Four-Manifolds,

Einstein Metrics, &

Differential Topology

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Stony Brook University

IMPA, 6/11/13

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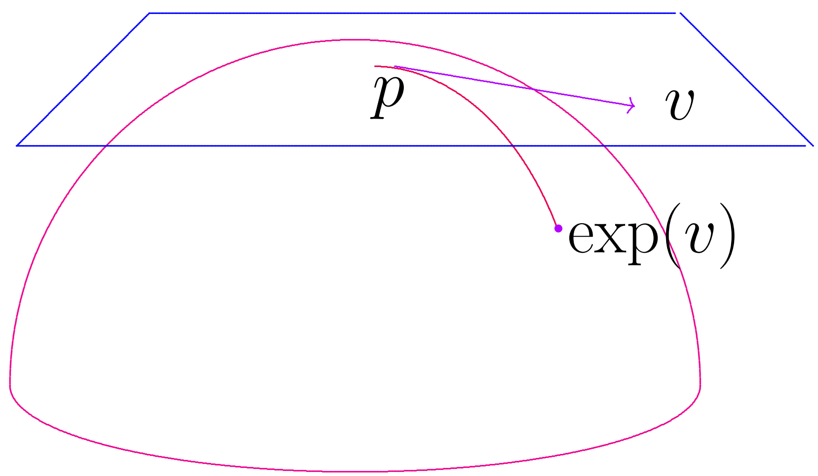

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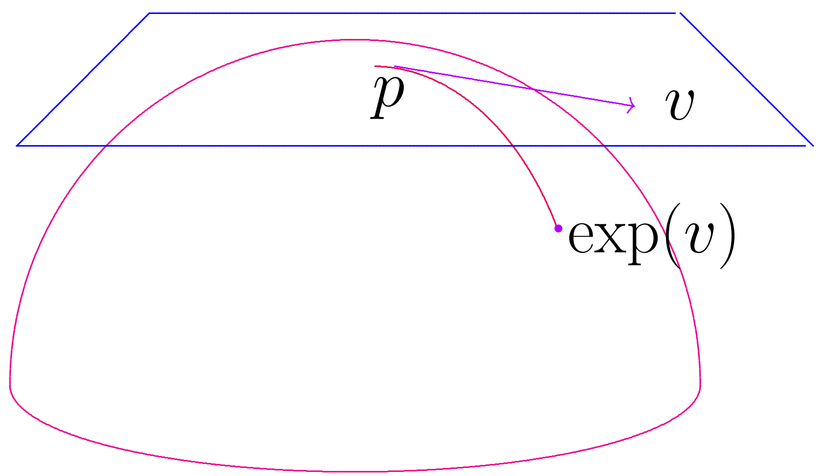

Now choosing $T_pM \xrightarrow{\cong} \mathbb{R}^n$ via some orthonormal basis gives us special coordinates on M.

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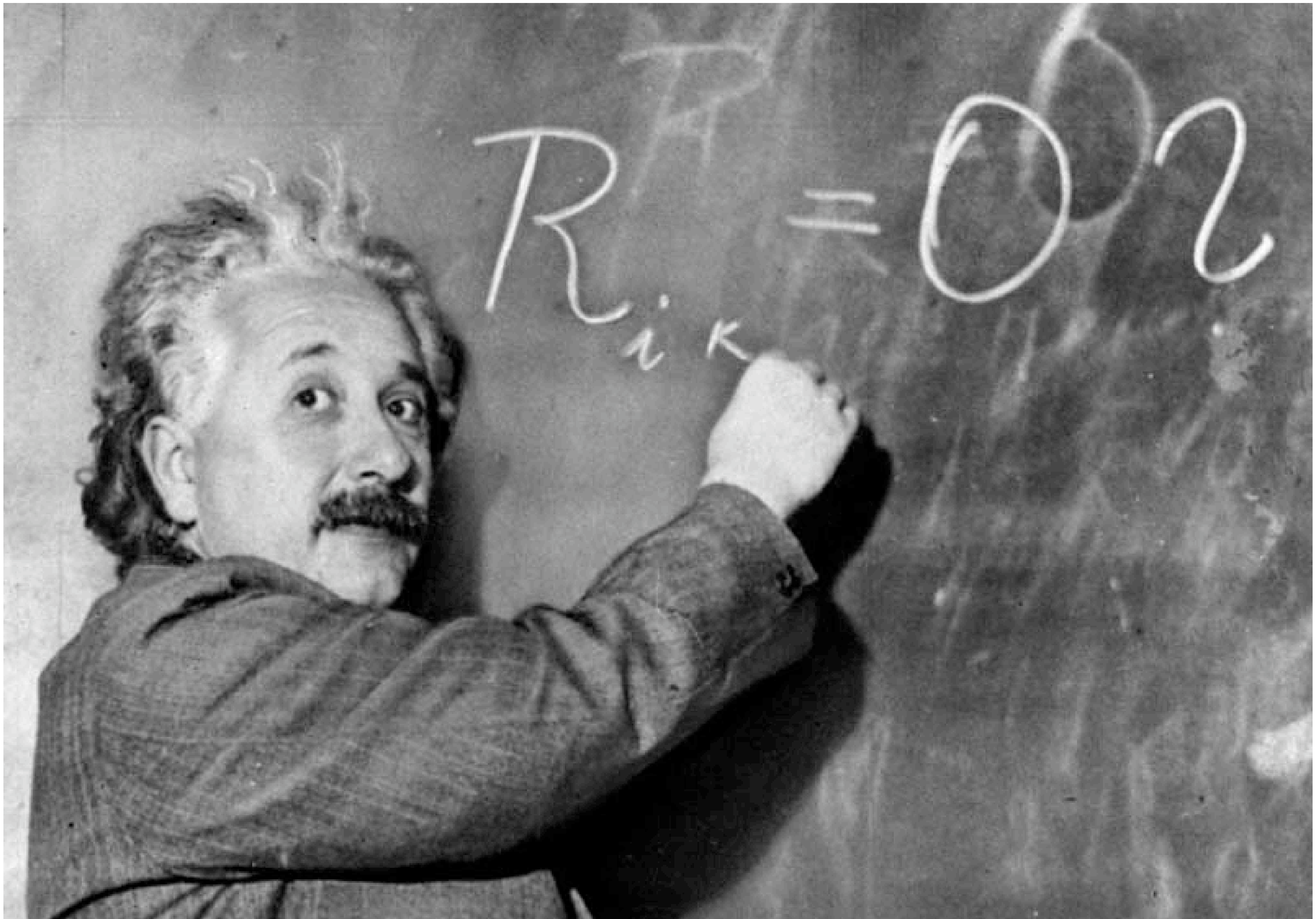

