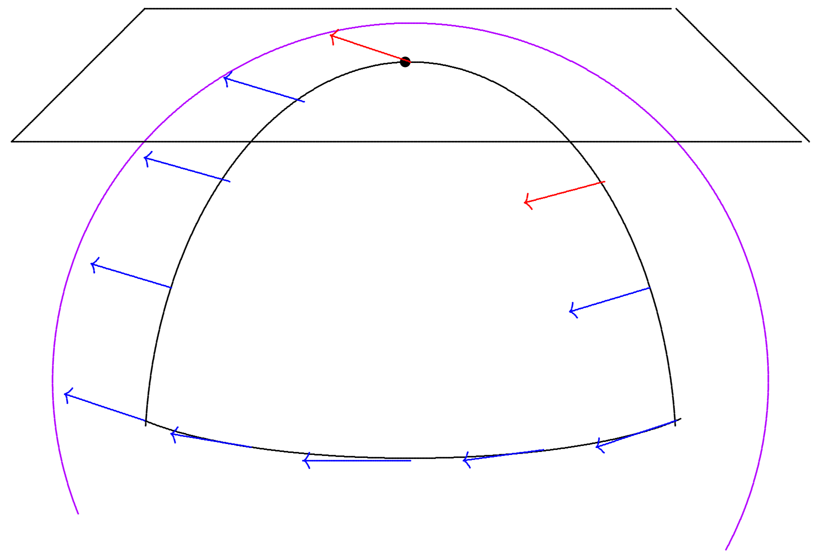

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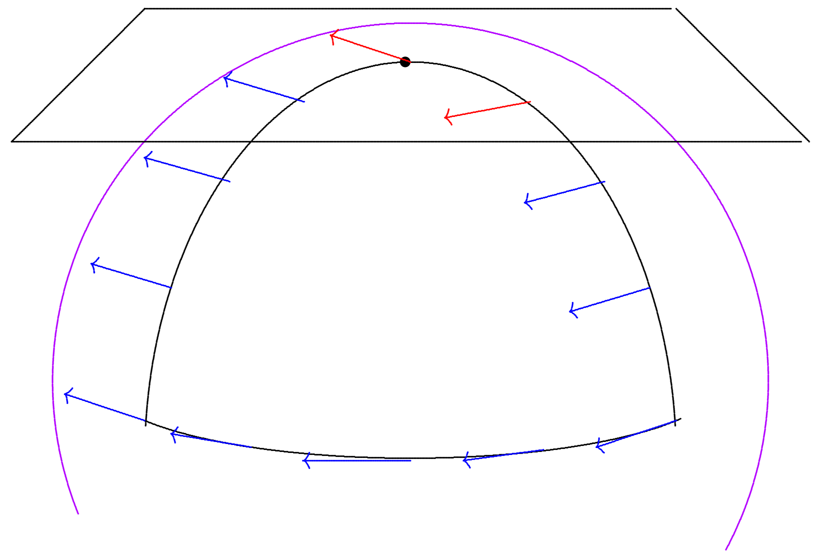

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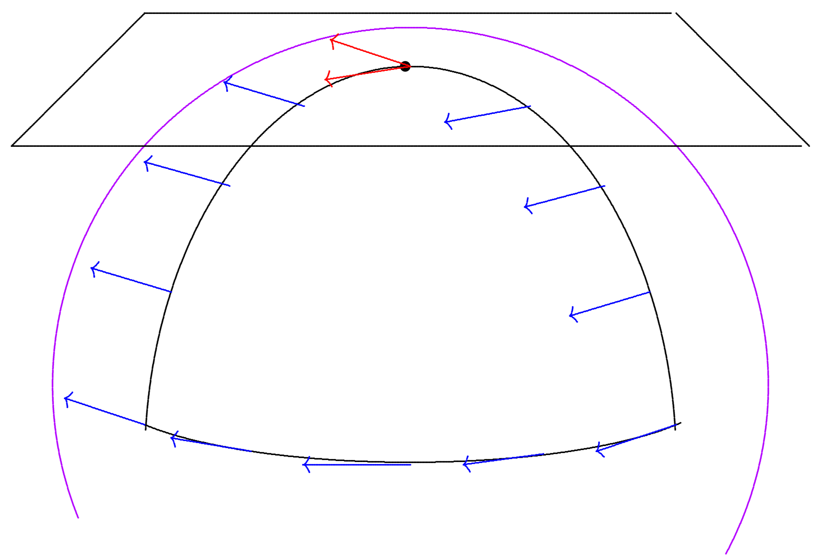

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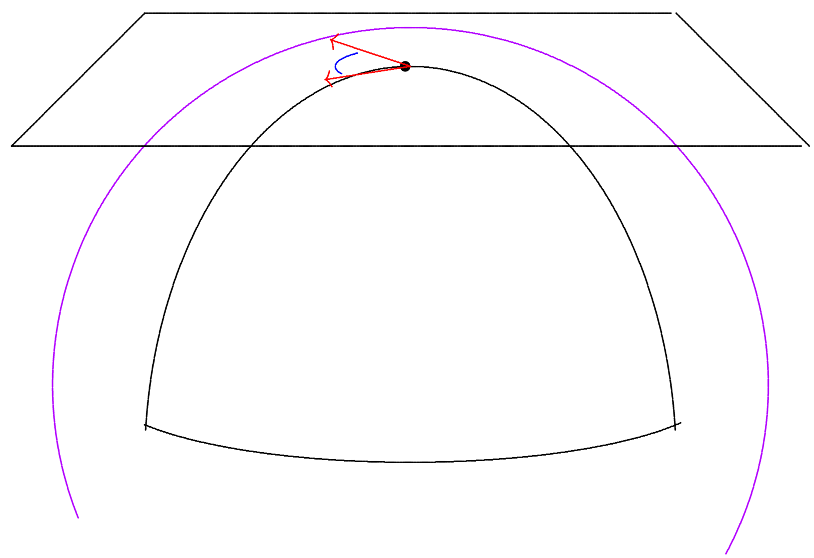
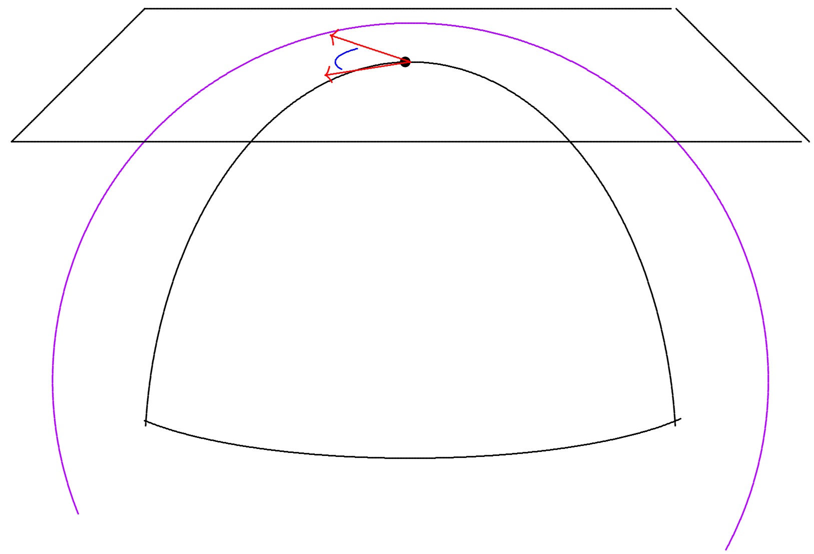


(M^n, g): holonomy $\subset \mathbf{O}(n)$



Kähler metrics:

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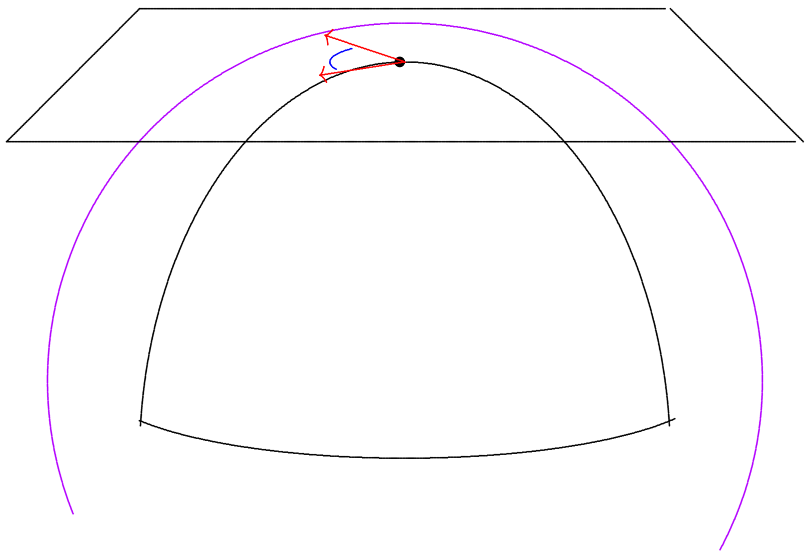

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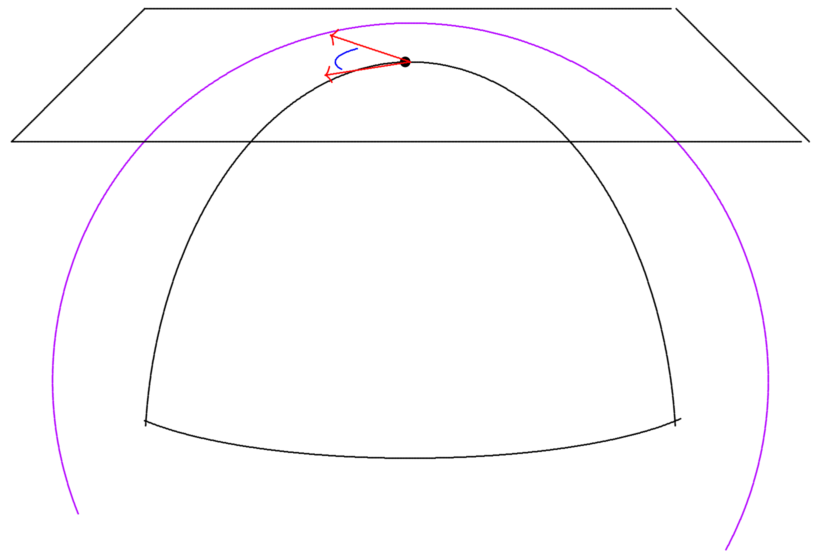

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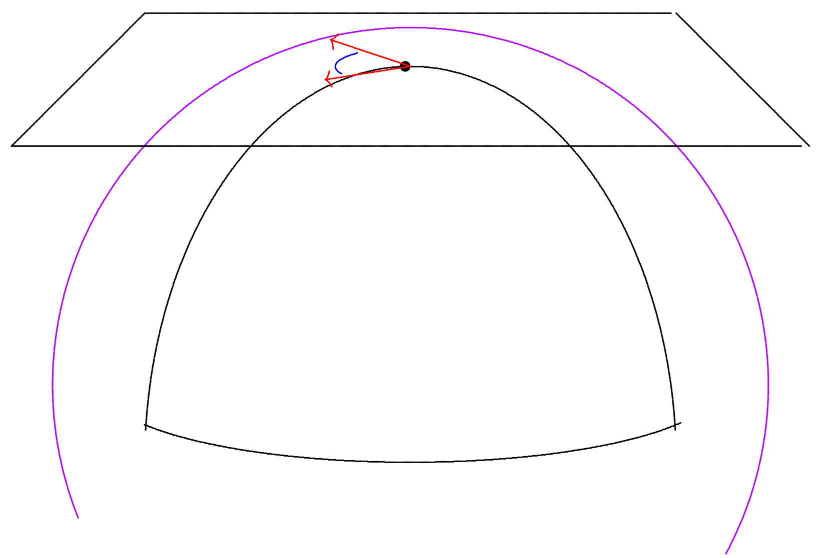

$$\mathbf{U}(m) := \mathbf{O}(2m) \cap \mathbf{GL}(m, \mathbb{C})$$

Ricci-flat Kähler metrics:

