Einstein Metrics,

Four-Manifolds, &

Gravitational Instantons

Claude LeBrun
Stony Brook University

European Doctorate School of
Differential Geometry,
Granada, Spain. June 25, 2024

Let (M^n, g) be a Riemannian n-manifold,

Let (M^n, g) be a Riemannian n-manifold, $p \in M$.

Let (M^n, g) be a Riemannian n-manifold, $p \in M$.
Metric defines locally shortest curves, called *geodesics*.

Let (M^n, g) be a Riemannian n-manifold, $p \in M$.
Metric defines locally shortest curves, called *geodesics*.
Following geodesics from p defines a map

$$\exp : T_pM \dashrightarrow M$$

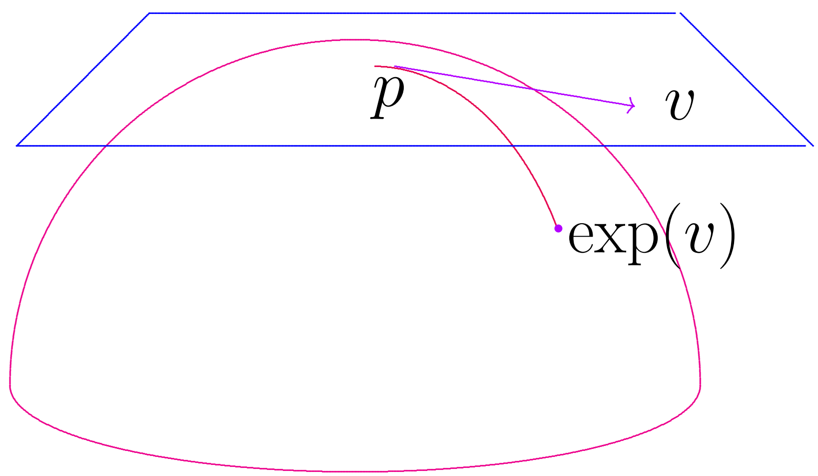
Let (M^n, g) be a Riemannian n-manifold, $p \in M$.
Metric defines locally shortest curves, called *geodesics*.
Following geodesics from p defines a map

$$\exp : T_pM \longrightarrow M$$

Let (M^n, g) be a Riemannian n-manifold, $p \in M$.
Metric defines locally shortest curves, called *geodesics*.
Following geodesics from p defines a map

$$\exp : T_pM \longrightarrow M$$

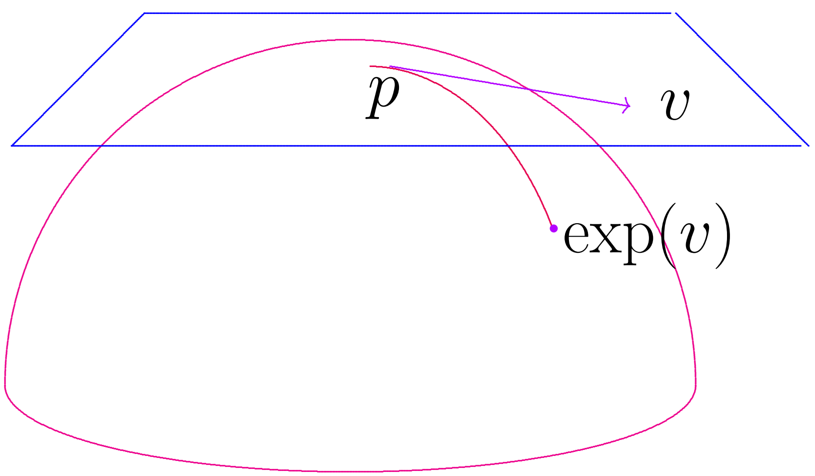
which is a diffeomorphism on a neighborhood of 0:



Let (M^n, g) be a Riemannian n-manifold, $p \in M$.
Metric defines locally shortest curves, called *geodesics*.
Following geodesics from p defines a map

$$\exp : T_pM \longrightarrow M$$

which is a diffeomorphism on a neighborhood of 0:



Now choosing $T_pM \xrightarrow{\cong} \mathbb{R}^n$ via some orthonormal basis gives us special coordinates on M.

In these "geodesic normal" coordinates the metric volume measure is given by

$$d\mu_g = \qquad d\mu_{\text{Euclidean}},$$

In these "geodesic normal" coordinates the metric volume measure is given by

$$d\mu_g = \Big[1 - \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big] d\mu_{\text{Euclidean}},$$

In these "geodesic normal" coordinates the metric volume measure is given by

$$d\mu_g = \left[1 - \frac{1}{6}\, r_{jk}\, x^j x^k + \qquad\qquad \right] d\mu_{\text{Euclidean}},$$

In these "geodesic normal" coordinates the metric volume measure is given by

$$d\mu_g = \left[1 - \frac{1}{6}\, r_{jk}\, x^j x^k + O(|x|^3)\right] d\mu_{\text{Euclidean}},$$

In these "geodesic normal" coordinates the metric volume measure is given by

$$d\mu_g = \left[1 - \tfrac{1}{6}\, r_{jk}\, x^j x^k + O(|x|^3)\right] d\mu_{\text{Euclidean}},$$

where r is the *Ricci tensor*

In these "geodesic normal" coordinates the metric volume measure is given by

$$d\mu_g = \left[1 - \frac{1}{6}\, r_{jk}\, x^j x^k + O(|x|^3)\right] d\mu_{\text{Euclidean}},$$

where r is the *Ricci tensor* $r_{jk} = \mathcal{R}^i{}_{jik}$.

In these "geodesic normal" coordinates the metric volume measure is given by

$$d\mu_g = \left[1 - \frac{1}{6}\, r_{jk}\, x^j x^k + O(|x|^3)\right] d\mu_{\text{Euclidean}},$$

where r is the *Ricci tensor* $r_{jk} = \mathcal{R}^i{}_{jik}$.

The *Ricci curvature*

In these "geodesic normal" coordinates the metric volume measure is given by

$$d\mu_g = \Big[1 - \tfrac{1}{6}\, r_{jk}\, x^j x^k + O(|x|^3)\Big] d\mu_{\text{Euclidean}},$$

where r is the *Ricci tensor* $r_{jk} = \mathcal{R}^i{}_{jik}$.

The *Ricci curvature* is by definition the function on the unit tangent bundle

In these "geodesic normal" coordinates the metric volume measure is given by

$$d\mu_g = \left[1 - \frac{1}{6}\, r_{jk}\, x^j x^k + O(|x|^3)\right] d\mu_{\text{Euclidean}},$$

where r is the *Ricci tensor* $r_{jk} = \mathcal{R}^i{}_{jik}$.

The *Ricci curvature* is by definition the function on the unit tangent bundle

$$STM = \{v \in TM \mid g(v,v) = 1\}$$

In these "geodesic normal" coordinates the metric volume measure is given by

$$d\mu_g = \left[1 - \frac{1}{6}\, r_{jk}\, x^j x^k + O(|x|^3)\right] d\mu_{\text{Euclidean}},$$

where r is the *Ricci tensor* $r_{jk} = \mathcal{R}^i{}_{jik}$.

The *Ricci curvature* is by definition the function on the unit tangent bundle

$$STM = \{v \in TM \mid g(v,v) = 1\}$$

given by

$$v \longmapsto r(v,v).$$

Definition. *A Riemannian metric* g

Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature*

Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

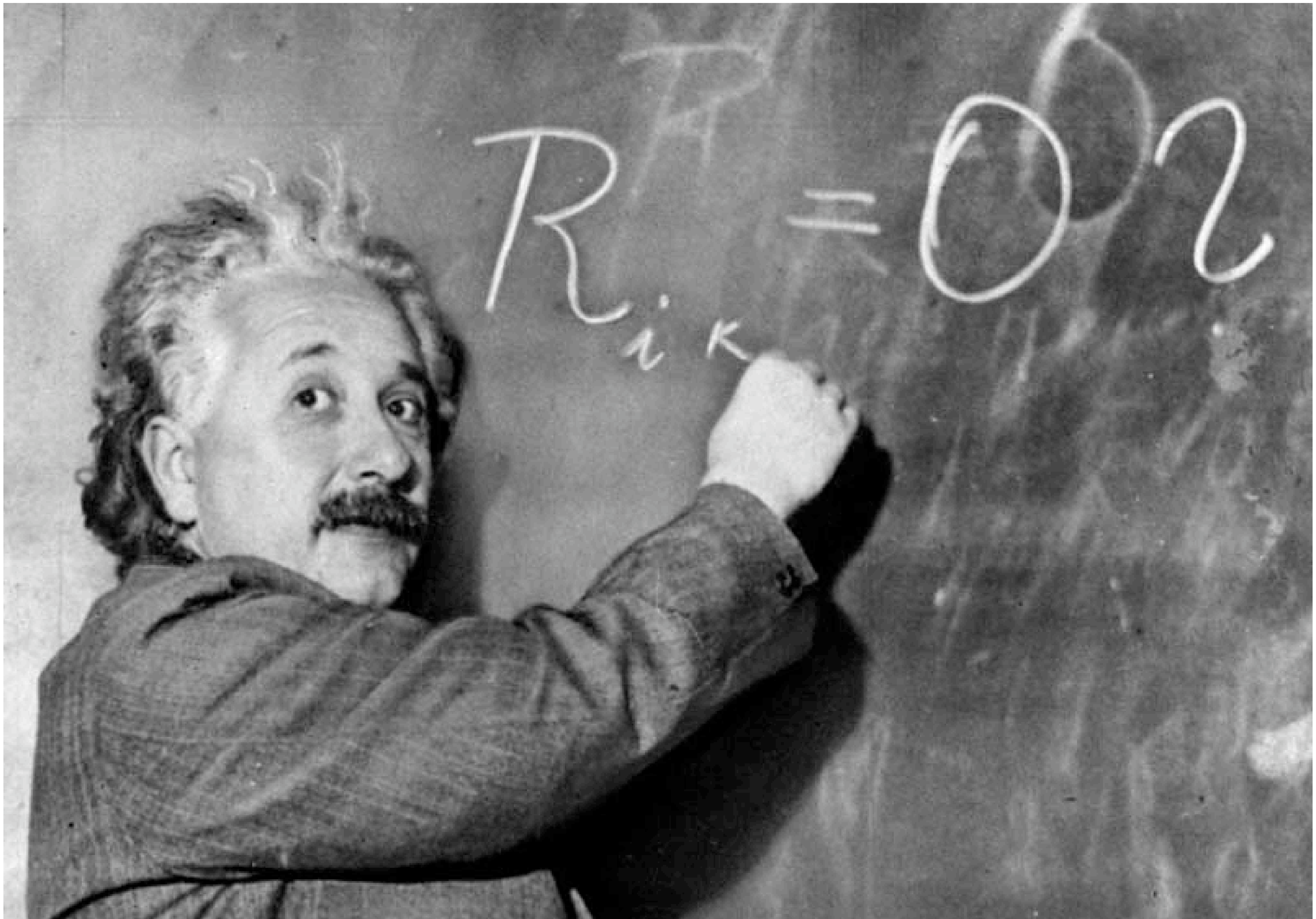
"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

Definition. *A Riemannian metric g is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.



Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

As punishment ...

Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Mathematicians call λ the Einstein constant.

Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Mathematicians call λ the Einstein constant.

Has same sign as the *scalar curvature*

$$s = r_j{}^j = \mathcal{R}^{ij}{}_{ij}.$$

Definition. *A Riemannian metric g is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

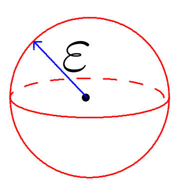
for some constant $\lambda \in \mathbb{R}$.

Mathematicians call λ the Einstein constant.

Has same sign as the *scalar curvature*

$$s = r_j^{\ j} = \mathcal{R}^{ij}{}_{ij}.$$

$$\frac{\mathrm{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s\,\frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Definition. *A Riemannian metric g is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

$n = 2, 3$: Einstein $\Longleftrightarrow$ constant sectional

Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

$n = 2, 3$: Einstein $\Longleftrightarrow$ constant sectional

$n \geq 4$: Einstein $\Leftarrow$, $\not\Rightarrow$ constant sectional

Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

Determined system:
same number of equations as unknowns.

Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

Determined system:
same number of equations as unknowns.

g_{jk}: $\frac{n(n+1)}{2}$ components.
r_{jk}: $\frac{n(n+1)}{2}$ components.

Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

Determined system:
same number of equations as unknowns.

g_{jk}: $\frac{n(n+1)}{2}$ components.
r_{jk}: $\frac{n(n+1)}{2}$ components.
$\mathcal{R}^j{}_{k\ell m}$: $\frac{n^2(n^2-1)}{12}$ components.

Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

Determined system:
same number of equations as unknowns.

Elliptic non-linear PDE after gauge fixing.

Definition. *A Riemannian metric g is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Generalizes constant sectional curvature condition, but weaker.

Determined system:
same number of equations as unknowns.

Elliptic non-linear PDE after gauge fixing.

$$\Delta x^j = 0 \Longrightarrow r_{jk} = \frac{1}{2}\Delta g_{jk} + \ell ots.$$

Question (Yamabe)**.** *Does every smooth compact simply-connected* n*-manifold admit an Einstein metric?*

Question (Yamabe). *Does every smooth compact simply-connected* n*-manifold admit an Einstein metric?*

What we know:

Question (Yamabe). *Does every smooth compact simply-connected n-manifold admit an Einstein metric?*

What we know:

- When $n = 2$: Yes! (Riemann)

Question (Yamabe). *Does every smooth compact simply-connected* n*-manifold admit an Einstein metric?*

What we know:

- When $n = 2$: Yes! (Riemann)
- When $n = 3$: $\Longleftrightarrow$ Poincaré conjecture.

Question (Yamabe). *Does every smooth compact simply-connected* n*-manifold admit an Einstein metric?*

What we know:

- When $n = 2$: Yes! (Riemann)
- When $n = 3$: $\Longleftrightarrow$ Poincaré conjecture.
  Hamilton, Perelman, ... Yes!

Question (Yamabe). *Does every smooth compact simply-connected* n*-manifold admit an Einstein metric?*

What we know:

- When $n = 2$: Yes! (Riemann)
- When $n = 3$: $\Longleftrightarrow$ Poincaré conjecture. Hamilton, Perelman, . . . Yes!
- When $n = 4$: No! (Hitchin)

Question (Yamabe). *Does every smooth compact simply-connected n-manifold admit an Einstein metric?*

What we know:

- When $n = 2$: Yes! (Riemann)
- When $n = 3$: $\Longleftrightarrow$ Poincaré conjecture.
  Hamilton, Perelman, ... Yes!
- When $n = 4$: No! (Hitchin)
- When $n = 5$: Yes?? (Boyer-Galicki-Kollár)

Question (Yamabe). *Does every smooth compact simply-connected n-manifold admit an Einstein metric?*

What we know:

- When $n = 2$: Yes! (Riemann)
- When $n = 3$: $\Longleftrightarrow$ Poincaré conjecture. Hamilton, Perelman, ... Yes!
- When $n = 4$: No! (Hitchin)
- When $n = 5$: Yes?? (Boyer-Galicki-Kollár)
- When $n \geq 6$, wide open. Maybe???

Dimension ≤ 3:

Dimension ≤ 3:

Einstein's equations are "locally trivial:"

Dimension ≤ 3:

Einstein's equations are "locally trivial:"

Einstein metrics have constant sectional curvature.

Dimension ≤ 3:

Einstein's equations are "locally trivial:"

Einstein metrics have constant sectional curvature.

$\Longrightarrow$ If M^3 carries Einstein metric, $\pi_2(M) = 0$.

Dimension ≤ 3:

Einstein's equations are "locally trivial:"

Einstein metrics have constant sectional curvature.

$\Longrightarrow$ If M^3 carries Einstein metric, $\pi_2(M) = 0$.

$\Longrightarrow$ Existence obstructed for connect sums $M^3 \# N^3$.

Dimension ≤ 3:

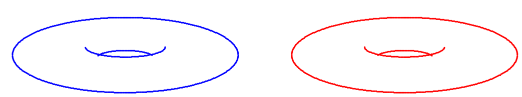
Einstein's equations are "locally trivial:"

Einstein metrics have constant sectional curvature.

$\Longrightarrow$ If M^3 carries Einstein metric, $\pi_2(M) = 0$.

$\Longrightarrow$ Existence obstructed for connect sums $M^3 \# N^3$.

Connected sum $\#$:



Dimension ≤ 3:

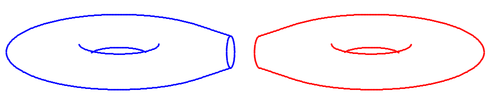
Einstein's equations are "locally trivial:"

Einstein metrics have constant sectional curvature.

$\Longrightarrow$ If M^3 carries Einstein metric, $\pi_2(M) = 0$.

$\Longrightarrow$ Existence obstructed for connect sums $M^3 \# N^3$.

Connected sum $\#$:



Dimension ≤ 3:

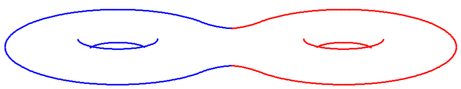
Einstein's equations are "locally trivial:"

Einstein metrics have constant sectional curvature.

$\Longrightarrow$ If M^3 carries Einstein metric, $\pi_2(M) = 0$.

$\Longrightarrow$ Existence obstructed for connect sums $M^3 \# N^3$.

Connected sum $\#$:



Dimension ≤ 3:

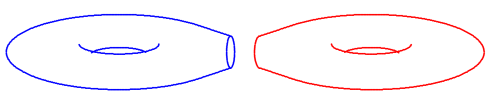
Einstein's equations are "locally trivial:"

Einstein metrics have constant sectional curvature.

$\Longrightarrow$ If M^3 carries Einstein metric, $\pi_2(M) = 0$.

$\Longrightarrow$ Existence obstructed for connect sums $M^3 \# N^3$.

Connected sum $\#$:



Dimension ≤ 3:

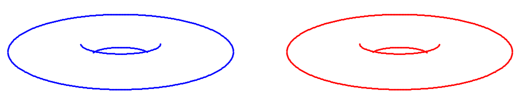
Einstein's equations are "locally trivial:"

Einstein metrics have constant sectional curvature.

$\Longrightarrow$ If M^3 carries Einstein metric, $\pi_2(M) = 0$.

$\Longrightarrow$ Existence obstructed for connect sums $M^3 \# N^3$.

Connected sum $\#$:



Dimension ≤ 3:

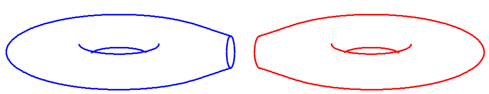
Einstein's equations are "locally trivial:"

Einstein metrics have constant sectional curvature.

$\Longrightarrow$ If M^3 carries Einstein metric, $\pi_2(M) = 0$.

$\Longrightarrow$ Existence obstructed for connect sums $M^3 \# N^3$.

Connected sum $\#$:



Dimension ≤ 3:

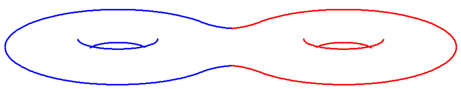
Einstein's equations are "locally trivial:"

Einstein metrics have constant sectional curvature.

$\Longrightarrow$ If M^3 carries Einstein metric, $\pi_2(M) = 0$.

$\Longrightarrow$ Existence obstructed for connect sums $M^3 \# N^3$.

Connected sum #:



Dimension ≤ 3:

Einstein's equations are "locally trivial:"

Einstein metrics have constant sectional curvature.

$\Longrightarrow$ If M^3 carries Einstein metric, $\pi_2(M) = 0$.

$\Longrightarrow$ Existence obstructed for connect sums $M^3 \# N^3$.

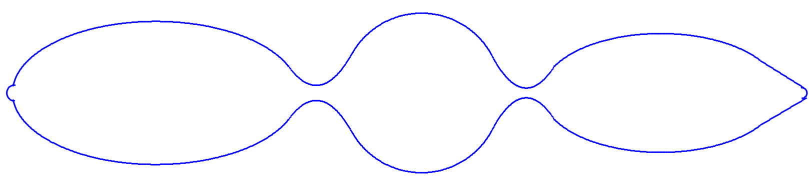
Dimension ≤ 3:

Einstein's equations are "locally trivial:"

Einstein metrics have constant sectional curvature.