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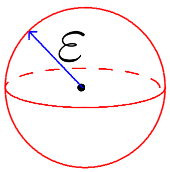

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$$\Delta x^j = 0 \Longrightarrow r_{jk} = \frac{1}{2}\Delta g_{jk} + \ell ots.$$

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Proposition. *If $n \geq 3$, A Riemannian n-manifold (M^n, g) is Einstein iff the trace-free part of its Ricci tensor vanishes:*

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Proof. Bianchi identity $\Longrightarrow \nabla \cdot \mathring{r} = (\frac{1}{2} - \frac{1}{n}) ds$.

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- When $n \geq 6$, wide open. Maybe???

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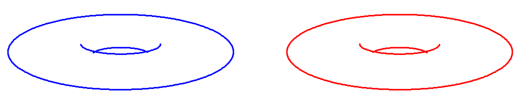

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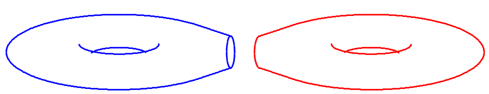

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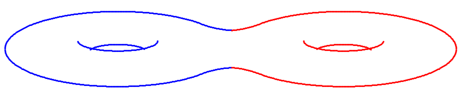

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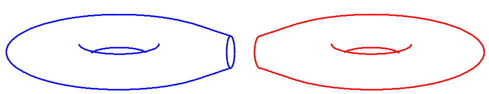

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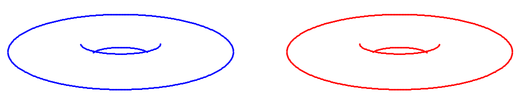

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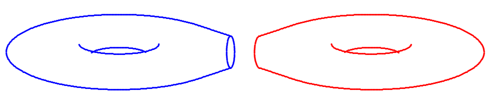

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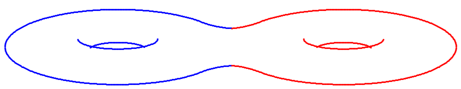

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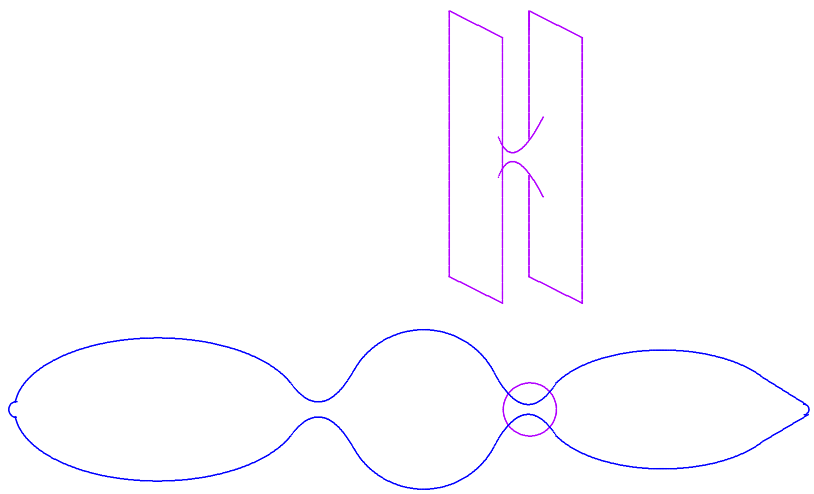