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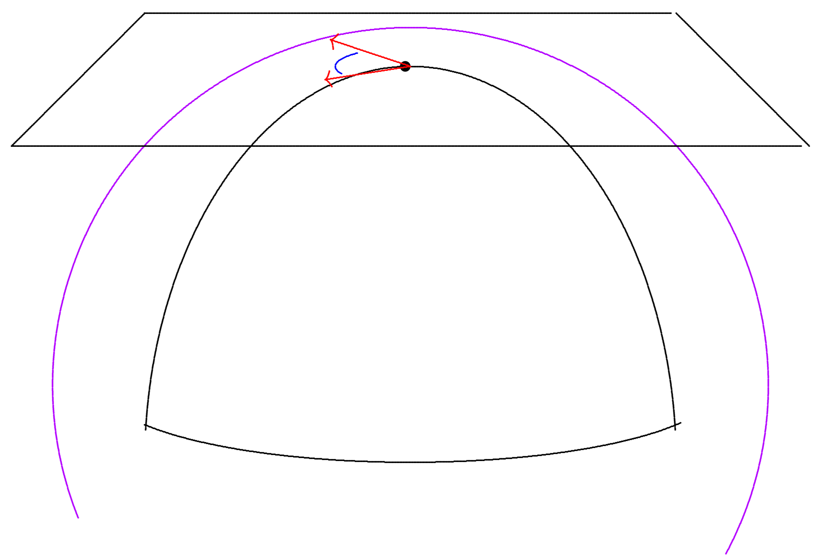

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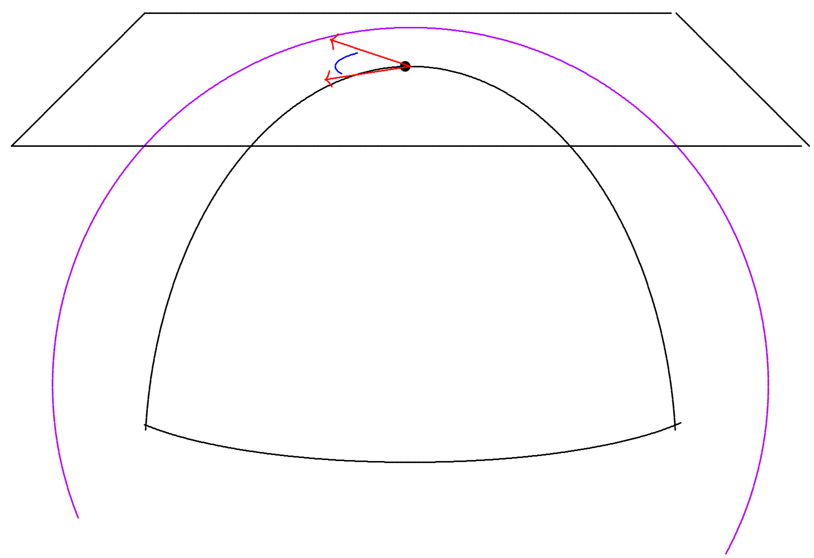

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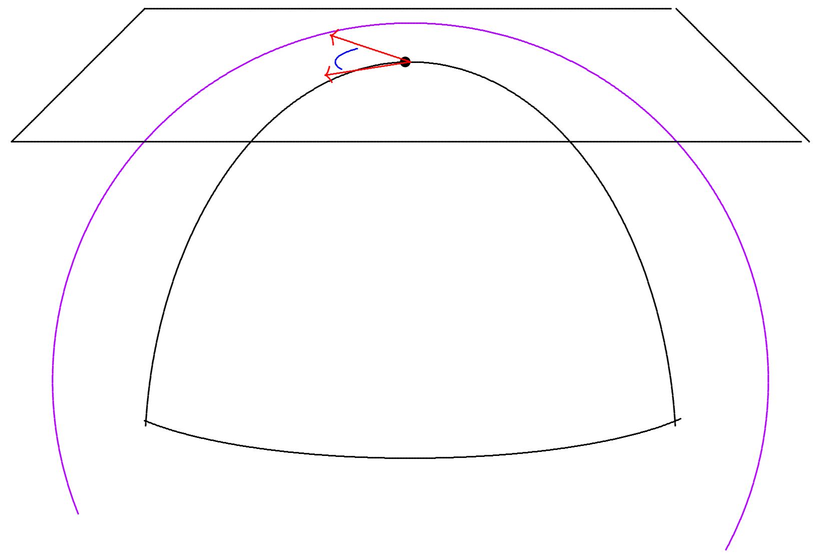
(M^{2m}, g): Ricci-flat Kähler $\Longleftarrow$ holonomy $\subset$ $\mathbf{SU}(m)$



$\mathbf{SU}(m) \subset \mathbf{U}(m)$: $\{A \mid \det A = 1\}$

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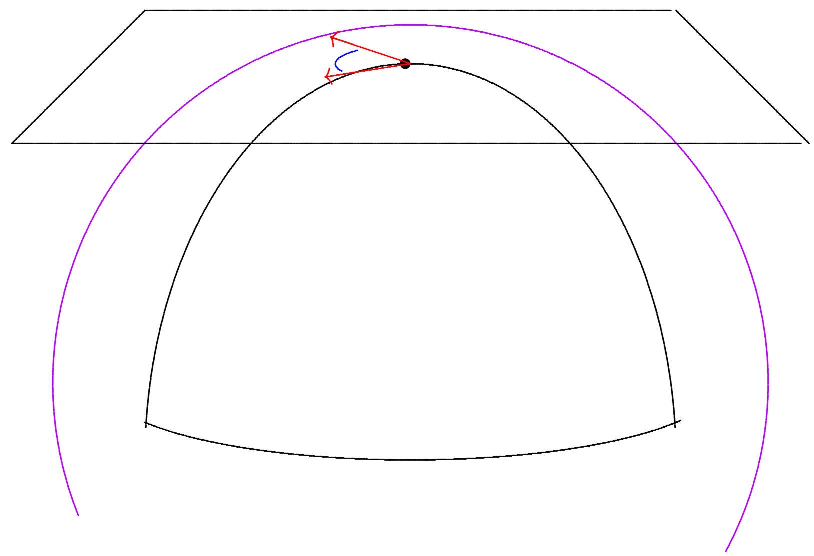

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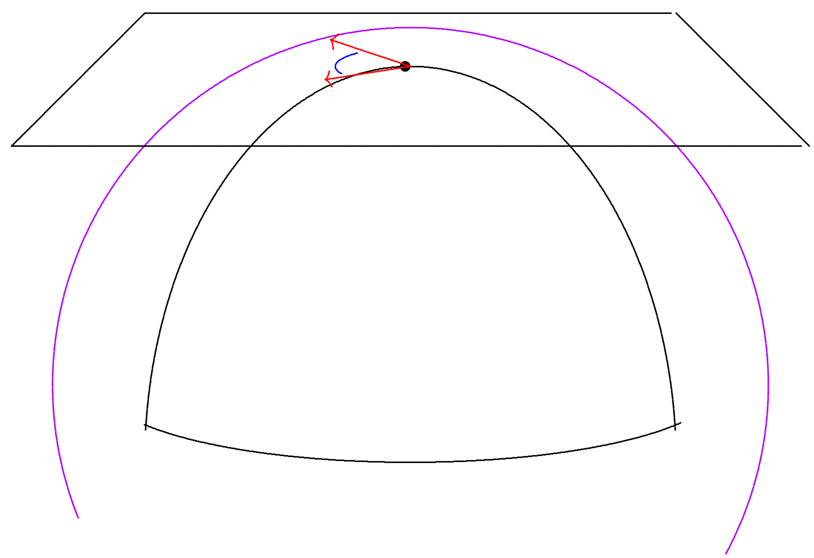

if M is simply connected.

Calabi-Yau metrics:

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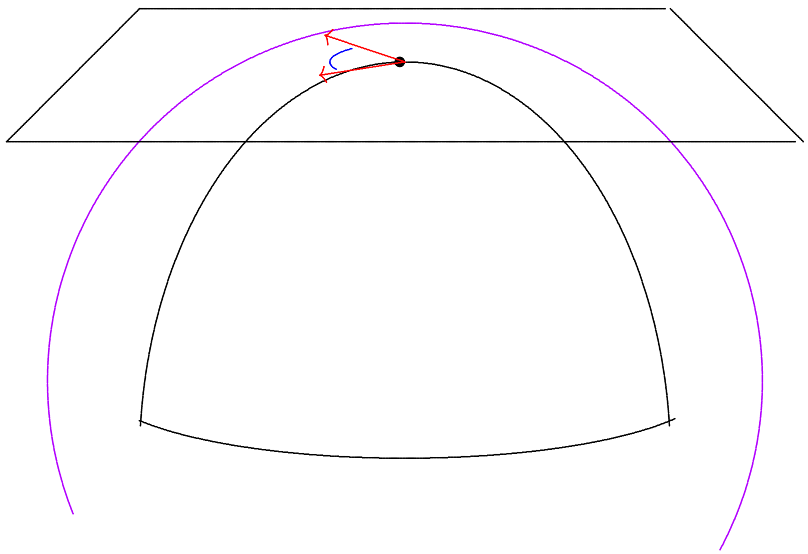

Hyper-Kähler metrics:

(M^{4k}, g)

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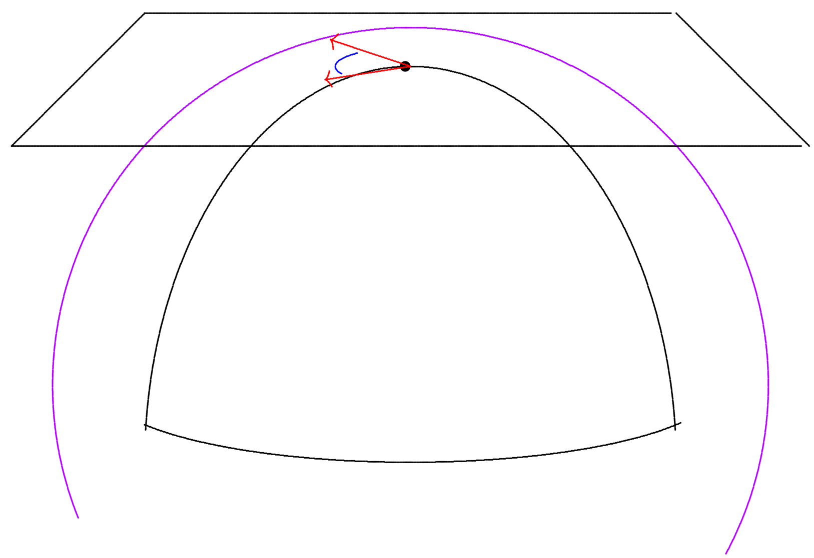

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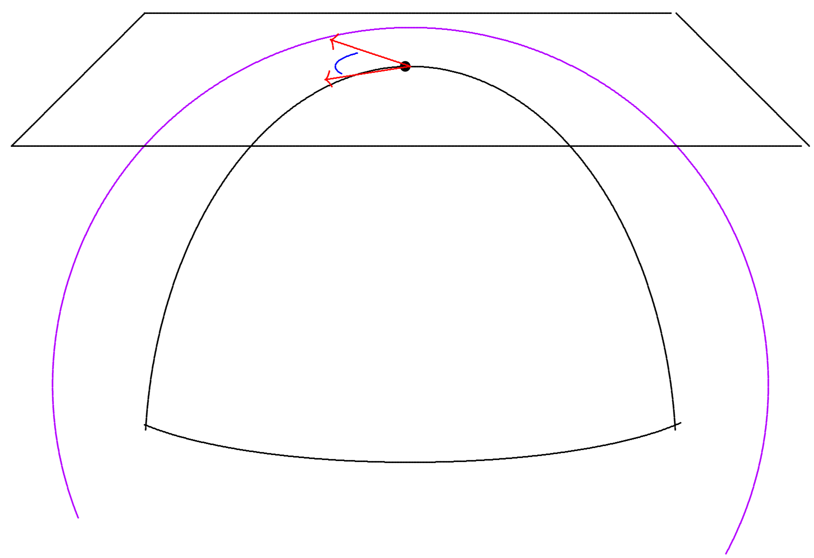