$\Longrightarrow$ If M^3 carries Einstein metric, $\pi_2(M) = 0$.

$\Longrightarrow$ Existence obstructed for connect sums $M^3 \# N^3$.



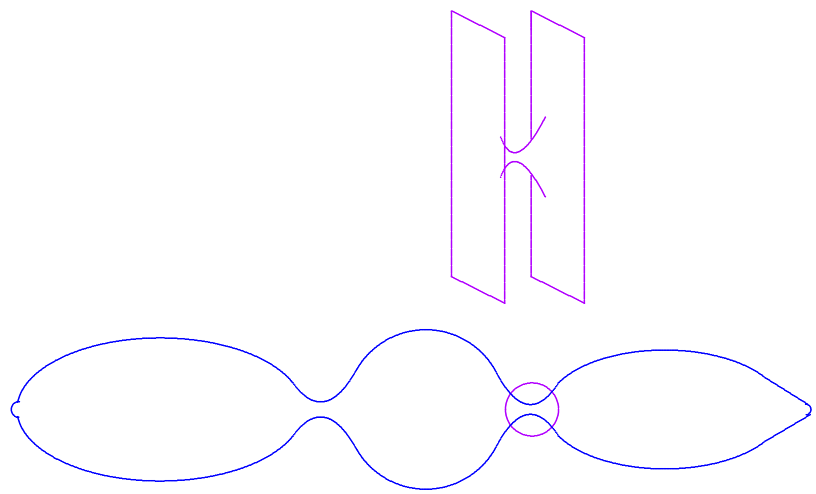
Dimension ≤ 3:

Einstein's equations are "locally trivial:"

Einstein metrics have constant sectional curvature.

$\Longrightarrow$ If M^3 carries Einstein metric, $\pi_2(M) = 0$.

$\Longrightarrow$ Existence obstructed for connect sums $M^3 \# N^3$.



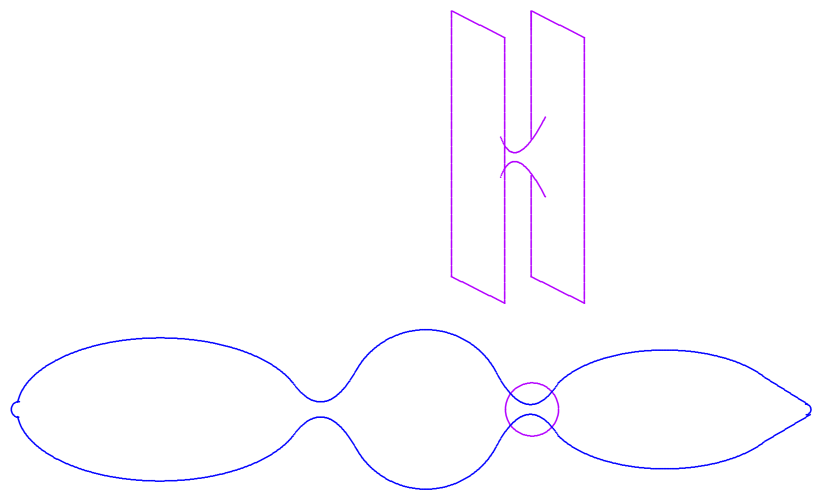
Dimension ≤ 3:

Einstein's equations are "locally trivial:"

Einstein metrics have constant sectional curvature.

$\Longrightarrow$ If M^3 carries Einstein metric, $\pi_2(M) = 0$.

$\Longrightarrow$ Existence obstructed for connect sums $M^3 \# N^3$.



Ricci flow pinches off S^2 necks.

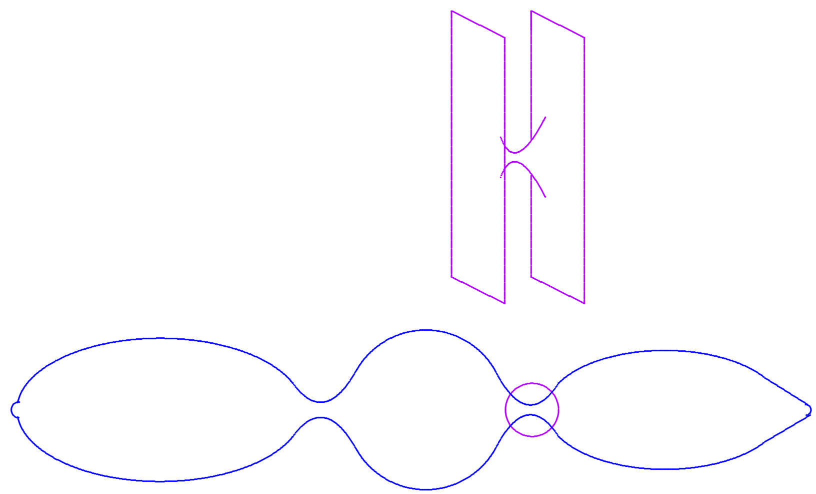
Dimension ≤ 3:

Einstein's equations are "locally trivial:"

Einstein metrics have constant sectional curvature.

$\Longrightarrow$ If M^3 carries Einstein metric, $\pi_2(M) = 0$.

$\Longrightarrow$ Existence obstructed for connect sums $M^3 \# N^3$.



Ricci flow pinches off S^2 necks.

First step in geometrization:

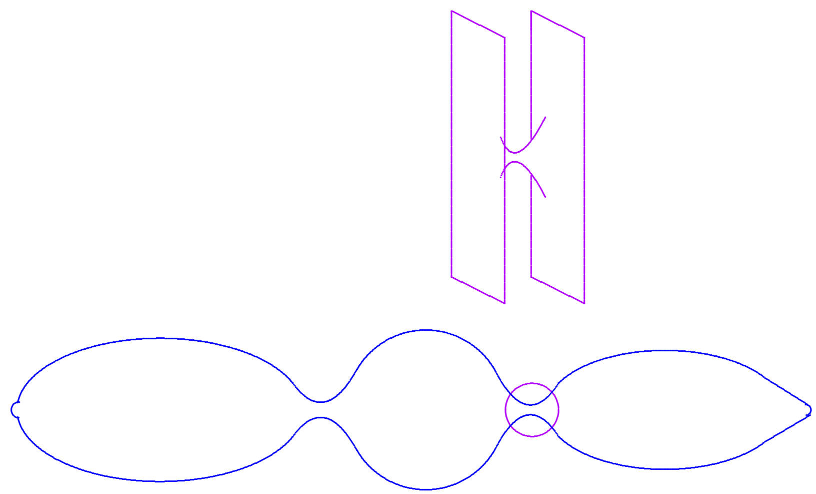
Dimension ≤ 3:

Einstein's equations are "locally trivial:"

Einstein metrics have constant sectional curvature.

$\Longrightarrow$ If M^3 carries Einstein metric, $\pi_2(M) = 0$.

$\Longrightarrow$ Existence obstructed for connect sums $M^3 \# N^3$.



Ricci flow pinches off S^2 necks.

First step in geometrization:

Prime Decomposition.

Dimension ≥ 5:

Dimension ≥ 5:

There are many known Einstein metrics on S^n, $n \geq 5$ which do not have constant curvature.

Dimension ≥ 5:

There are many known Einstein metrics on S^n, $n \geq 5$ which do not have constant curvature.

In fact, the moduli space of Einstein metrics on S^5 has infinitely many connected components, because $\exists$ sequences unit-volume Einstein metrics with $\lambda \rightarrow 0^+$.

(Böhm, Collins-Székelyhidi)

Dimension ≥ 5:

There are many known Einstein metrics on S^n, $n \geq 5$ which do not have constant curvature.

In fact, the moduli space of Einstein metrics on S^5 has infinitely many connected components, because $\exists$ sequences unit-volume Einstein metrics with $\lambda \to 0^+$.

(Böhm, Collins-Székelyhidi)

Moduli Spaces of Einstein metrics

Dimension ≥ 5:

There are many known Einstein metrics on S^n, $n \geq 5$ which do not have constant curvature.

In fact, the moduli space of Einstein metrics on S^5 has infinitely many connected components, because $\exists$ sequences unit-volume Einstein metrics with $\lambda \to 0^+$.

(Böhm, Collins-Székelyhidi)

Moduli Spaces of Einstein metrics

$$\mathscr{E}(M) = \{\text{Einstein } h\}/(\text{Diffeos} \times \mathbb{R}^+)$$

Dimension ≥ 5:

There are many known Einstein metrics on S^n, $n \geq 5$ which do not have constant curvature.

In fact, the moduli space of Einstein metrics on S^5 has infinitely many connected components, because $\exists$ sequences unit-volume Einstein metrics with $\lambda \to 0^+$.

(Böhm, Collins-Székelyhidi)

Dimension ≥ 5:

There are many known Einstein metrics on S^n, $n \geq 5$ which do not have constant curvature.

In fact, the moduli space of Einstein metrics on S^5 has infinitely many connected components, because $\exists$ sequences unit-volume Einstein metrics with $\lambda \to 0^+$.

(Böhm, Collins-Székelyhidi)

Connected sums $(S^2 \times S^3) \# \cdots \# (S^2 \times S^3)$ admit Einstein metrics for arbitrarily many summands. Moduli spaces highly disconnected.

Dimension ≥ 5:

There are many known Einstein metrics on S^n, $n \geq 5$ which do not have constant curvature.

In fact, the moduli space of Einstein metrics on S^5 has infinitely many connected components, because $\exists$ sequences unit-volume Einstein metrics with $\lambda \to 0^+$.

(Böhm, Collins-Székelyhidi)

Connected sums $(S^2 \times S^3) \# \cdots \# (S^2 \times S^3)$ admit Einstein metrics for arbitrarily many summands. Moduli spaces highly disconnected.

Similar results for most simply connected spin 5-manifolds. (Boyer-Galicki-Kollár, et al.)