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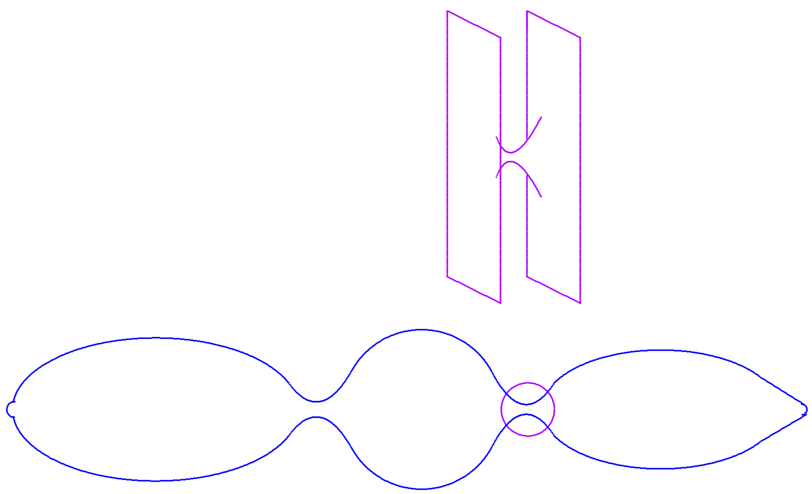

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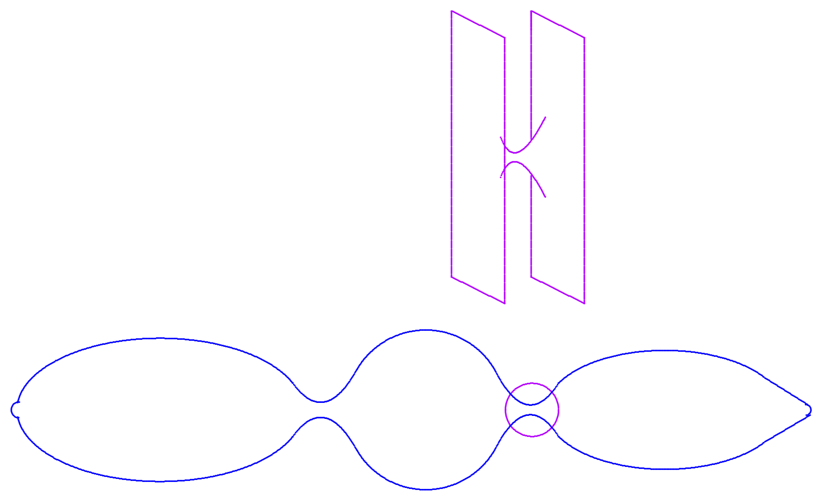

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Similar results for most simply connected spin 5-manifolds. (Boyer, Galicki, Kollar, et al.)

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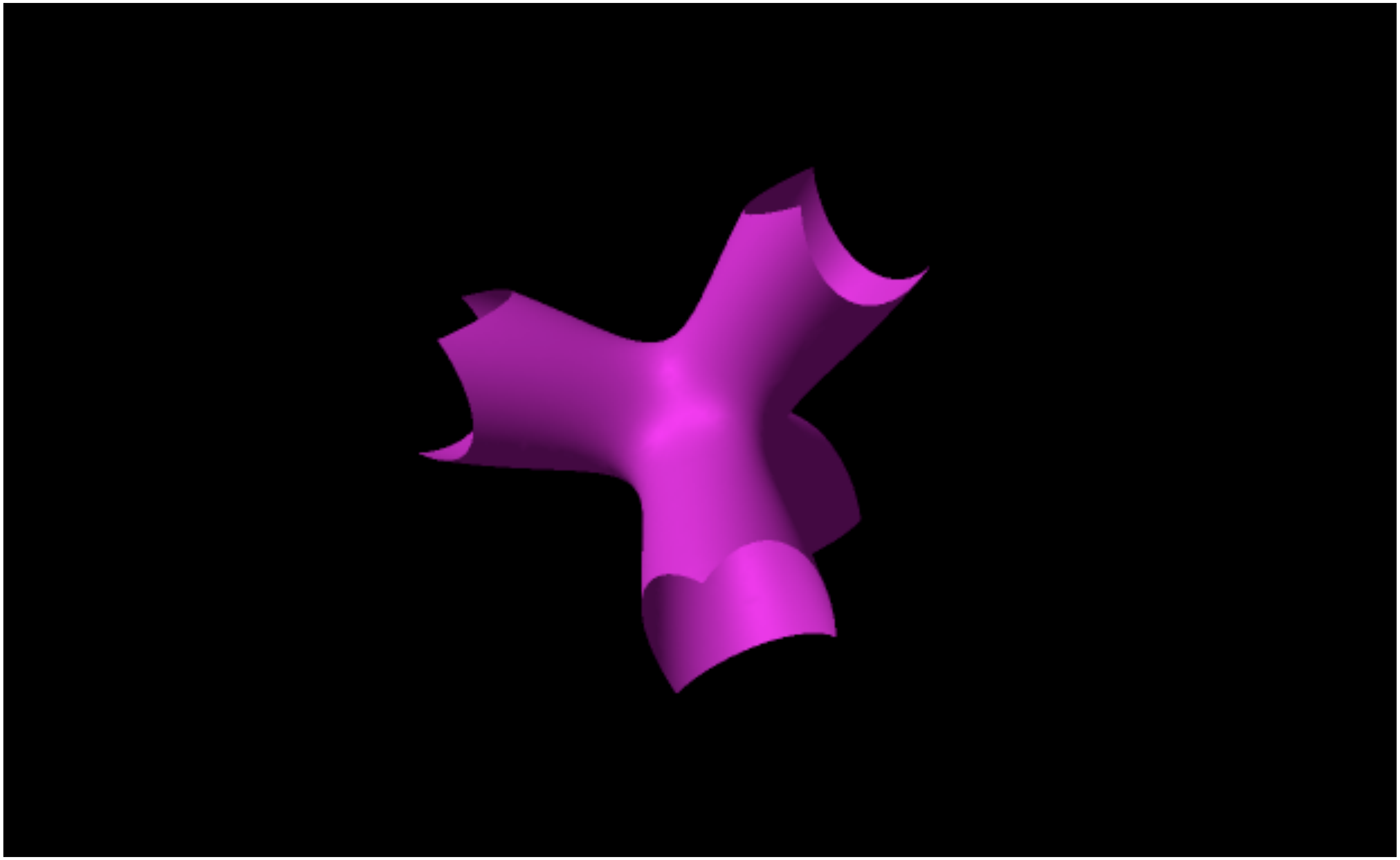