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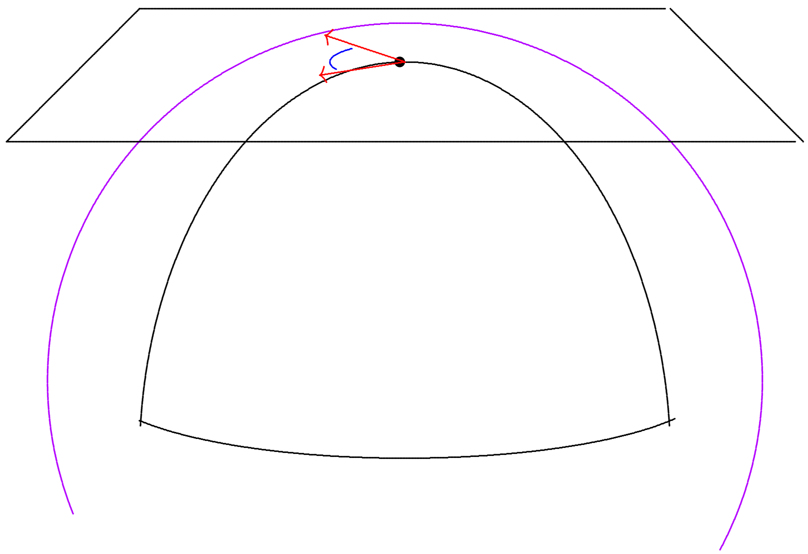

$$\mathbf{Sp}(k) := \mathbf{O}(4k) \cap \mathbf{GL}(\ell, \mathbb{H})$$

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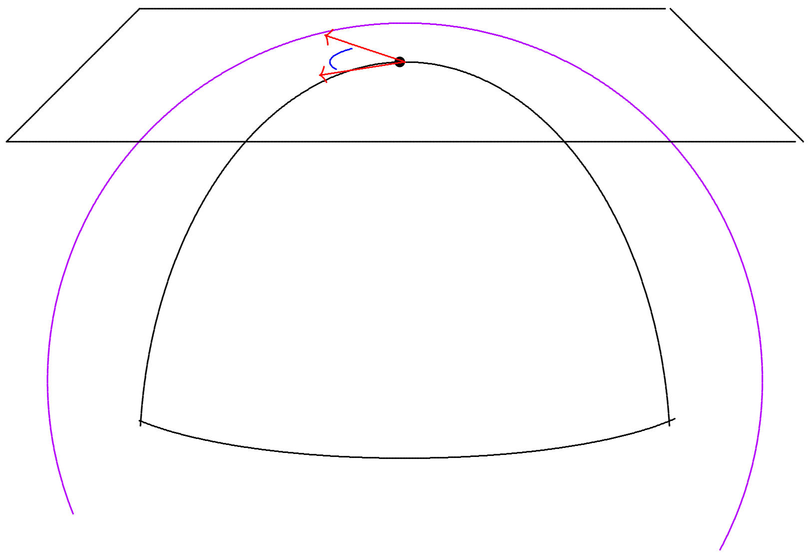

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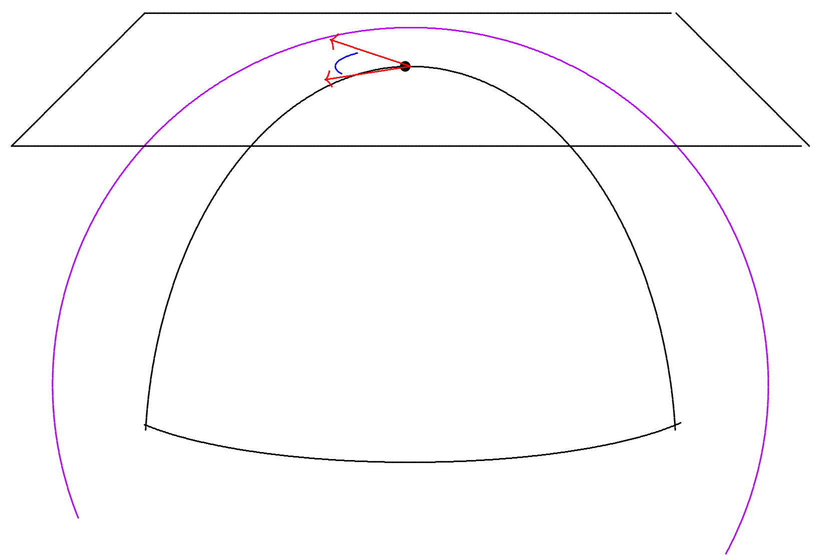

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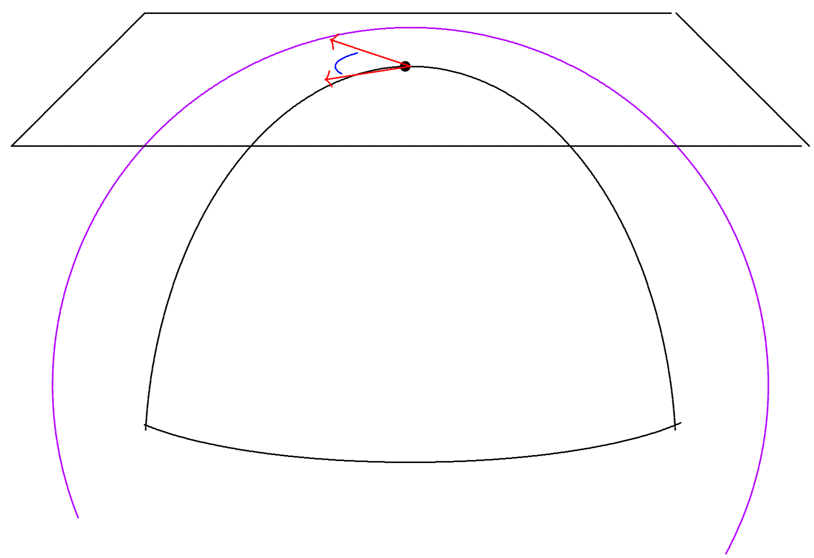

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in many ways! (For example, permute i, j, $k \ldots$)

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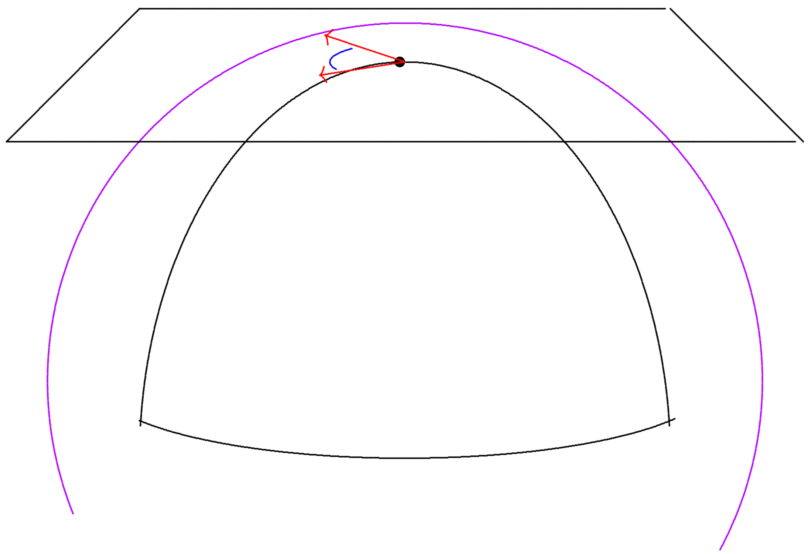

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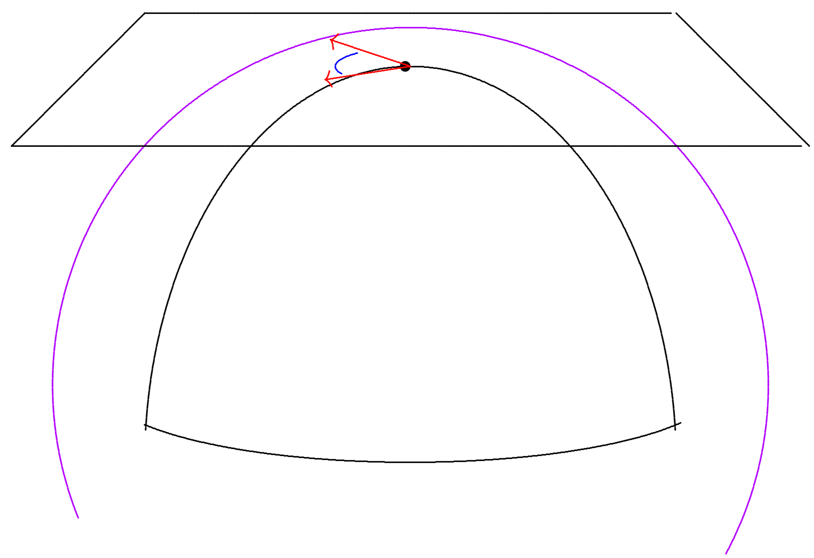

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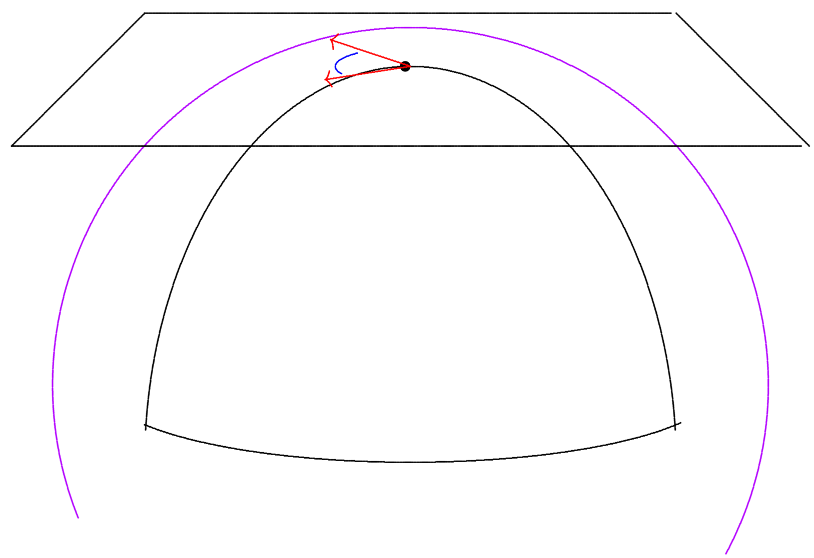

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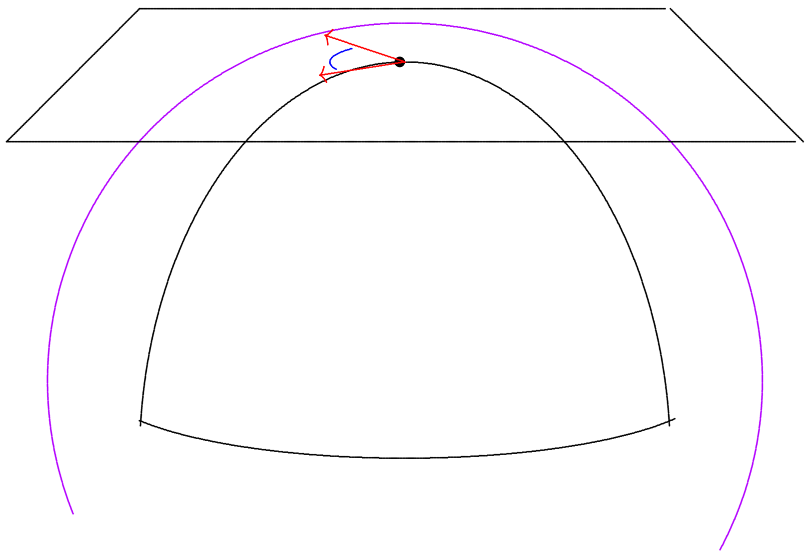

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For (M^4, g):

hyper-Kähler $\Longleftrightarrow$ Calabi-Yau.