Dimension 4:

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

$K3$ = Kummer-Kähler-Kodaira

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

A. Weil:

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus* T^4 *is flat.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on* $K3$ *is Ricci-flat Kähler.*

A. Weil: harder to climb than $K2$!

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus* T^4 *is flat.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on* $K3$ *is Ricci-flat Kähler.*

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.
(Kodaira, Yau, Siu, Kobayashi-Todorov)

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.
(Kodaira, Yau, Siu, Kobayashi-Todorov)

Theorem (Besson-Courtois-Gallot). *There is only one Einstein metric on compact hyperbolic 4-manifold $\mathcal{H}^4/\Gamma$, up to scale and diffeos.*

Dimension 4:

Theorem (Berger). *Any Einstein metric on 4-torus T^4 is flat.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.

Theorem (Hitchin). *Any Einstein metric on $K3$ is Ricci-flat Kähler.*

$\Longrightarrow$ Moduli space of Einstein metrics is connected.
(Kodaira, Yau, Siu, Kobayashi-Todorov)

Theorem (Besson-Courtois-Gallot). *There is only one Einstein metric on compact hyperbolic 4-manifold $\mathcal{H}^4/\Gamma$, up to scale and diffeos.*

Theorem (L). *There is only one Einstein metric on compact complex-hyperbolic 4-manifold $\mathbb{C}\mathcal{H}_2/\Gamma$, up to scale and diffeos.*

Four Dimensions is Exceptional

Four Dimensions is Exceptional

"Goldilocks Zone" for Einstein metrics:

Four Dimensions is Exceptional

"Goldilocks Zone" for Einstein metrics:

Not too rigid.

Four Dimensions is Exceptional

“Goldilocks Zone” for Einstein metrics:

Not too rigid. Not too flexible.

Four Dimensions is Exceptional

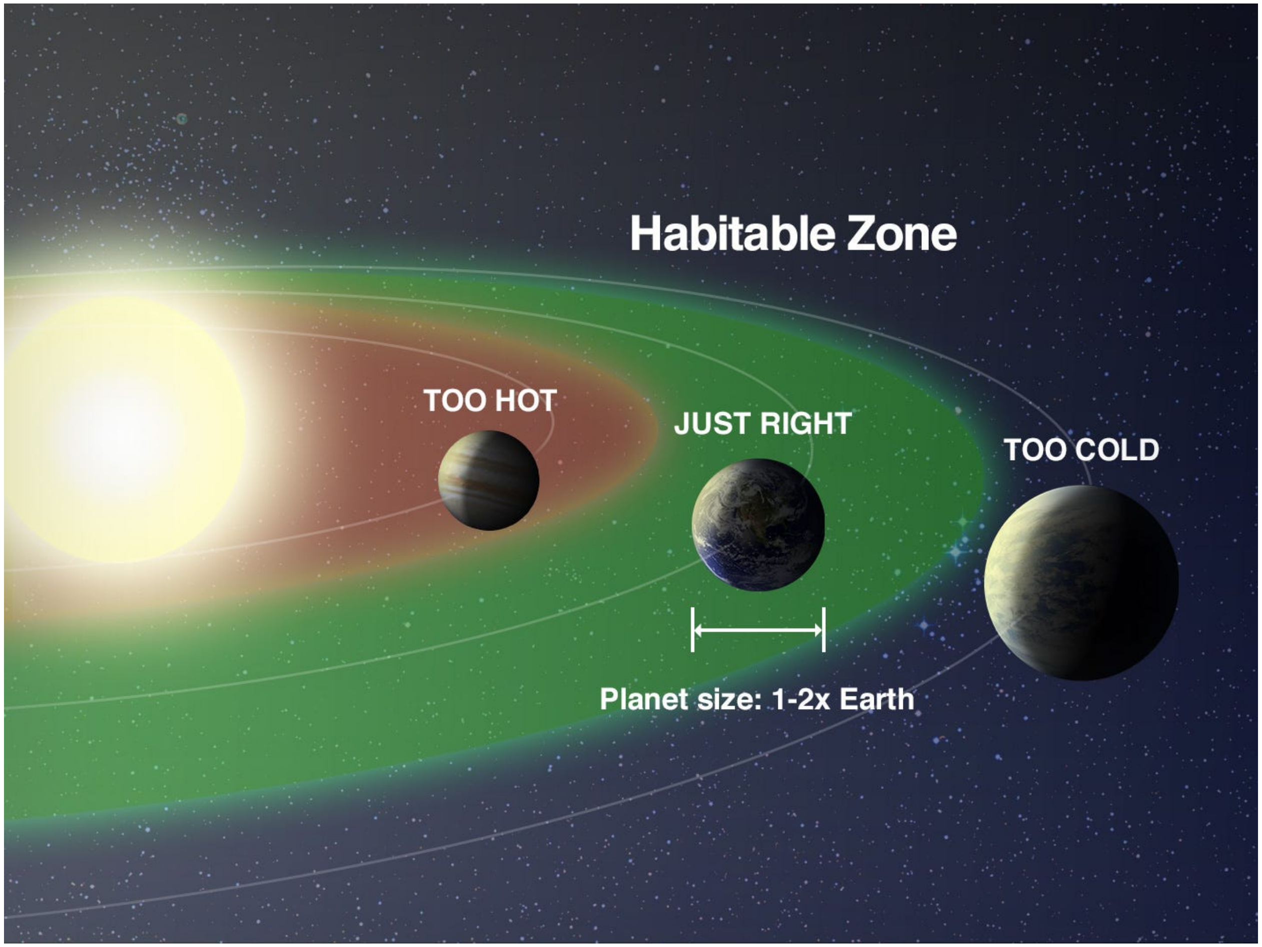
“Goldilocks Zone” for Einstein metrics:

Not too rigid. Not too flexible. Just right!

Four Dimensions is Exceptional

“Goldilocks Zone” for Einstein metrics:

Not too rigid. Not too flexible. Just right!



Four Dimensions is Exceptional

"Goldilocks Zone" for Einstein metrics:

Not too rigid. Not too flexible. Just right!

Four Dimensions is Exceptional

“Goldilocks Zone” for Einstein metrics:

Not too rigid. Not too flexible. Just right!

Four Dimensions is Exceptional

Four Dimensions is Exceptional

When $n = 4$, existence for Einstein depends delicately on smooth structure.

Four Dimensions is Exceptional

When $n = 4$, existence for Einstein depends delicately on smooth structure.

There are topological 4-manifolds which admit an Einstein metric for one smooth structure, but not for others.

Four Dimensions is Exceptional

When $n = 4$, existence for Einstein depends delicately on smooth structure.

There are topological 4-manifolds which admit an Einstein metric for one smooth structure, but not for others.

But might allow for geometrization of 4-manifolds by decomposition into Einstein and collapsed pieces.

Four Dimensions is Exceptional

When $n = 4$, existence for Einstein depends delicately on smooth structure.

There are topological 4-manifolds which admit an Einstein metric for one smooth structure, but not for others.

But might allow for geometrization of 4-manifolds by decomposition into Einstein and collapsed pieces.

Enough rigidity apparently still holds in dimension four to call this a geometrization.

Four Dimensions is Exceptional

When $n = 4$, existence for Einstein depends delicately on smooth structure.

There are topological 4-manifolds which admit an Einstein metric for one smooth structure, but not for others.

But might allow for geometrization of 4-manifolds by decomposition into Einstein and collapsed pieces.

Enough rigidity apparently still holds in dimension four to call this a geometrization.

By contrast, high-dimensional Einstein metrics too common; have little to do with geometrization.

What's so special about dimension 4?

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g),

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are (± 1)-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

What's so special about dimension 4?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are (± 1)-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

$$\star^2 = 1.$$

Λ^+ self-dual 2-forms.

Λ^- anti-self-dual 2-forms.

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^+	$W_+ + \frac{s}{12}$	$\mathring{r}$
Λ^-	$\mathring{r}$	$W_- + \frac{s}{12}$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^{+}	$W_+ + \frac{s}{12}$	$\mathring{r}$
Λ^{-}	$\mathring{r}$	$W_- + \frac{s}{12}$

where

$$\begin{aligned} s &= \text{scalar curvature} \\ \mathring{r} &= \text{trace-free Ricci curvature} \\ W_+ &= \text{self-dual Weyl curvature} \\ W_- &= \text{anti-self-dual Weyl curvature} \end{aligned}$$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^{+}	$W_+ + \frac{s}{12}$	$\mathring{r}$
Λ^{-}	$\mathring{r}$	$W_- + \frac{s}{12}$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W_+ = self-dual Weyl curvature *(conformally invariant)*

W_- = anti-self-dual Weyl curvature 〃

Thus (M^4, g) Einstein $\Longleftrightarrow$

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

commutes with

$$\star : \Lambda^2 \to \Lambda^2 :$$

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array} \right).$$

Thus (M^4, g) Einstein $\Longleftrightarrow$

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

commutes with

$$\star : \Lambda^2 \to \Lambda^2 :$$

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & 0 \\ \hline 0 & W_- + \frac{s}{12} \end{array} \right) .$$

Thus (M^4, g) Einstein $\Longleftrightarrow$

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

commutes with

$$\star : \Lambda^2 \to \Lambda^2 :$$

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array} \right).$$

Thus (M^4, g) Einstein $\Longleftrightarrow$

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

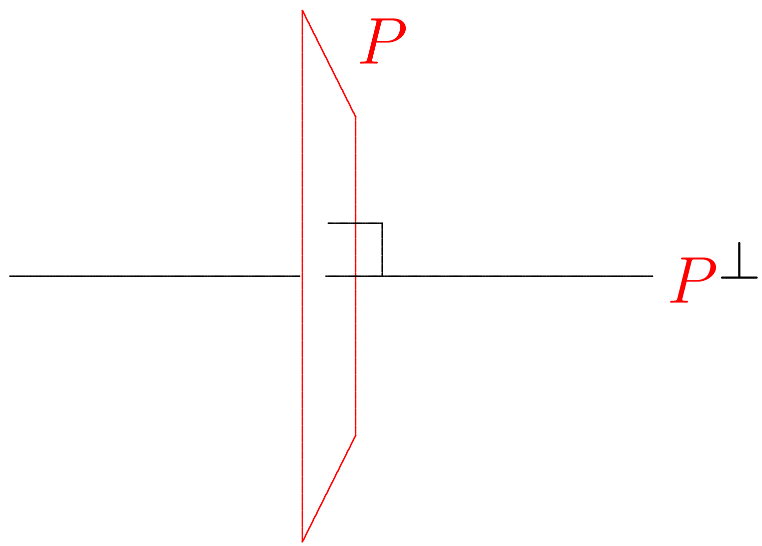
commutes with

$$\star : \Lambda^2 \to \Lambda^2 :$$

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & 0 \\ \hline 0 & W_- + \frac{s}{12} \end{array} \right).$$