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By contrast, high-dimensional Einstein metrics too common; have little to do with geometrization.

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Λ^+ self-dual 2-forms.

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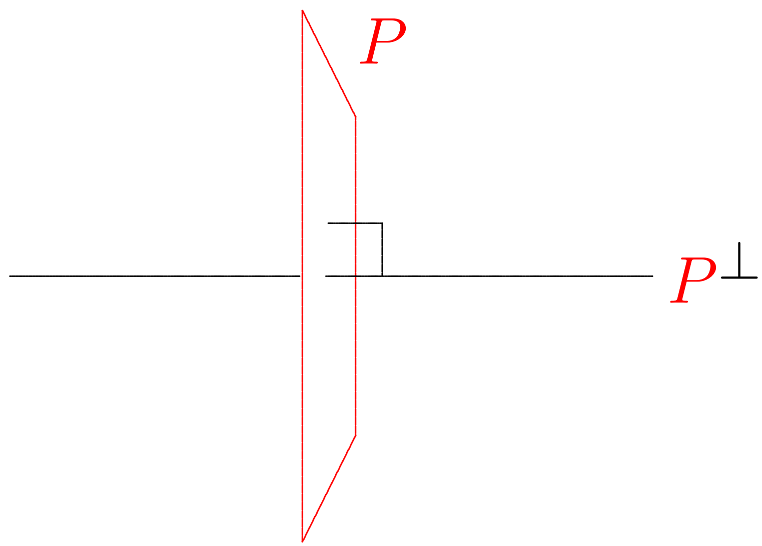

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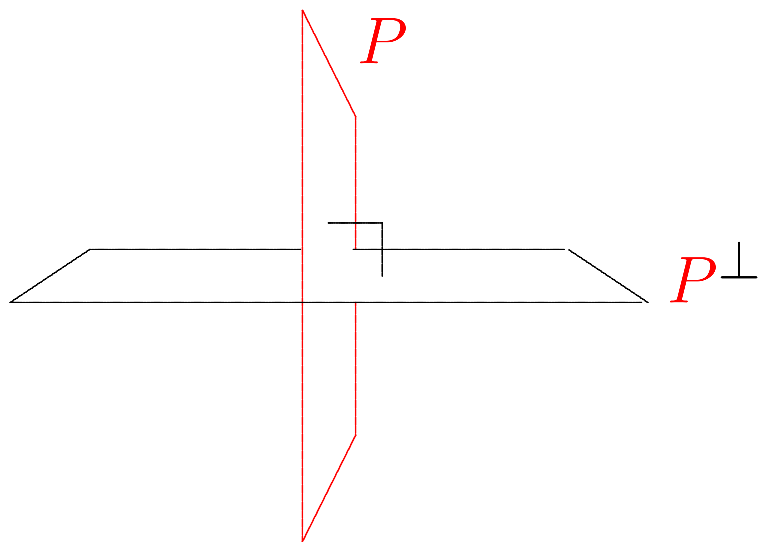

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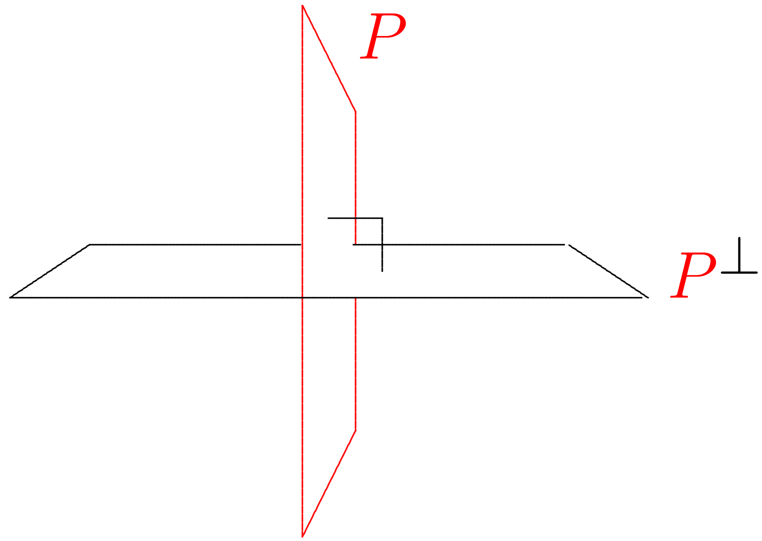

$$K(P) = K(P^{\perp})$$

(M, g) compact oriented Riemannian.