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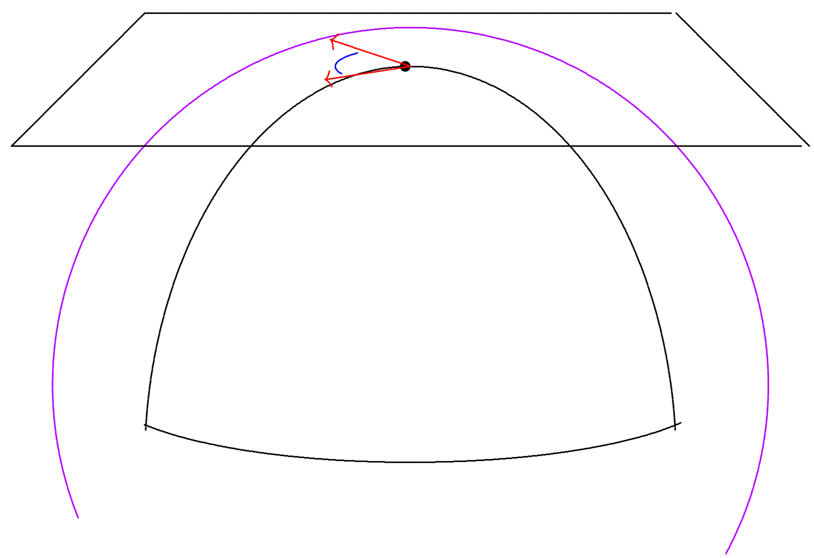

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When (M^4, g) simply connected:

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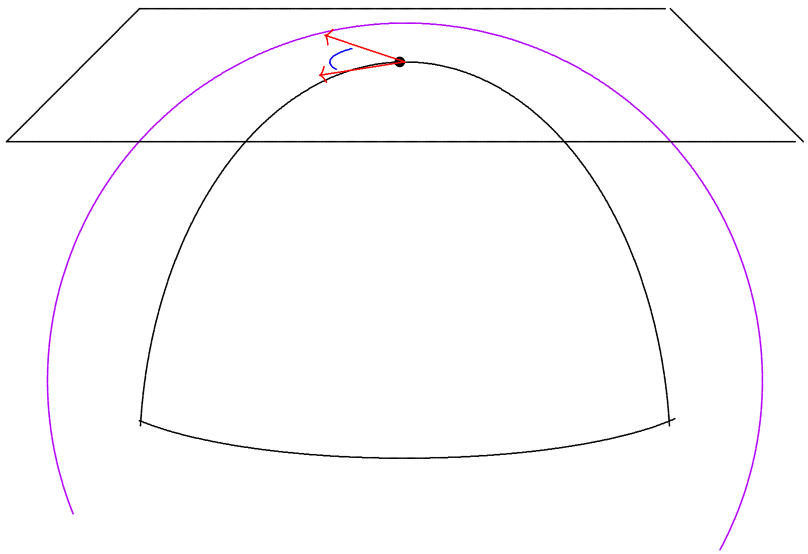

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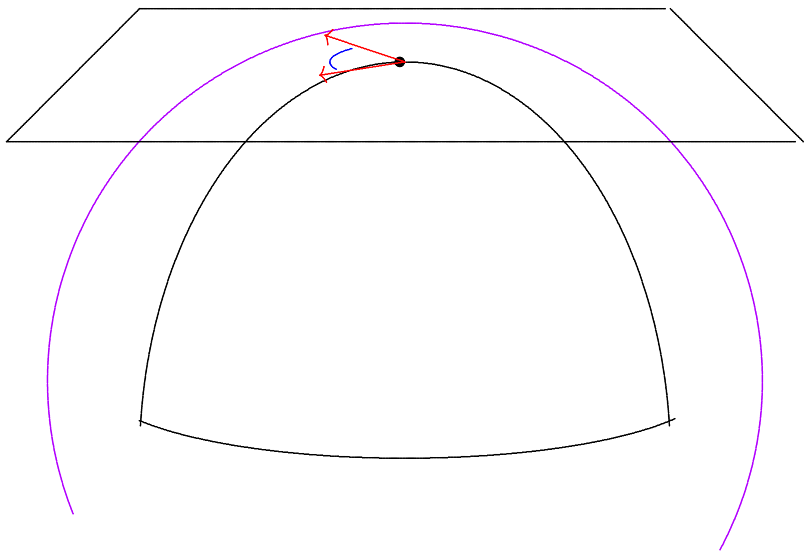

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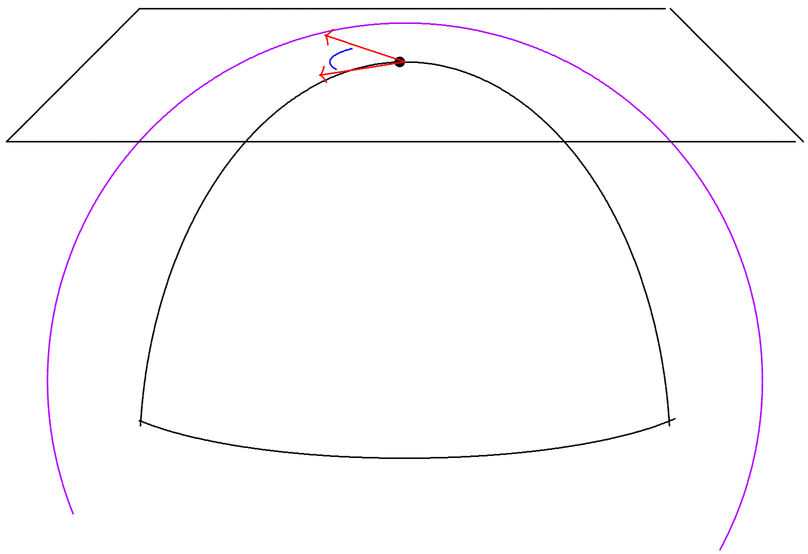

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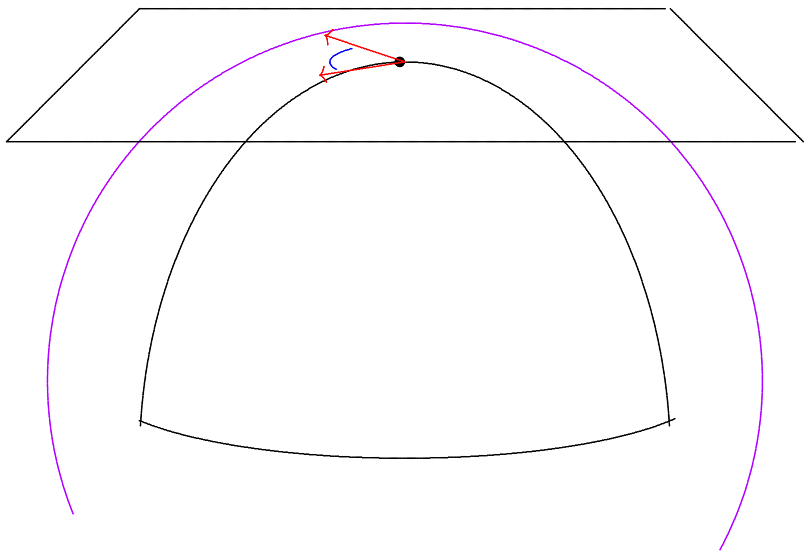
(M^{2m}, g): holonomy



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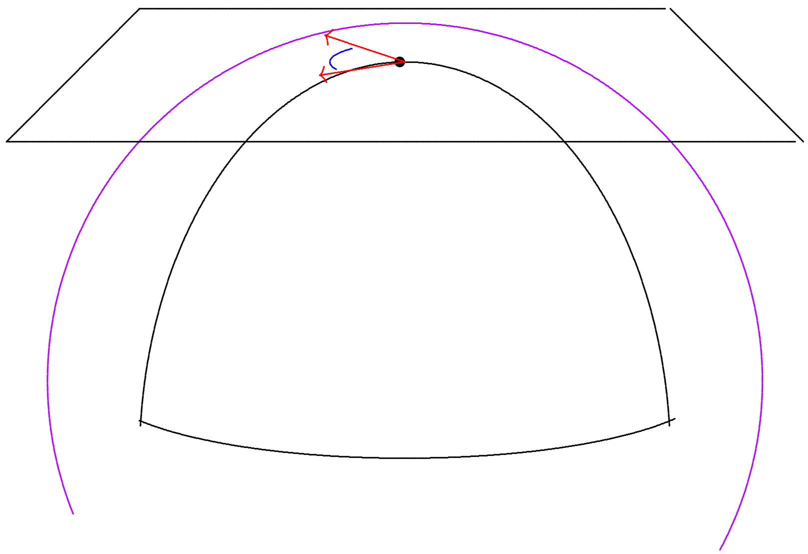

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"Kähler class"

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ω is also defines a symplectic structure on M.

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Some Suggestive Questions. *If* (M^4, ω) *is a symplectic* 4*-manifold, when does* M^4 *admit an Einstein metric* g *(unrelated to* ω*)? What if we also require* $\lambda \geq 0$*?*

Theorem (L '09)**.**

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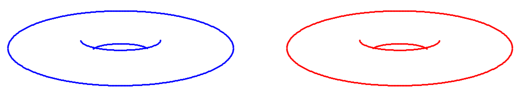

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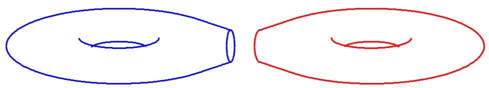

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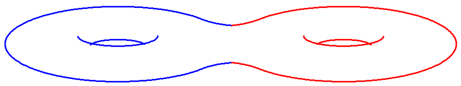

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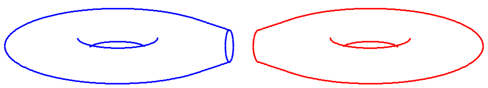

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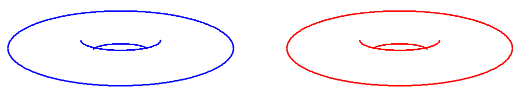

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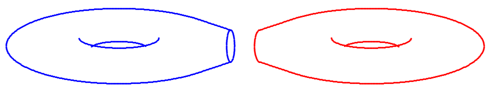

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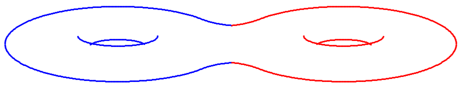

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Del Pezzo surfaces,
K3 surface, Enriques surface,
Abelian surface, Hyper-elliptic surfaces.