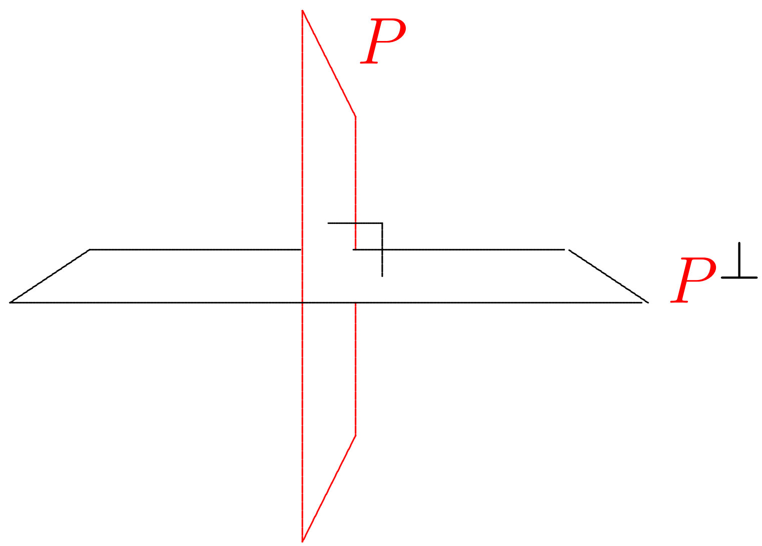
Corollary. *A Riemannian 4-manifold* (M, g) *is Einstein* $\Longleftrightarrow$ *sectional curvatures are equal for any pair of perpendicular 2-planes.*

Corollary. *A Riemannian 4-manifold* (M, g) *is Einstein* $\Longleftrightarrow$ *sectional curvatures are equal for any pair of perpendicular 2-planes.*



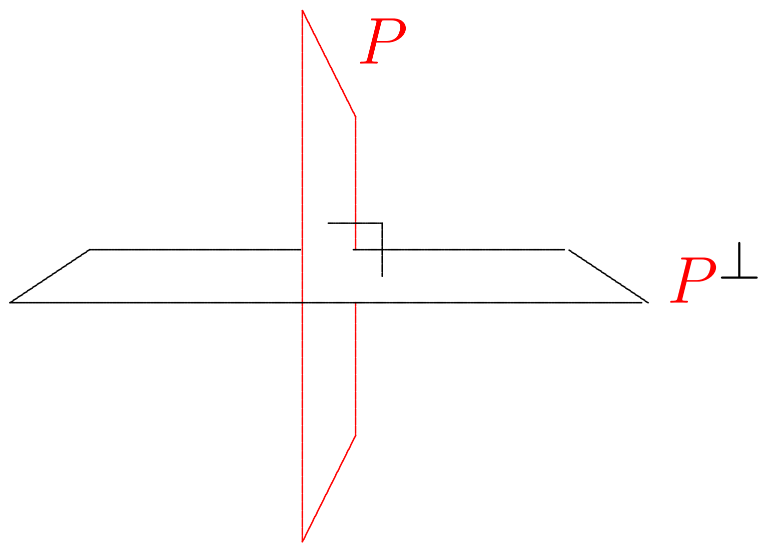
$T_x M$

Corollary. *A Riemannian 4-manifold* (M, g) *is Einstein* $\Longleftrightarrow$ *sectional curvatures are equal for any pair of perpendicular 2-planes.*



T_xM

Corollary. *A Riemannian 4-manifold* (M, g) *is Einstein* $\Longleftrightarrow$ *sectional curvatures are equal for any pair of perpendicular 2-planes.*



$$K(P) = K(P^{\perp})$$

(M^4, g) compact oriented Riemannian.

(M^4, g) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

(M^4, g) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + \qquad\qquad\qquad\qquad \right) d\mu$$

(M^4, g) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + |W_+|^2 \qquad\qquad\qquad\qquad \right) d\mu$$

(M^4, g) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 \qquad\qquad \right) d\mu$$

(M^4, g) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu$$

(M^4, g) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu$$

for Euler-characteristic $\chi(M) = \sum_j (-1)^j b_j(M)$.

4-dimensional Thom-Hirzebruch signature formula

4-dimensional Thom-Hirzebruch signature formula

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

4-dimensional Thom-Hirzebruch signature formula

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

for signature $\tau(M) = b_+(M) - b_-(M)$.

4-dimensional Thom-Hirzebruch signature formula

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

for signature $\tau(M) = b_+(M) - b_-(M)$.

Thom's contribution:

$$\tau(M) = \frac{1}{3} \langle p_1(TM), [M] \rangle$$

4-dimensional Thom-Hirzebruch signature formula

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

for signature $\tau(M) = b_+(M) - b_-(M)$.

Thom's contribution:

$$\tau(M) = \frac{1}{3}\langle p_1(TM), [M]\rangle$$

Hirzebruch generalized this to dimensions $4k$.

4-dimensional Thom-Hirzebruch signature formula

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

for signature $\tau(M) = b_+(M) - b_-(M)$.

Defined in terms of intersection pairing

Defined in terms of intersection pairing

$$\begin{aligned} H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) &\longrightarrow \mathbb{R} \\ (\,[\varphi]\,,\,[\psi]\,) &\longmapsto \int_M \varphi \wedge \psi \end{aligned}$$

Defined in terms of intersection pairing

$$\begin{aligned} H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) &\longrightarrow \mathbb{R} \\ ([\varphi], [\psi]) &\longmapsto \int_M \varphi \wedge \psi \end{aligned}$$

Diagonalize:

$$\begin{bmatrix} +1 & & & & & \\ & \ddots & & & & \\ & & +1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix}.$$

Defined in terms of intersection pairing

$$\begin{array}{ccc} H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) & \longrightarrow & \mathbb{R} \\ (\ [\varphi]\ ,\ [\psi]\ ) & \longmapsto & \int_M \varphi \wedge \psi \end{array}$$

Diagonalize: (over $\mathbb{R}$)

$$\begin{bmatrix} +1 & & & & & \\ & \ddots & & & & \\ & & +1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix}.$$

Defined in terms of intersection pairing

$$
\begin{array}{ccc}
H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) & \longrightarrow & \mathbb{R} \\
(\ [\varphi]\ ,\ [\psi]\ ) & \longmapsto & \int_M \varphi \wedge \psi
\end{array}
$$

Diagonalize:

$$
\begin{bmatrix}
+1 & & & & & \\
& \ddots & & & & \\
& & +1 & & & \\
& & & -1 & & \\
& & & & \ddots & \\
& & & & & -1
\end{bmatrix}.
$$

Defined in terms of intersection pairing

$$\begin{aligned} H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) &\longrightarrow \mathbb{R} \\ ([\varphi] , [\psi]) &\longmapsto \int_M \varphi \wedge \psi \end{aligned}$$

Diagonalize:

$$\left[\begin{array}{cccccc} \underbrace{\begin{array}{ccc} +1 & & \\ & \ddots & \\ & & +1 \end{array}}_{b_+(M)} & \\ & b_-(M) \left\{ \begin{array}{ccc} -1 & & \\ & \ddots & \\ & & -1 \end{array} \right. \end{array} \right].$$

Defined in terms of intersection pairing

$$\begin{array}{ccc} H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) & \longrightarrow & \mathbb{R} \\ (\, [\varphi] \, , \, [\psi] \,) & \longmapsto & \int_M \varphi \wedge \psi \end{array}$$

Diagonalize:

$$\left[\begin{array}{cccccc} +1 & & & & & \\ & \ddots & & & & \\ & & +1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{array} \right] .$$

with $b_+(M)$ entries $+1$ and $b_-(M)$ entries -1.

$$b_2(M) = b_+(M) + b_-(M)$$

Defined in terms of intersection pairing

$$H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) \longrightarrow \mathbb{R}$$
$$(\,[\varphi]\,,\,[\psi]\,) \longmapsto \int_M \varphi \wedge \psi$$

Diagonalize:

$$\begin{bmatrix} \underbrace{\begin{matrix} +1 & & \\ & \ddots & \\ & & +1 \end{matrix}}_{b_+(M)} & \\ & b_-(M)\left\{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}\right. \end{bmatrix}.$$

$$\tau(M) = b_+(M) - b_-(M)$$

"Signature" of M.

Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

$$\text{Einstein} \Rightarrow \qquad = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu_g$$

Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

$$\text{Einstein} \Rightarrow \qquad = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu_g$$

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array}\right)$$

Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu_g$$

$$\text{Einstein} \Rightarrow \qquad = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 \right) d\mu_g$$

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array} \right)$$

Curvature Λ^+ Curvature Λ^-

Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

Einstein $\Rightarrow$

$$= \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu_g$$

Theorem (Hitchin-Thorpe Inequality)**.** *If smooth compact oriented M^4 admits Einstein g, then*

$$(2\chi + 3\tau)(M) \geq 0,$$

Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

$$\text{Einstein} \Rightarrow \qquad = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu_g$$

Theorem (Hitchin-Thorpe Inequality). *If smooth compact oriented M^4 admits Einstein g, then*

$$(2\chi + 3\tau)(M) \geq 0,$$

with equality only if (M, g) is locally hyper-Kähler.

Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

Einstein $\Rightarrow$ $$= \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu_g$$

Theorem (Hitchin-Thorpe Inequality). *If smooth compact oriented M^4 admits Einstein g, then*

$$(2\chi + 3\tau)(M) \geq 0,$$

with equality only if (M, g) is locally hyper-Kähler.

$\Longleftrightarrow$ Λ^+ is flat.

Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

$$\text{Einstein} \Rightarrow \quad = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu_g$$

Theorem (Hitchin-Thorpe Inequality). *If smooth compact oriented M^4 admits Einstein g, then*

$$(2\chi + 3\tau)(M) \geq 0,$$

with equality only if (M, g) is locally hyper-Kähler. The latter case happens only if M finitely covered by flat T^4 or Calabi-Yau $K3$.