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Here $b_\pm(M) = \max \dim$ subspaces $\subset H^2(M, \mathbb{R})$ on which intersection pairing

$$\begin{array}{ccc} H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) & \longrightarrow & \mathbb{R} \\ ([\varphi] , [\psi]) & \mapsto & \int_M \varphi \wedge \psi \end{array}$$

is positive (resp. negative) definite.

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Typically, one homeotype $\longleftrightarrow$ ∞ many diffeotypes.

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Convention:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

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Certainly true of all examples in this lecture!

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Today's Main Question. *If (M^4, J) is a compact complex surface, when does M^4 admit an Einstein metric g (unrelated to J)?*

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Kähler if the 2-form

$$\omega = g(J\cdot, \cdot)$$

is closed:

$$d\omega = 0.$$

But we do not assume this!

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Only two metrics arise in non-Kähler case!

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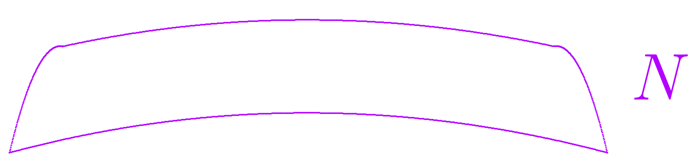

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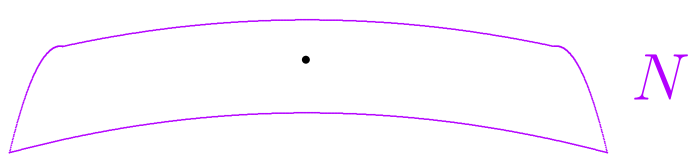

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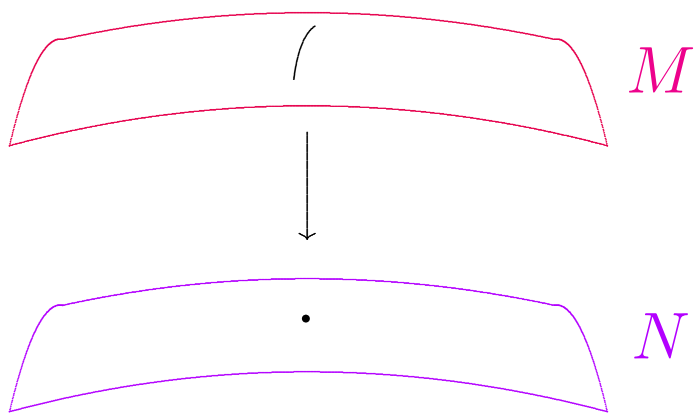

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in which new $\mathbb{CP}_1$ has self-intersection -1.

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$\Longrightarrow$: Hitchin-Thorpe inequality, easy Seiberg-Witten.

Theorem. *Suppose that* M *is a smooth compact oriented* 4*-manifold which admits an integrable complex structure* J*. Then* M *also admits an Einstein metric* g *with* $\lambda \geq 0$ *if and only if*

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Similarly when M symplectic instead of complex.

Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

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Theorem (Hitchin-Thorpe Inequality). *If smooth compact oriented M^4 admits Einstein g, then*

$$(2\chi + 3\tau)(M) \geq 0,$$

with equality only if (M, g) finitely covered by flat T^4 or Calabi-Yau $K3$.

Seiberg-Witten theory:

generalized Kähler geometry of non-Kähler 4-manifolds.

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But $\bar{\partial} + \bar{\partial}^*$ does generalize:

spin^c Dirac operator, preferred connection on L.

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Every unitary connection A on L induces spin^c Dirac operator

$$D_A : \Gamma(\mathbb{V}_+) \to \Gamma(\mathbb{V}_-)$$

generalizing $\bar\partial + \bar\partial^*$.

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Non-linear, but elliptic once ‘gauge-fixing’

$$d^*(A - A_0) = 0$$

imposed to eliminate automorphisms of $L \to M$.

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If, in addition, ${c_1}^2 > 0$,
$\Longrightarrow \nexists g$ with $s \geq 0$.

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A complex surface M is of general type $\Longleftrightarrow$ its minimal model X satisfies

$$\begin{aligned} {c_1}^2(X) &> 0 \\ c_1 \cdot [\omega] &< 0 \end{aligned}$$

for some Kähler class $[\omega]$.

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where X is the minimal model of M.

Moreover, equality holds in either case iff $M = X$, and g is Kähler-Einstein with $\lambda < 0$.

Theorem (L '01). *Let* X *be a minimal surface of general type, and let*

$$M = X \# k \overline{\mathbb{CP}_2}.$$

Then M *cannot admit an Einstein metric if*

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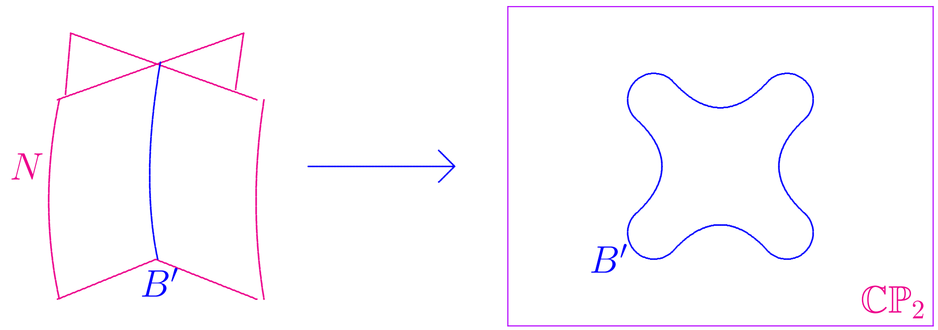
(Better than Hitchin-Thorpe by a factor of 3.)