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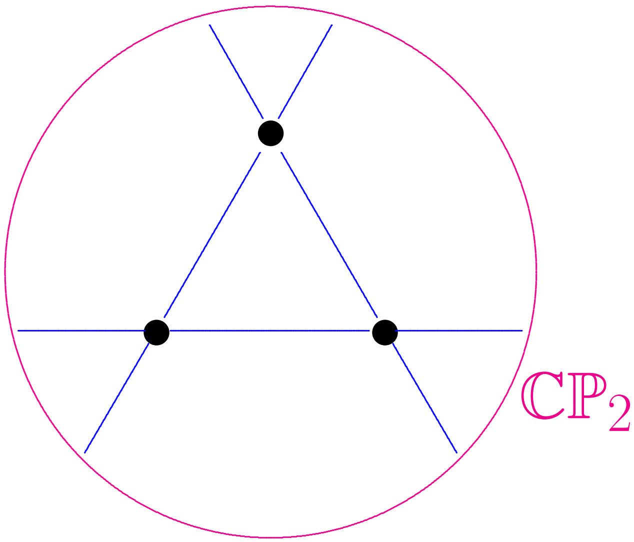

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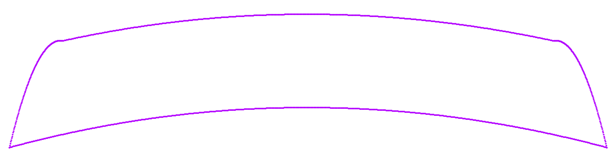
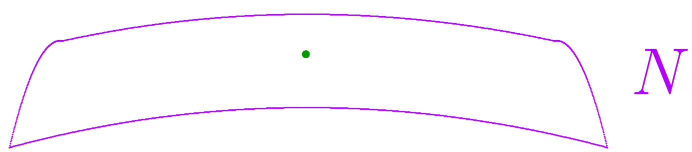
If N is a complex surface,



Blowing up:

If N is a complex surface, may replace $p \in N$



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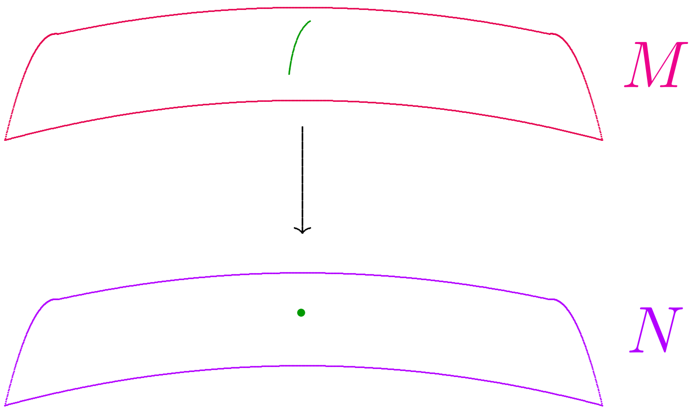

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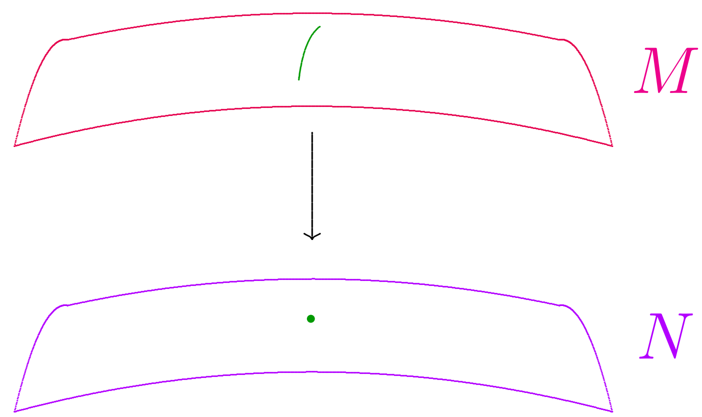
If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$



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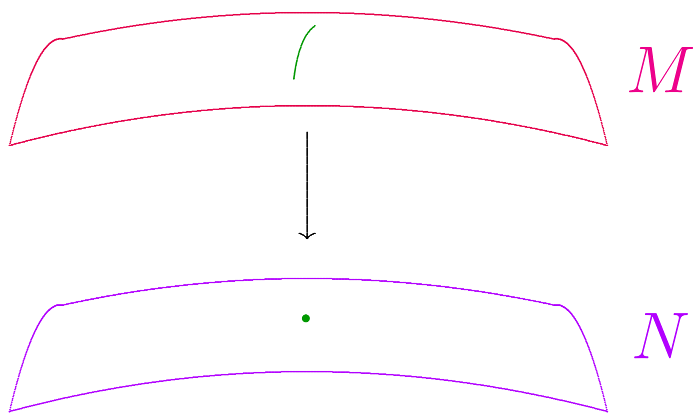
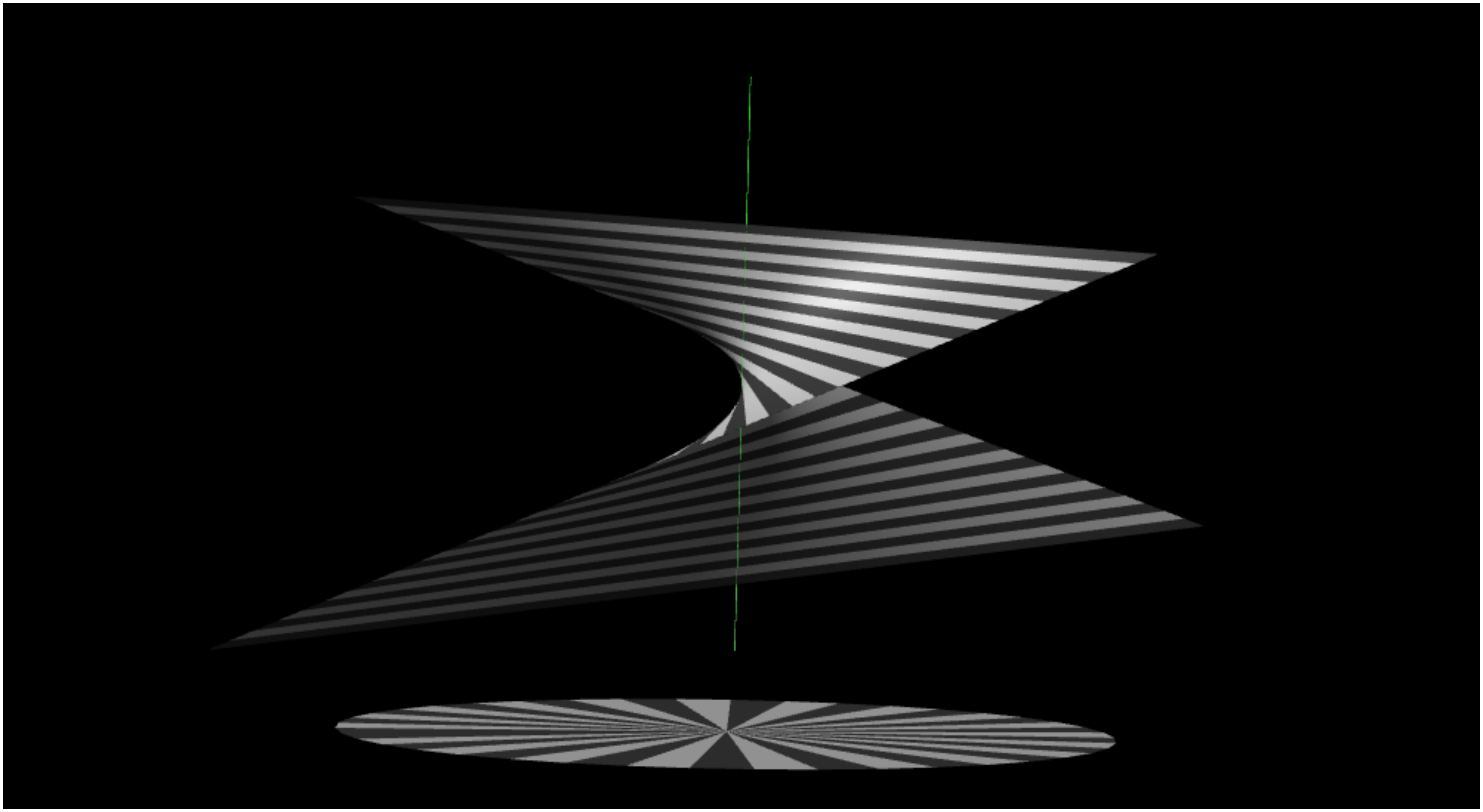
in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.





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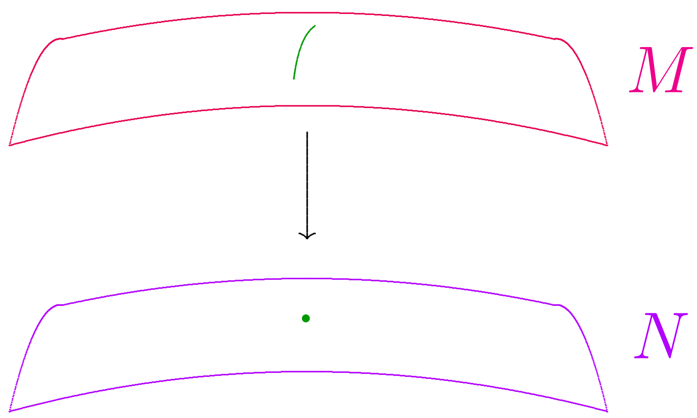

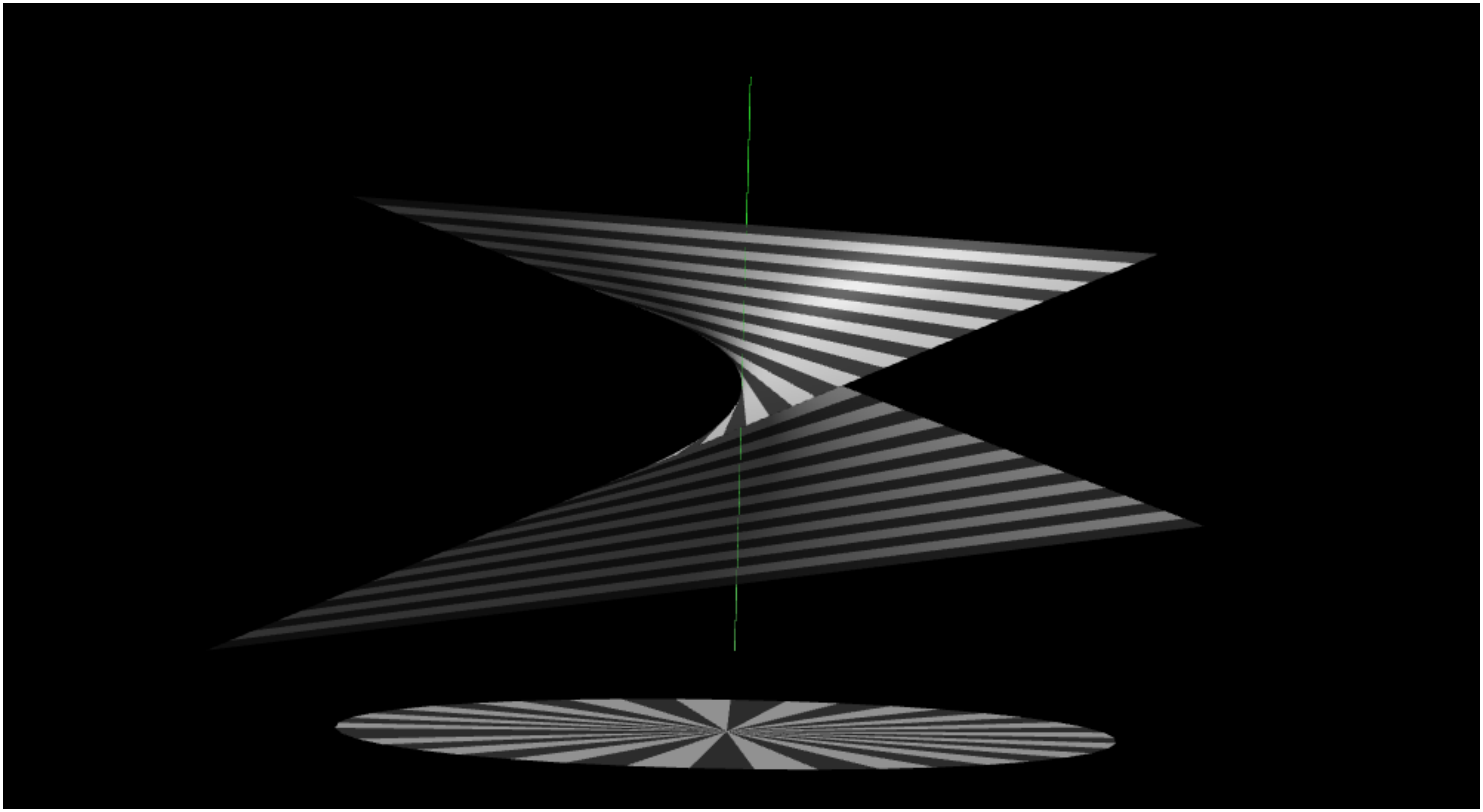

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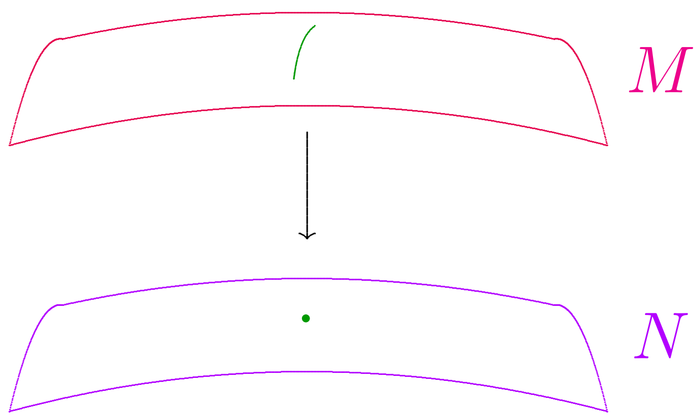