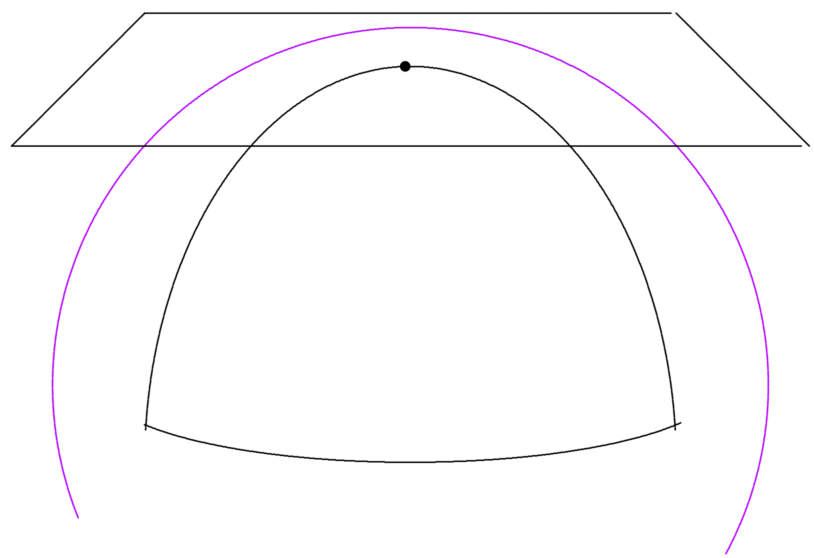
Hyper-Kähler?

Hyper-Kähler? Kähler?

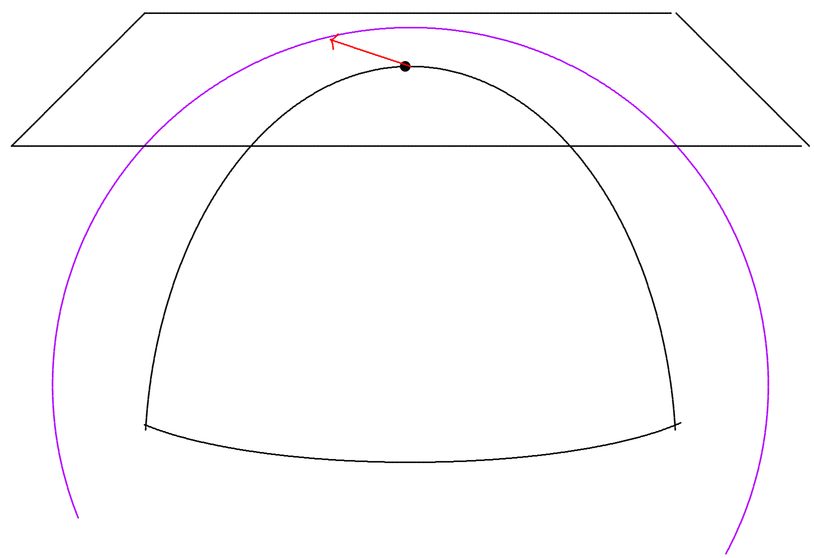
Hyper-Kähler? Kähler? Calabi-Yau?

(M^n, g): holonomy

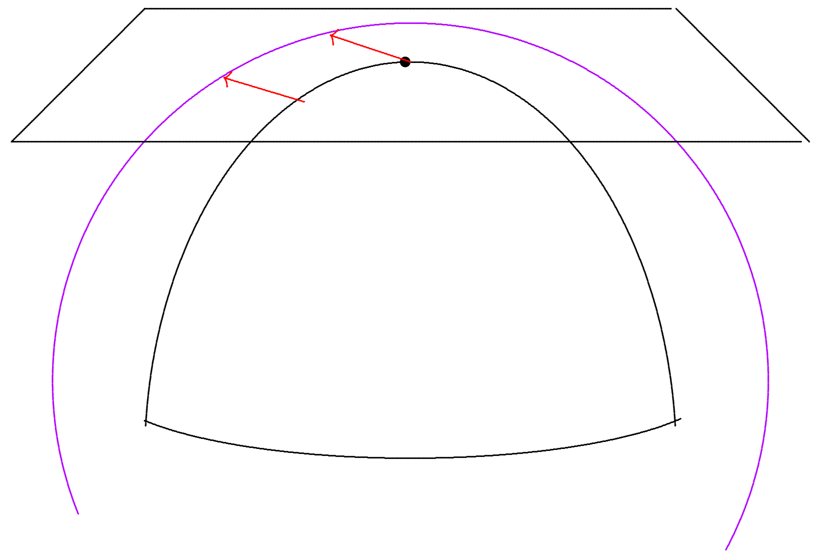
(M^n, g): holonomy



(M^n, g): holonomy

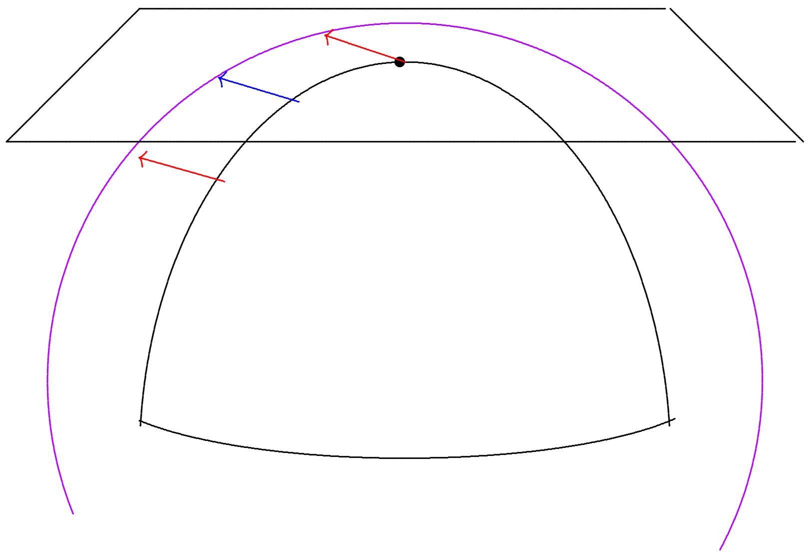


(M^n, g): holonomy



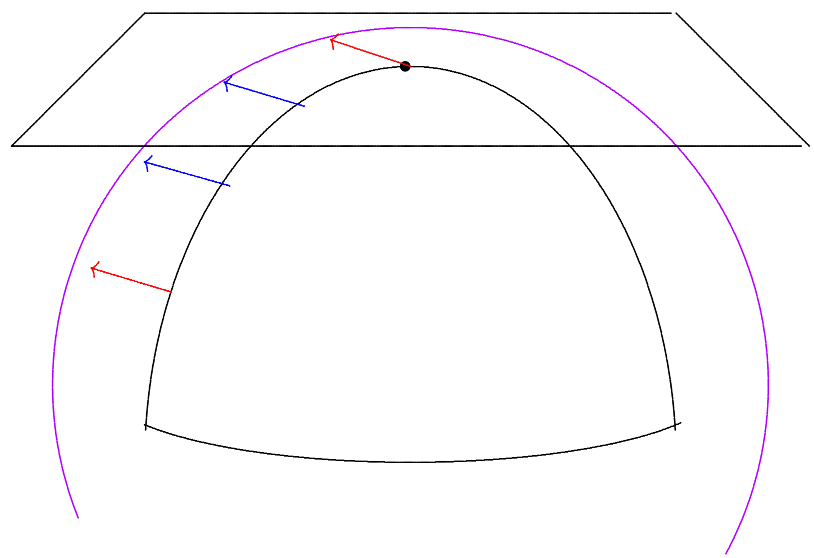
(M^n, g):

holonomy

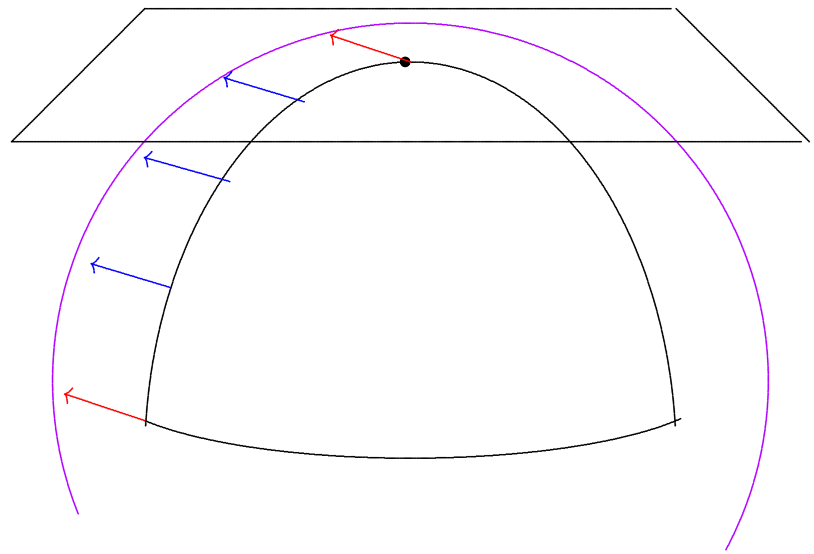


(M^n, g):