So being "very" non-minimal is an obstruction.

Example.

Example. Let N be double branched cover $\mathbb{CP}_2$, ramified at a smooth octic:



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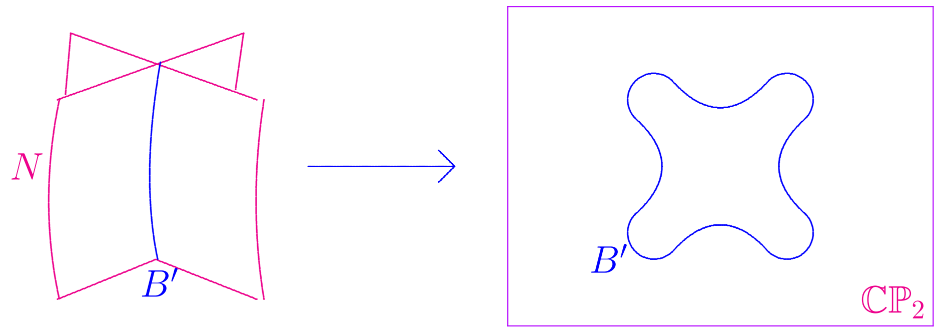

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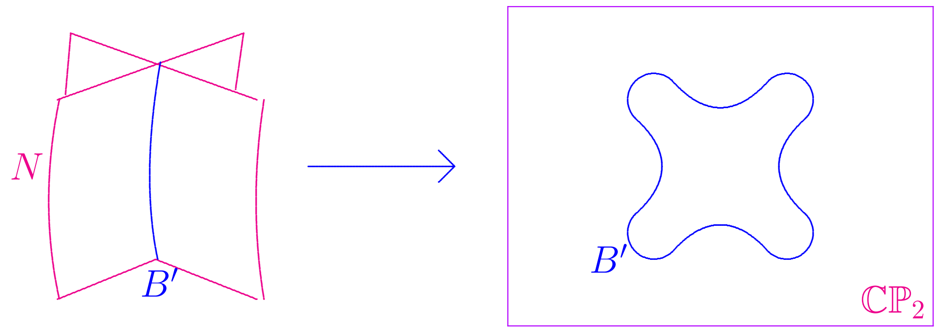
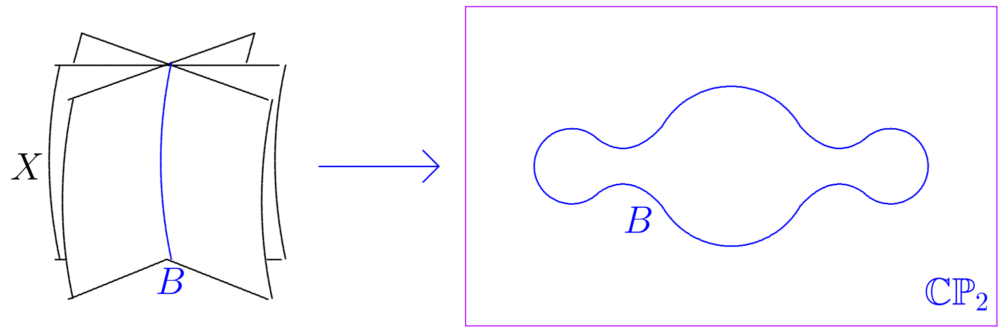


$c_1 < 0 \implies N$ carries an Einstein metric.

Now let X be a triple cyclic cover $\mathbb{CP}_2$, ramified at a smooth sextic



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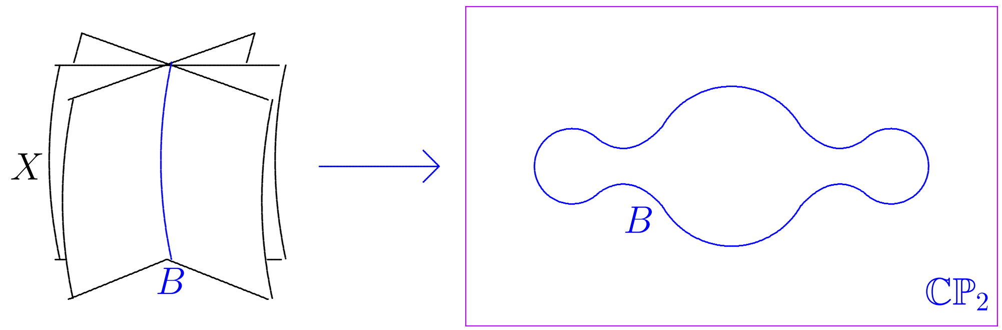