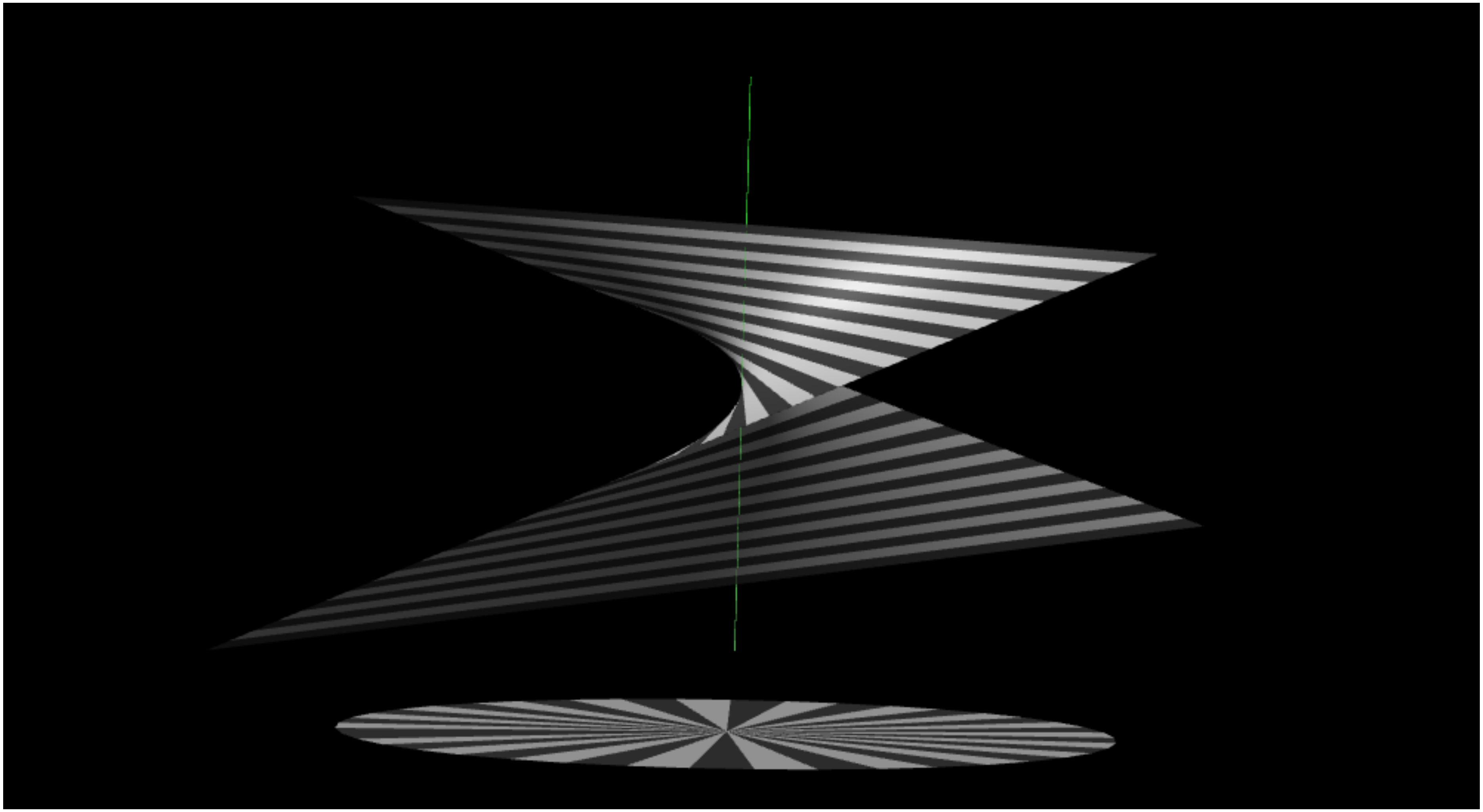

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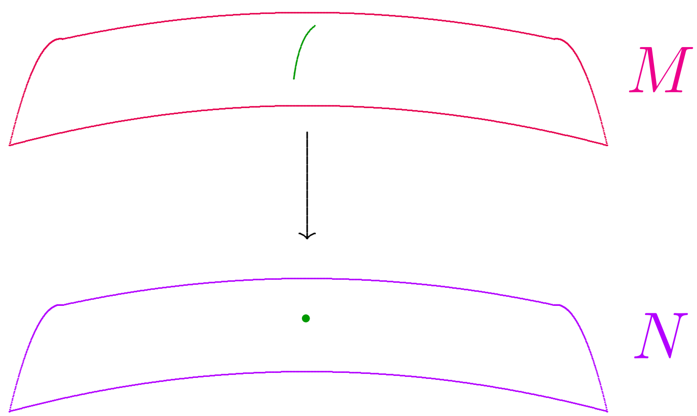
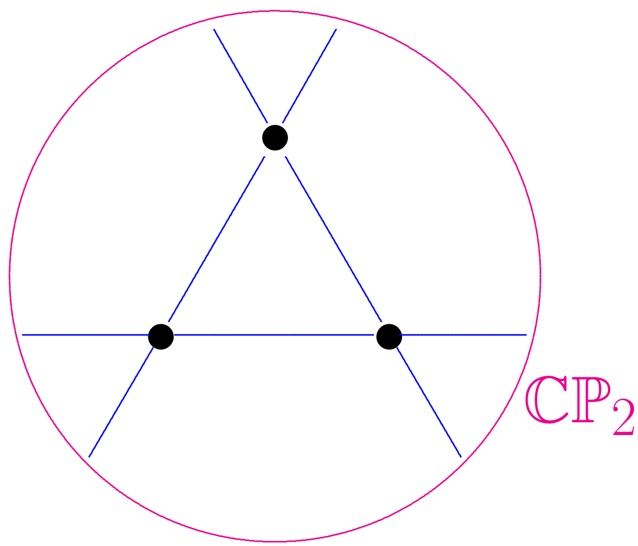


Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



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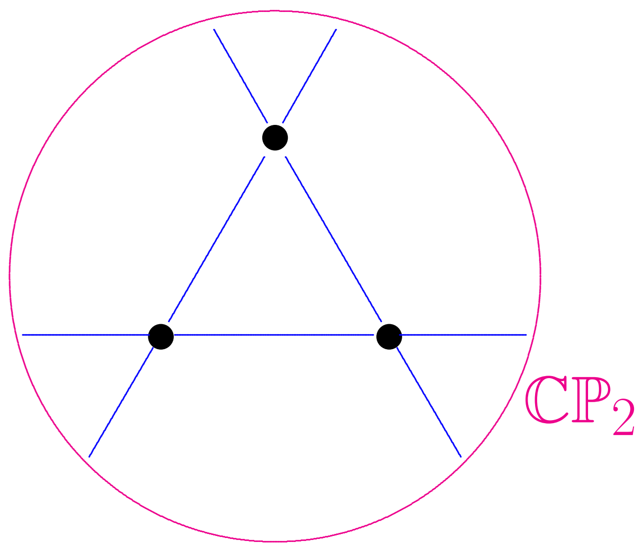

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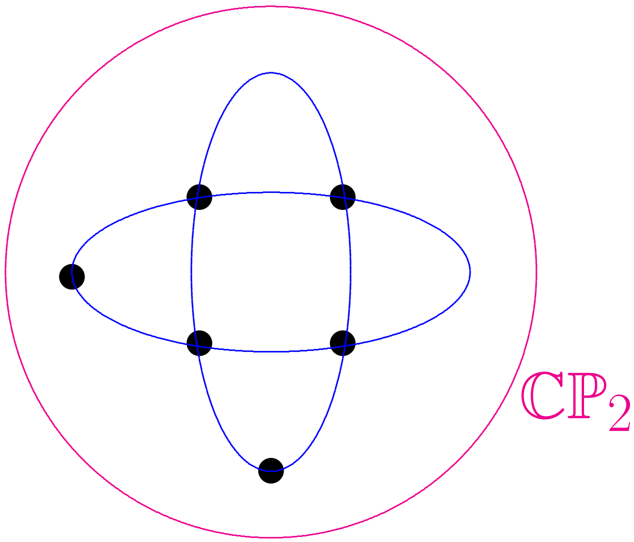

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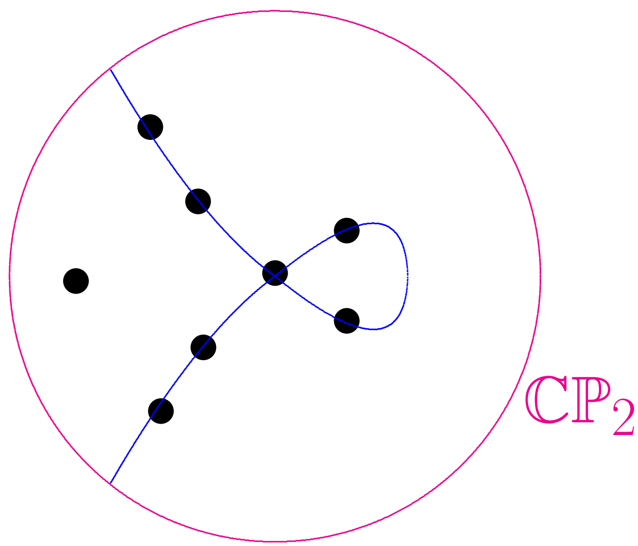

No 3 on a line, no 6 on conic, no 8 on nodal cubic.

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$$g = g(J\cdot, J\cdot)$$

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L '97: Such Einstein metrics are necessarily conformal to extremal Kähler metrics, in the sense of Calabi.

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Existence: Page

1978

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Existence: Page-Derdziński,

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Existence: Page-Derdziński, Siu, Tian-Yau, Tian,

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Existence: Page-Derdziński, Siu, Tian-Yau, Tian, Odaka-Spotti-Sun,

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Nice characterizations of known Einstein metrics.