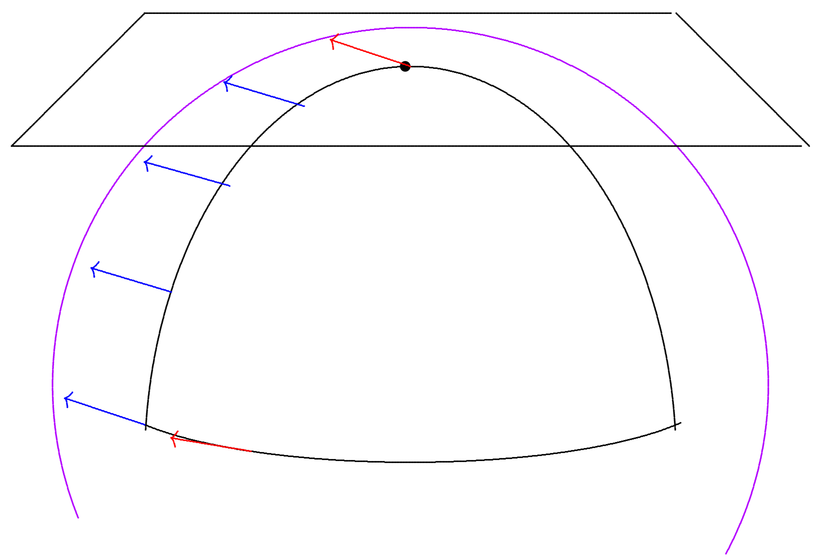
holonomy



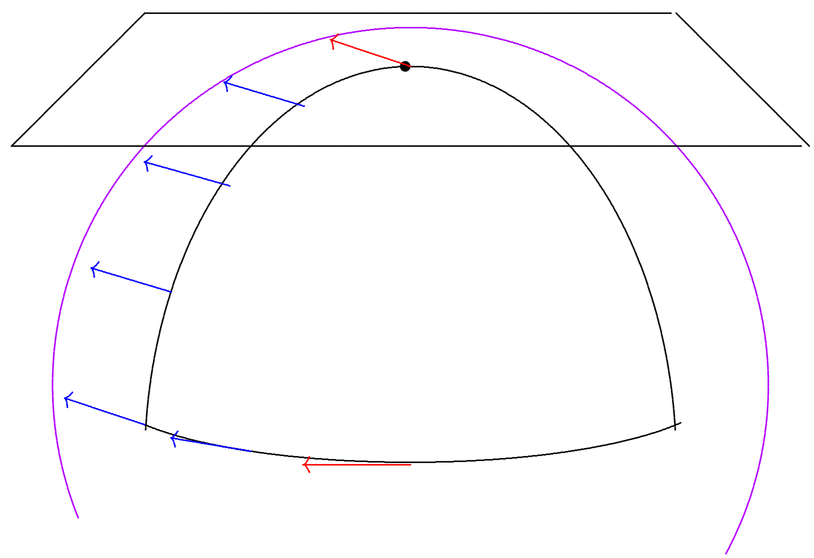
(M^n, g): holonomy



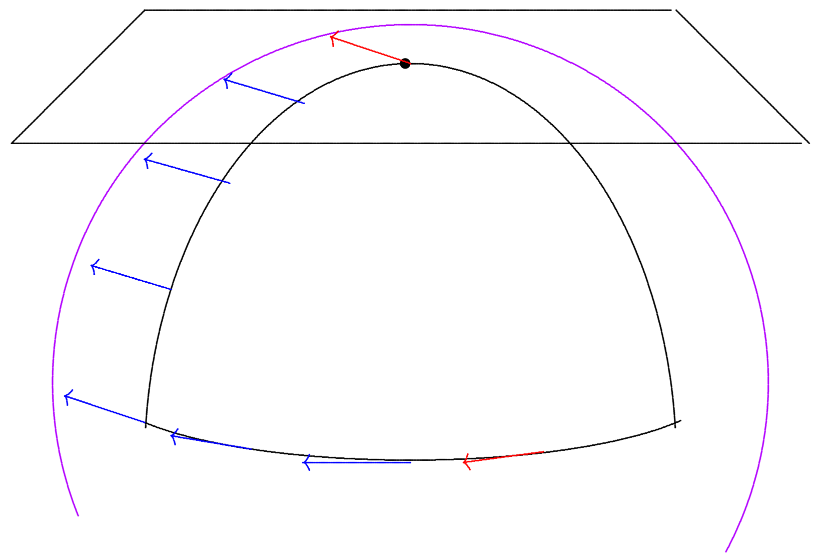
(M^n, g): holonomy



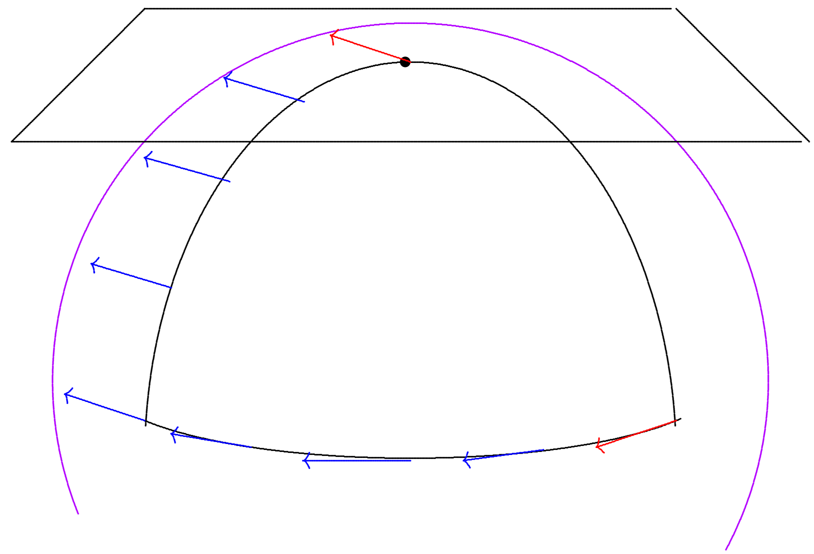
(M^n, g): holonomy



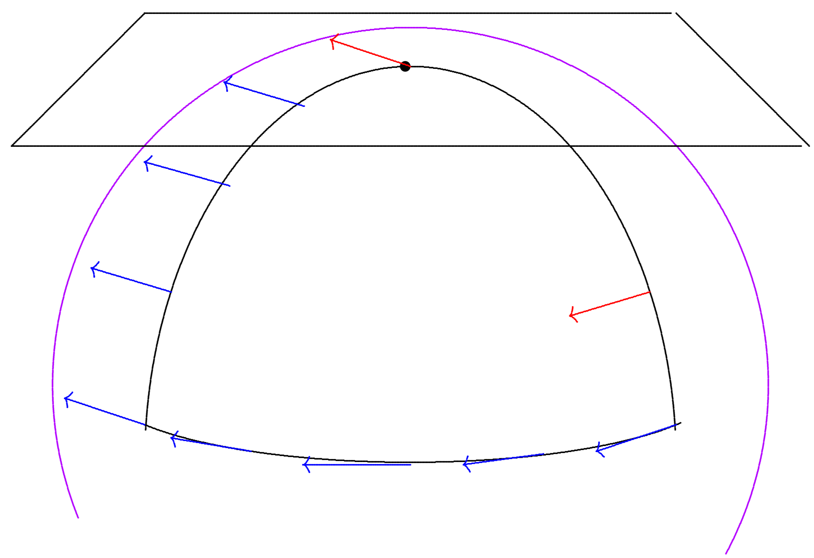
(M^n, g): holonomy



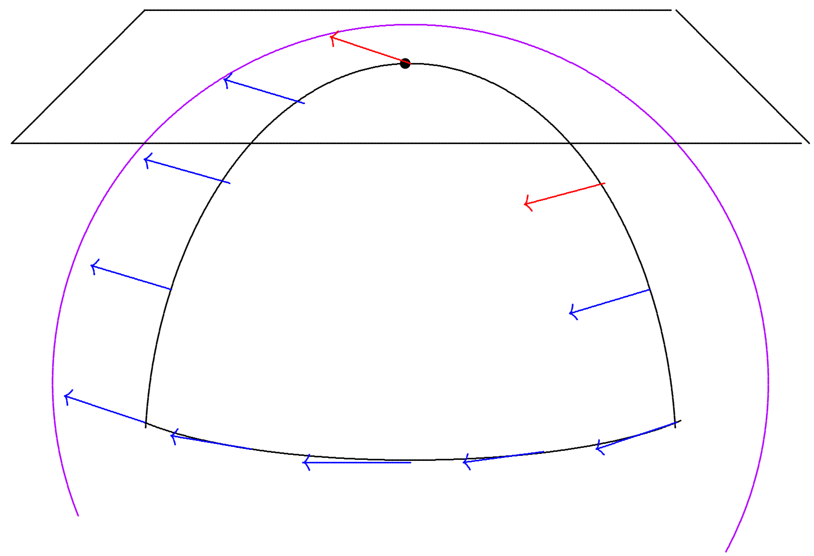
(M^n, g): holonomy



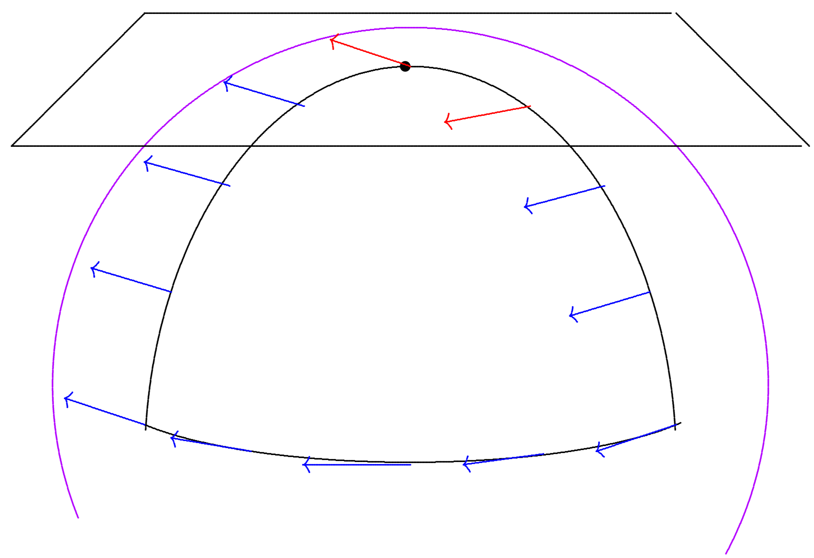
(M^n, g): holonomy



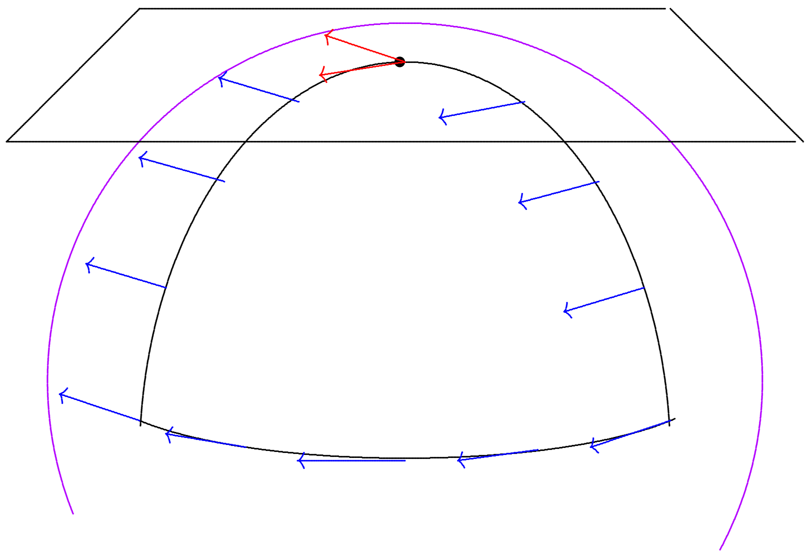
(M^n, g): holonomy



(M^n, g): holonomy

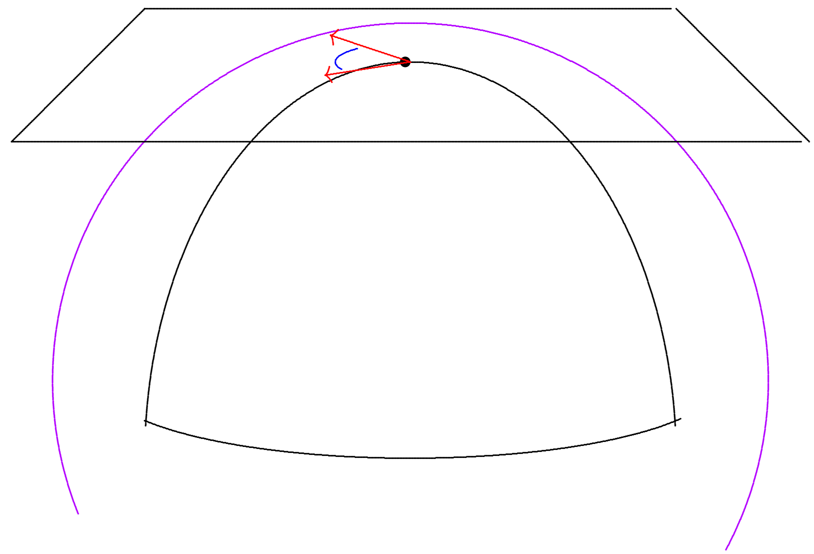