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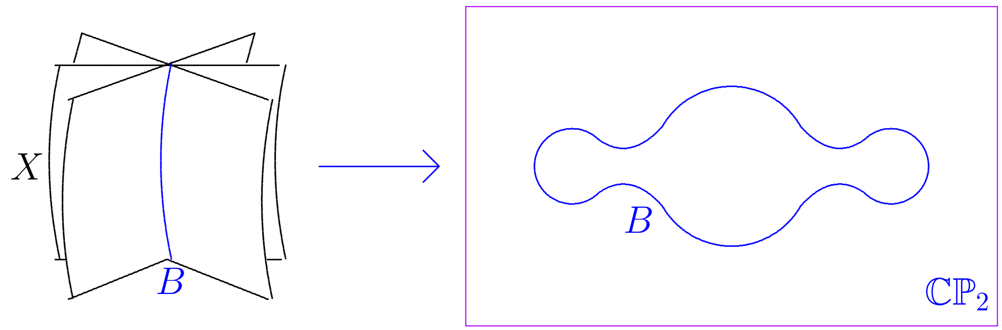

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Then

$$\begin{aligned} c_1^2(X) &= 3 \\ k &= 1 \end{aligned}$$

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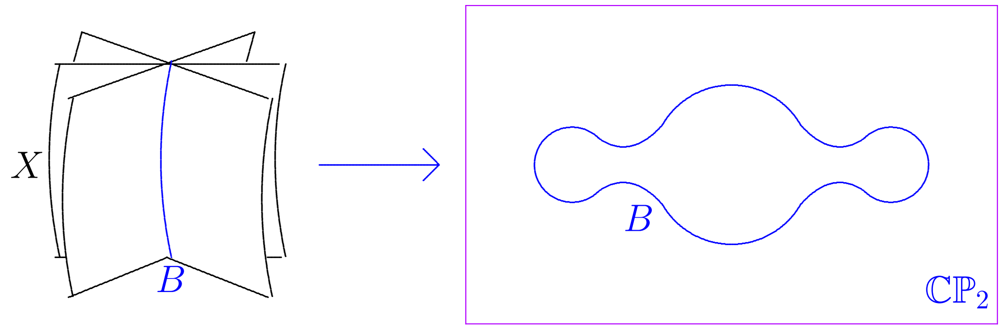

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So Theorem $\Longrightarrow$ *no* Einstein metric on M.

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Moral: Existence depends on diffeotype!

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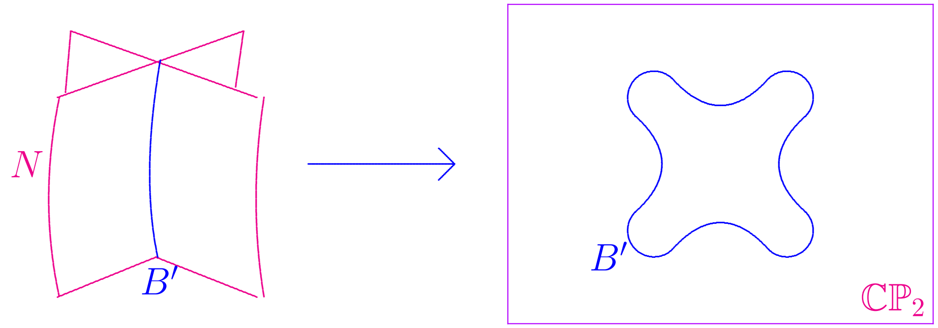

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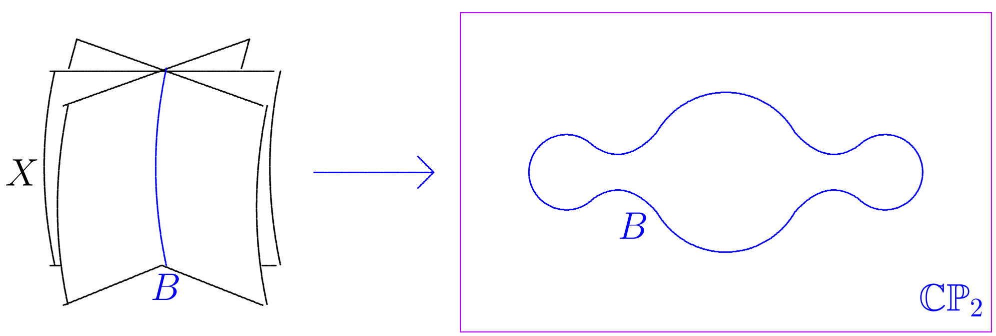

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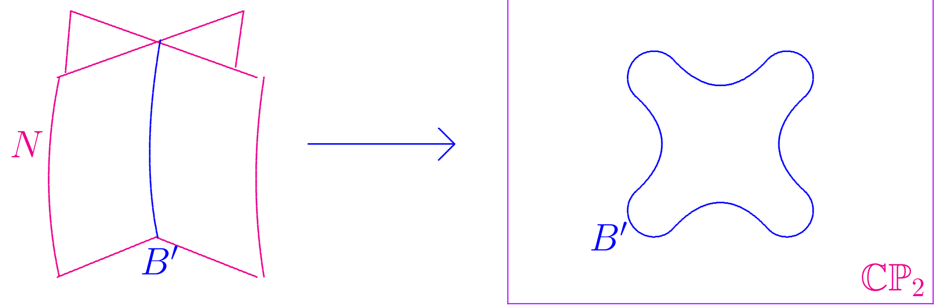