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Peng Wu proposed one beautiful characterization, in terms of an open condition on

$$W_+ : \Lambda^+ \to \Lambda^+.$$

Wu's criterion:

$$\det(W_+) > 0.$$

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Corollary. *Every simply-connected compact oriented Einstein (M^4, h) with $\det(W_+) > 0$ is diffeomorphic to a del Pezzo surface.*

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Corollary. *Every simply-connected compact oriented Einstein* (M^4, h) *with* $\det(W_+) > 0$ *is diffeomorphic to a del Pezzo surface. Conversely, every del Pezzo* M^4 *carries Einstein* h *with* $\det(W_+) > 0$*, and these sweep out exactly one connected component of moduli space* $\mathscr{E}(M)$*.*

We've focused on compact Einstein manifolds.

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But non-compact, complete solutions are often key to proving theorems about compact ones.

Joint work with

Olivier Biquard
Sorbonne Université

and

Paul Gauduchon
École Polytechnique

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e-print:
arXiv:2310.14387 [math.DG]

Definition. *A gravitational instanton is a*

Definition. *A gravitational instanton is a complete,*

Definition. *A gravitational instanton is a complete, non-compact,*

Definition. *A gravitational instanton is a complete, non-compact, non-flat,*

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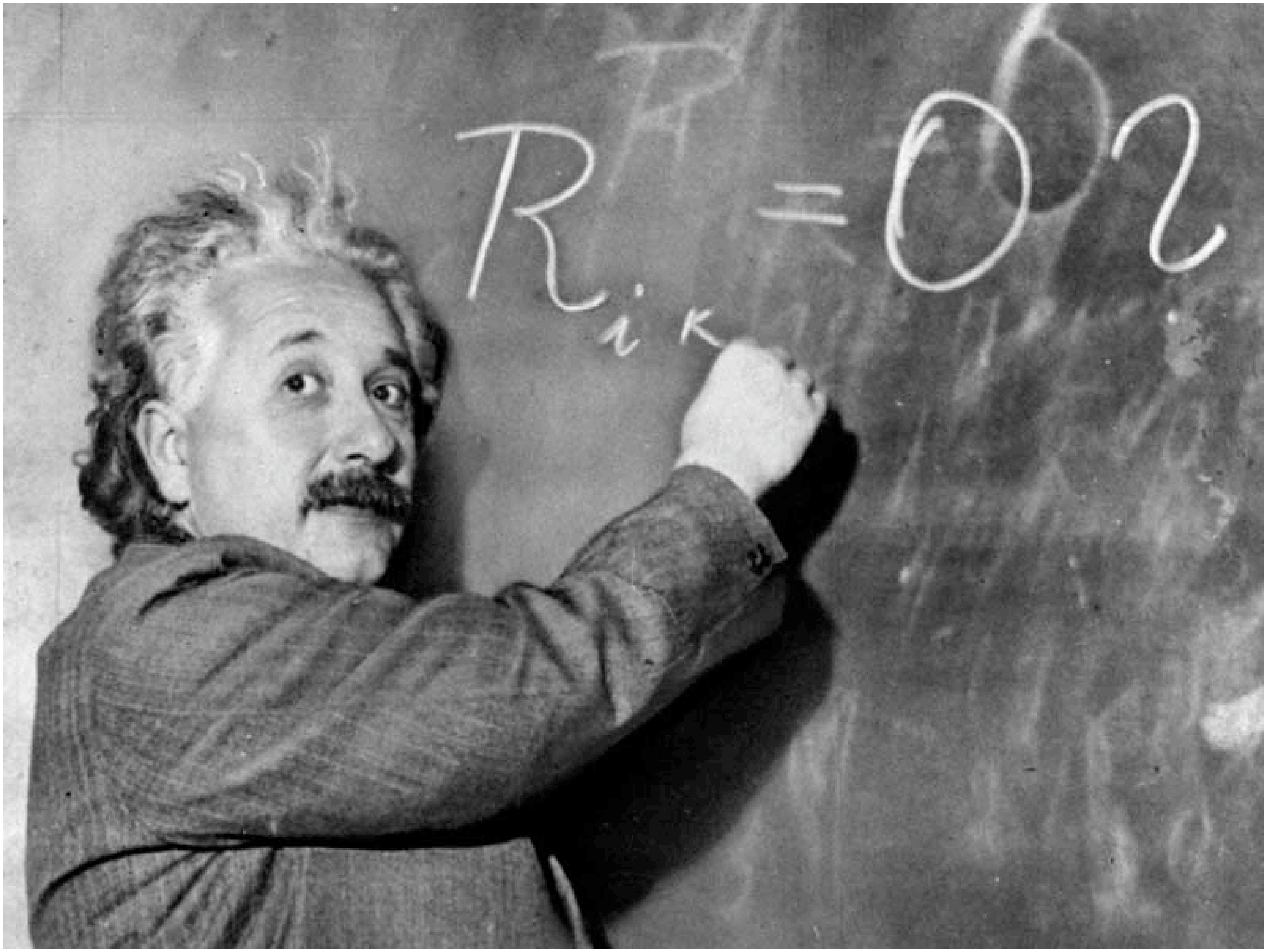

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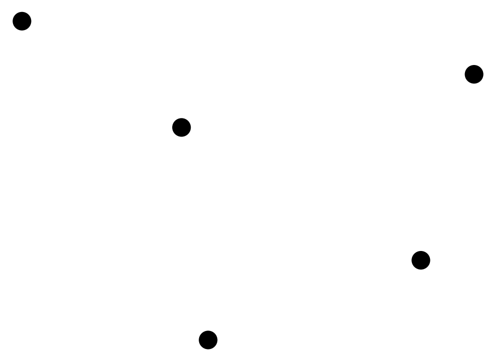
Data: ℓ points in $\mathbb{R}^3$ and a constant $\kappa^2 \geq 0$.



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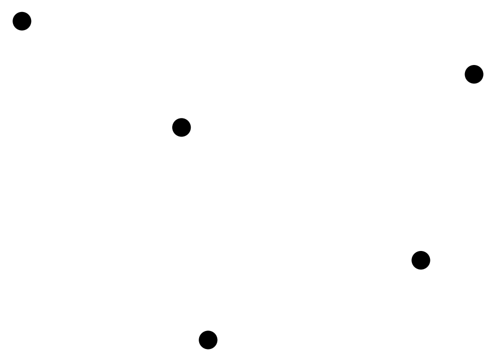
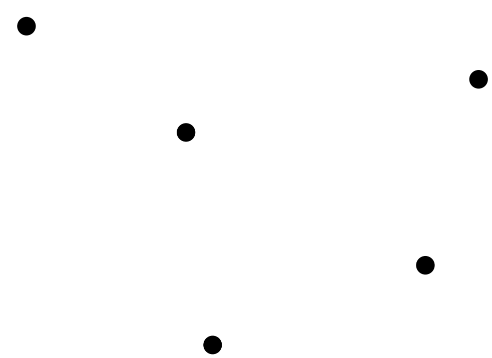
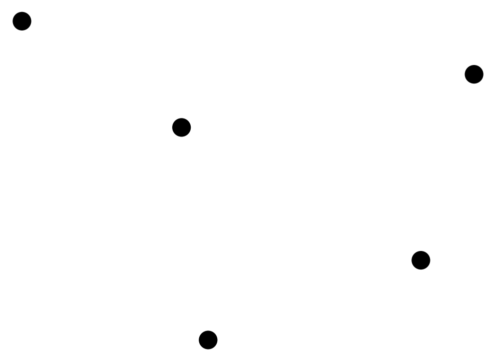


Data: ℓ points in $\mathbb{R}^3$ and κ^2



Data: ℓ points in $\mathbb{R}^3$ and $\kappa^2 \Longrightarrow V$ with $\Delta V = 0$

$$V = \kappa^2 + \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$



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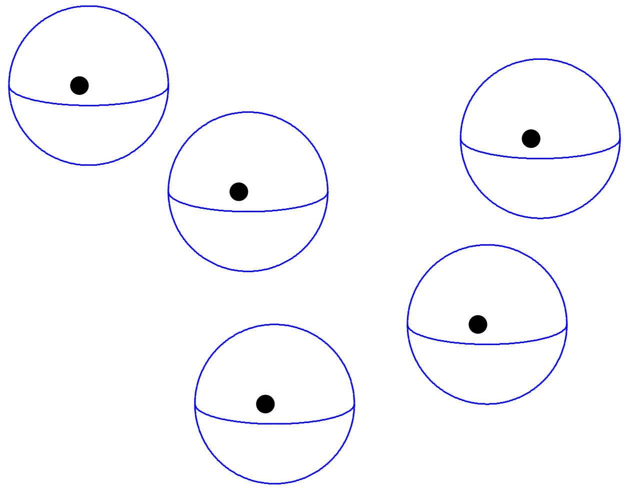

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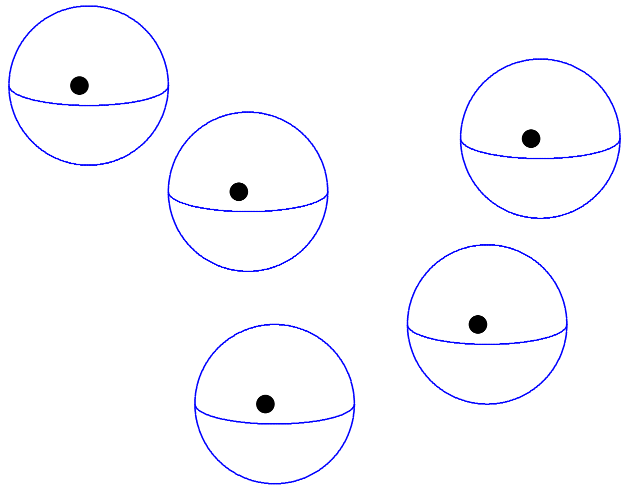

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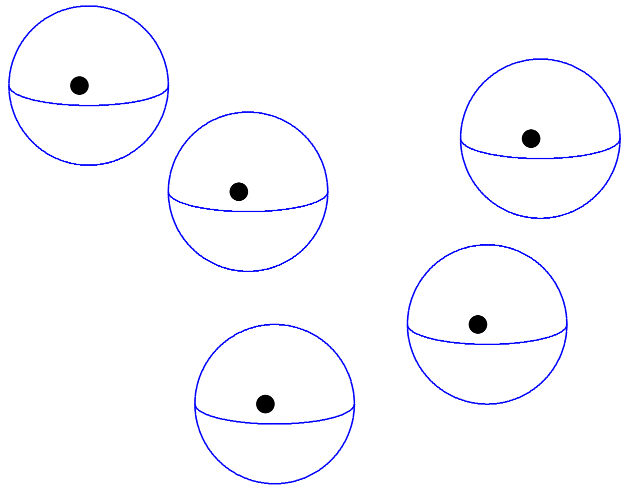

Data: ℓ points in $\mathbb{R}^3$. $\Longrightarrow V$ with $\Delta V = 0$

$$g = Vh + V^{-1}\theta^2$$

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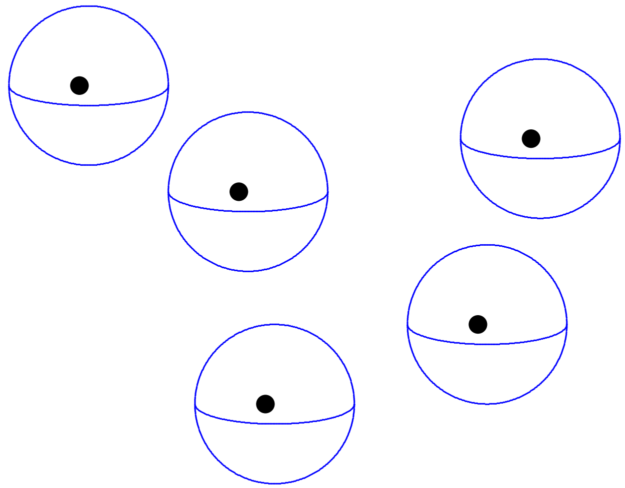

Data: ℓ points in $\mathbb{R}^3$. $\Longrightarrow V$ with $\Delta V = 0$

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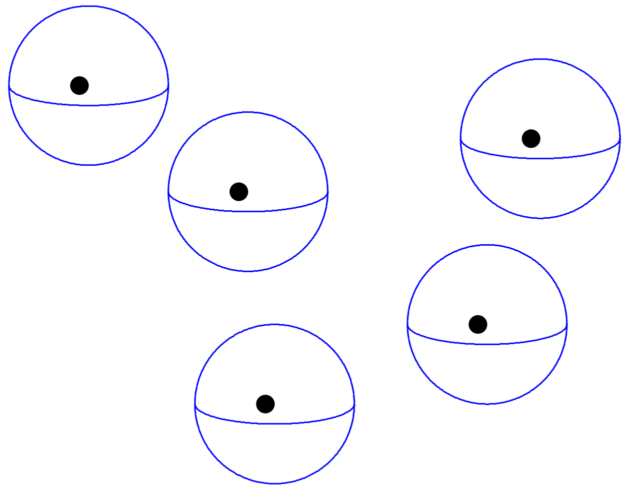

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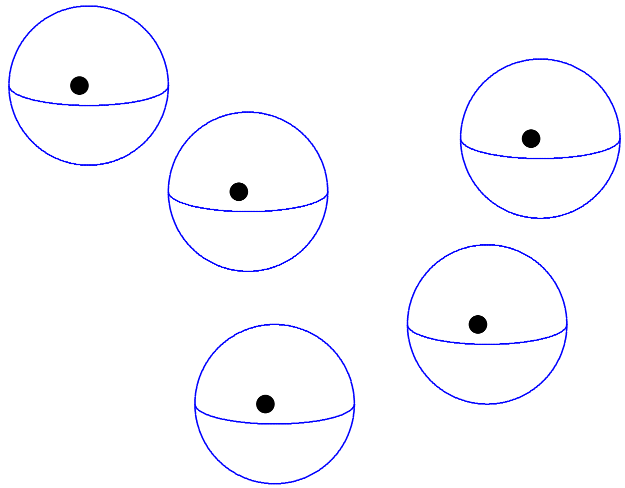

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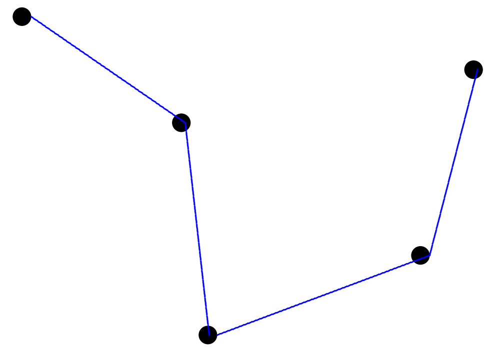

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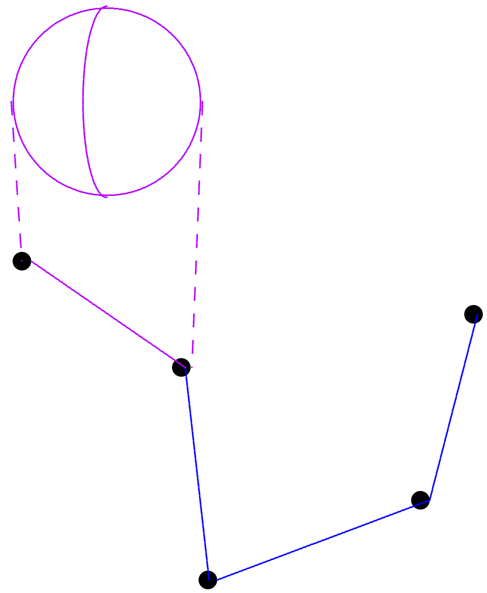

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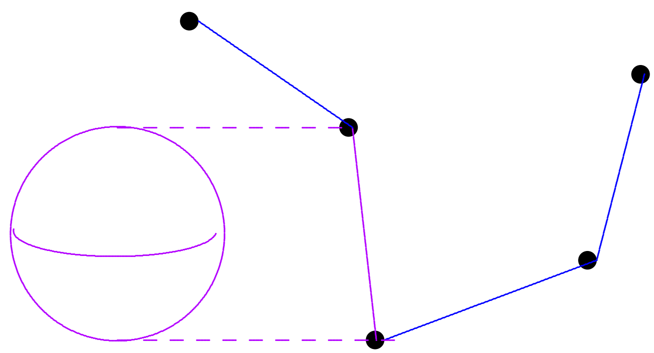

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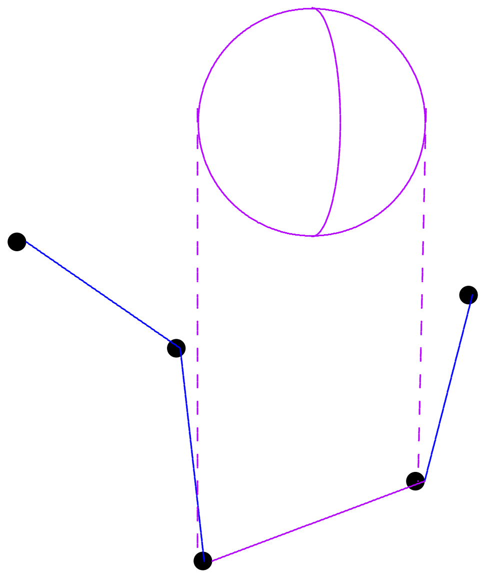

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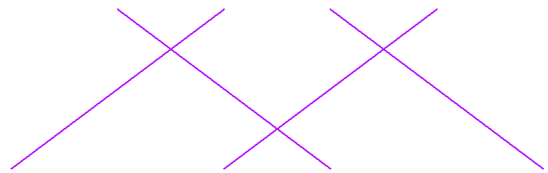

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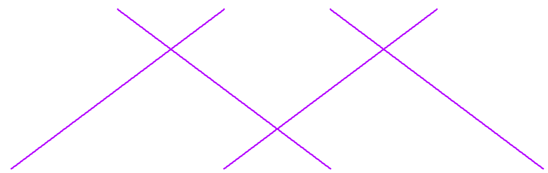

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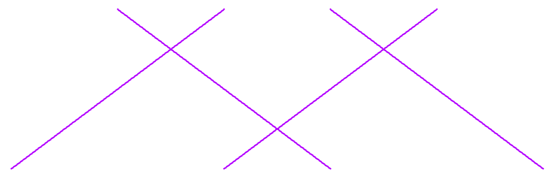

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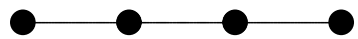

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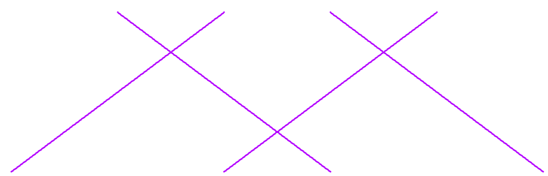

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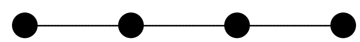

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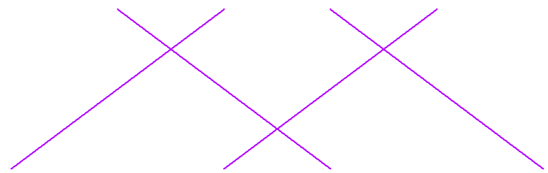

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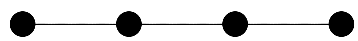

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Plumb together k copies of T^*S^2
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Gibbons and Hawking were unaware of all this!

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This last property distinguishes the ALF spaces from other classes of gravitational instantons:

ALG, ALH, ALG*, ALH*, . . .

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This J determines opposite orientation from the hyper-Kähler complex structures.

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Non-Kähler, but conformally Kähler!

Hawking also explored non-hyper-Kähler examples. . .

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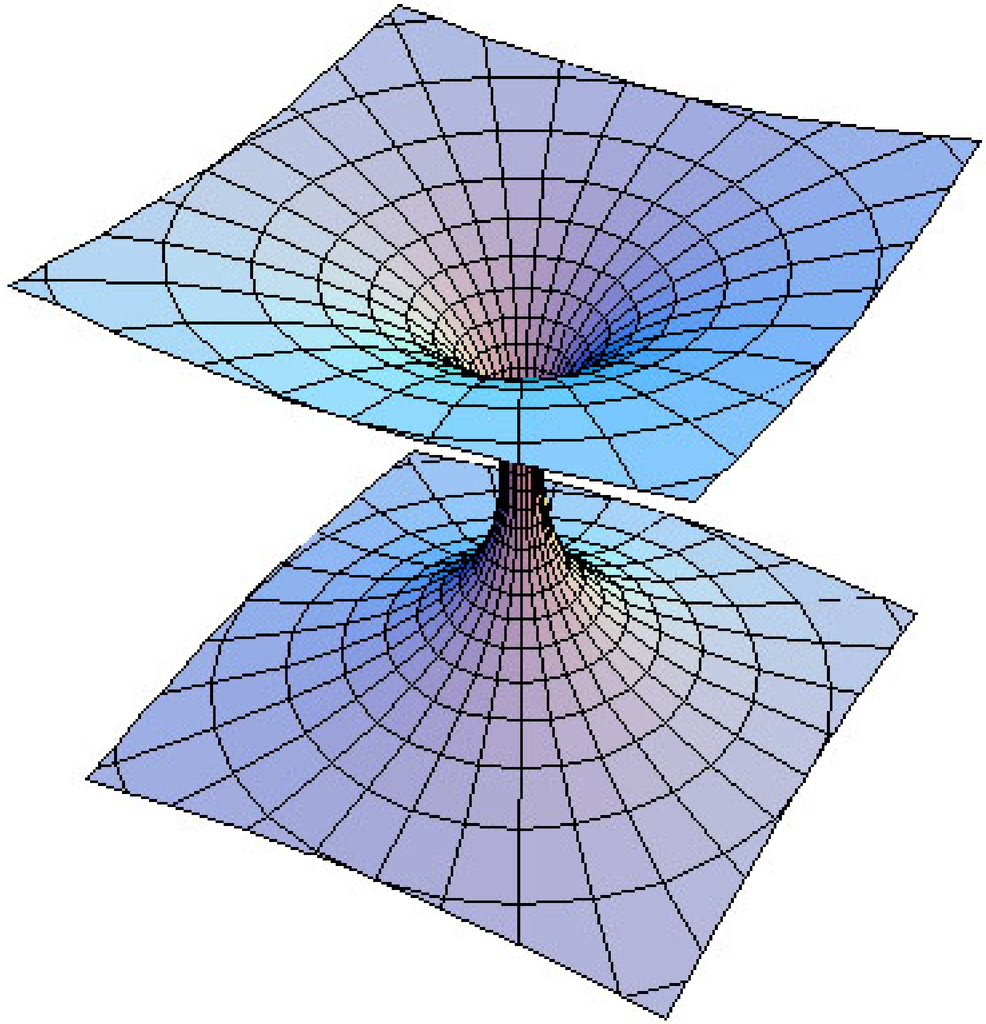
This makes g into a Ricci-flat metric on $\mathbb{R}^2 \times S^2$.
Makes h into extremal Kähler metric on $\mathbb{C} \times \mathbb{CP}_1$.



$$\mathbb{R} \times S^2 \subset \mathbb{R}^2 \times S^2$$

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This might lend some credence to the aphorism…

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$$\Longrightarrow \quad \mathrm{Vol}(B_\rho) \sim \mathrm{const} \cdot \rho^3$$

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Y. Chen & E. Teo, 2011

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Each of the metrics g in question is conformal to a complete extremal Kähler metric with $s > 0$.

This implies that they always satisfy Peng Wu's criterion

$$\det(W^+) > 0,$$

allowing one to generalize methods first explored in the compact case.

Theorem A.

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This optimal result combines **Theorem A** with a result of Mingyang Li, `arXiv:2310.13197`.

Thanks for the invitation!

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