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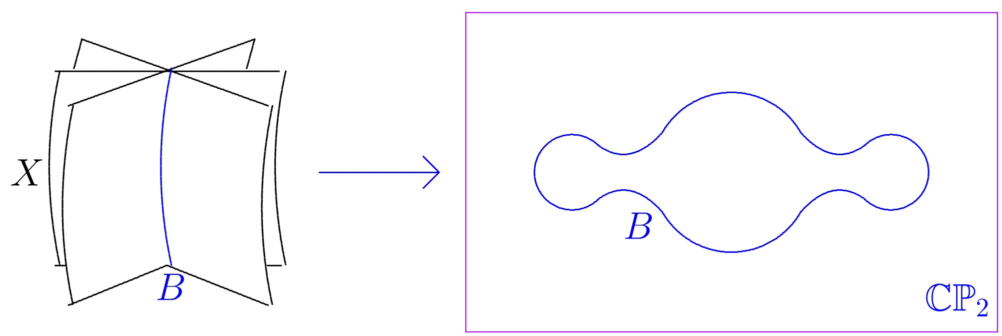

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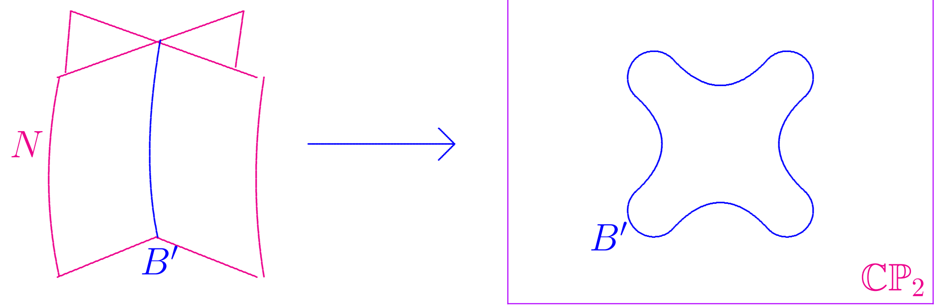

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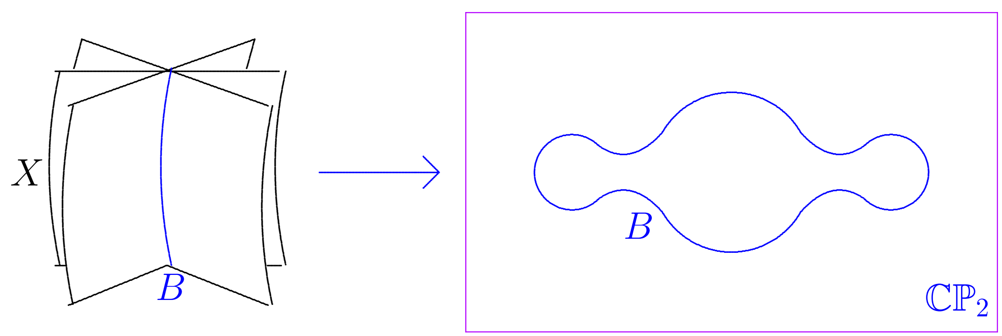

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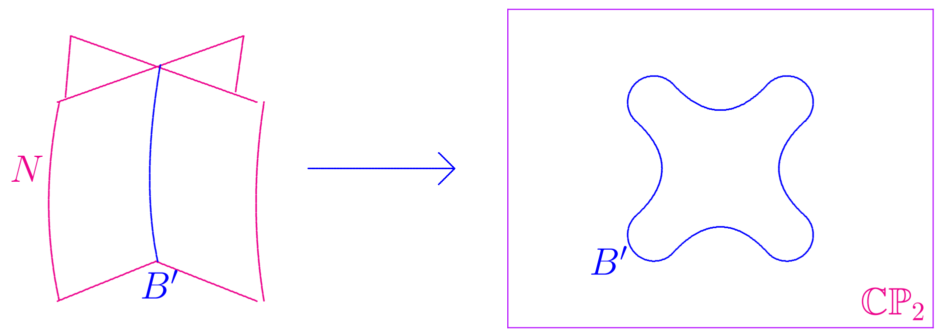

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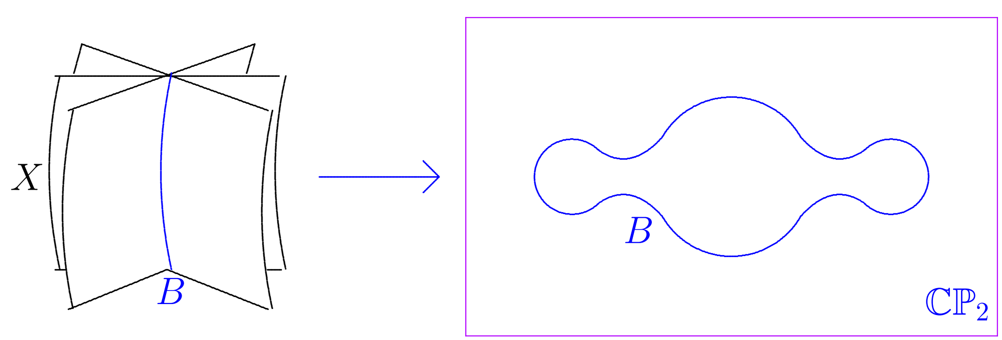

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