(M^n, g): holonomy

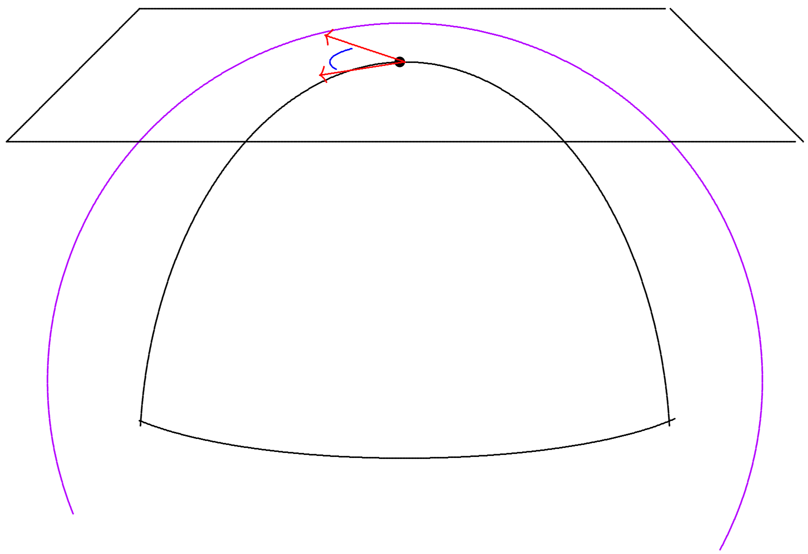


(M^n, g):

holonomy

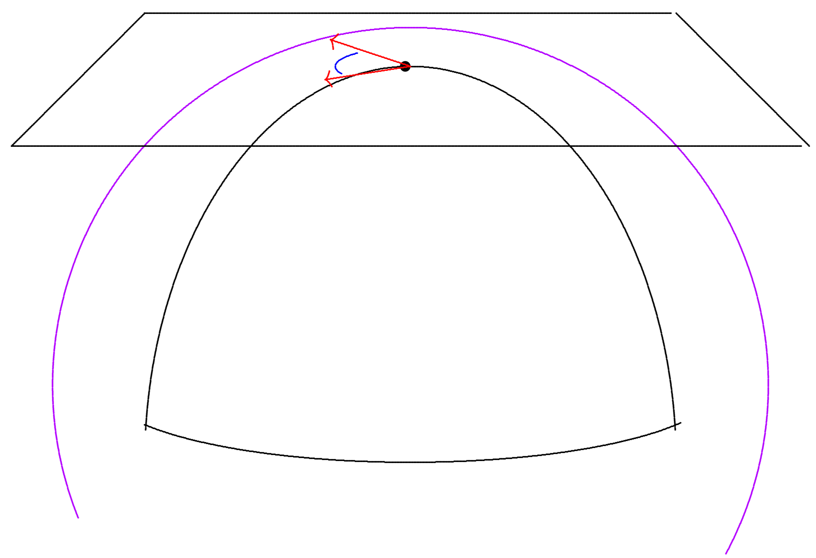


(M^n, g): holonomy $\subset \mathbf{O}(n)$



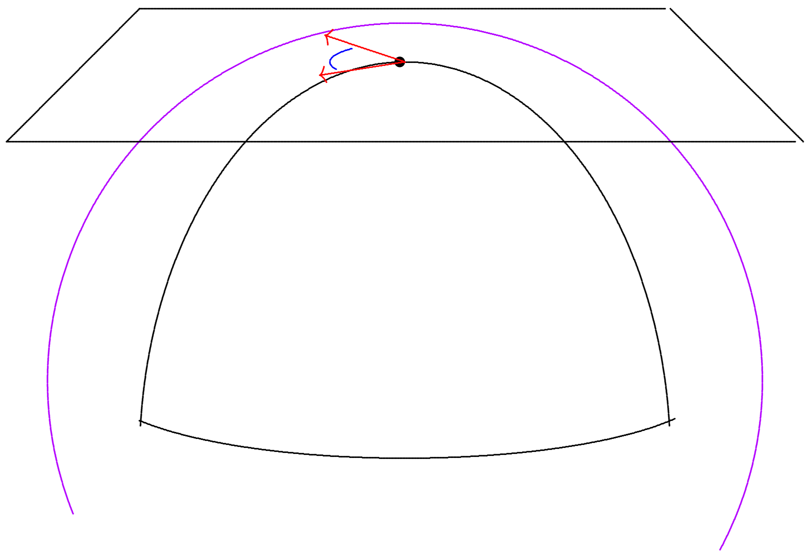
Kähler metrics:

(M^{2m}, g): holonomy



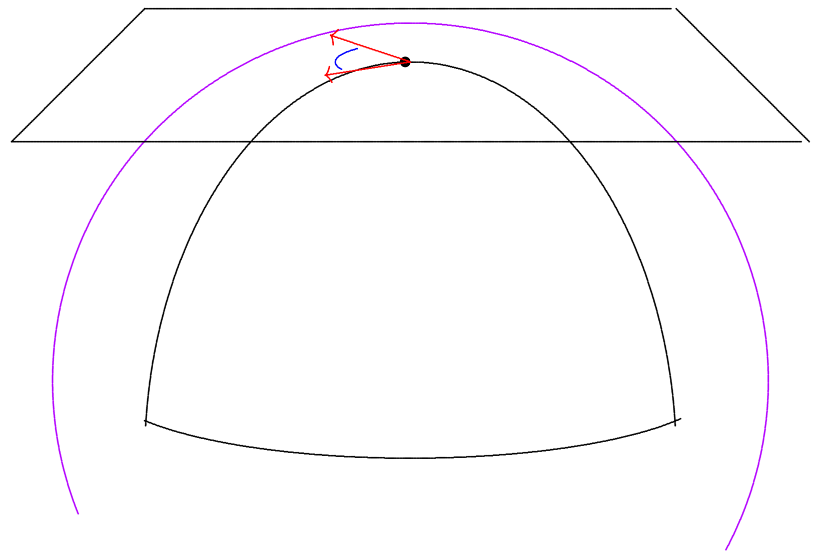
Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$



Kähler metrics:

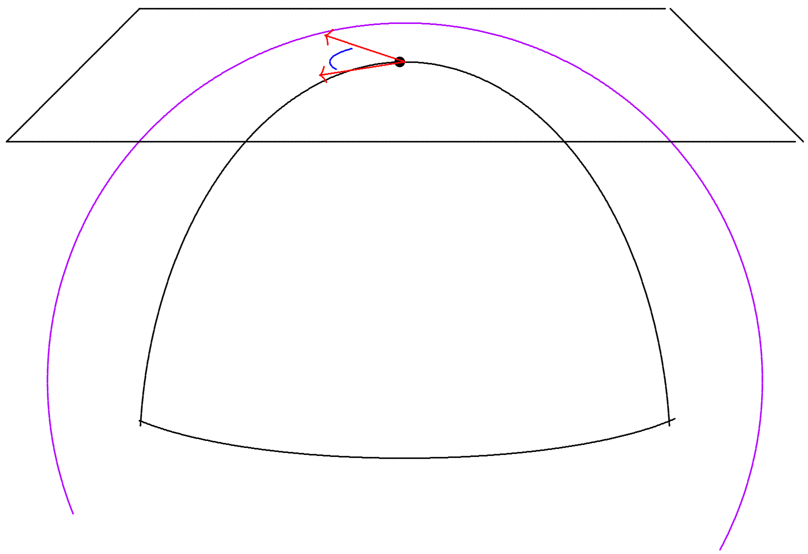
(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$



$$\mathbf{U}(m) := \mathbf{O}(2m) \cap \mathbf{GL}(m, \mathbb{C})$$

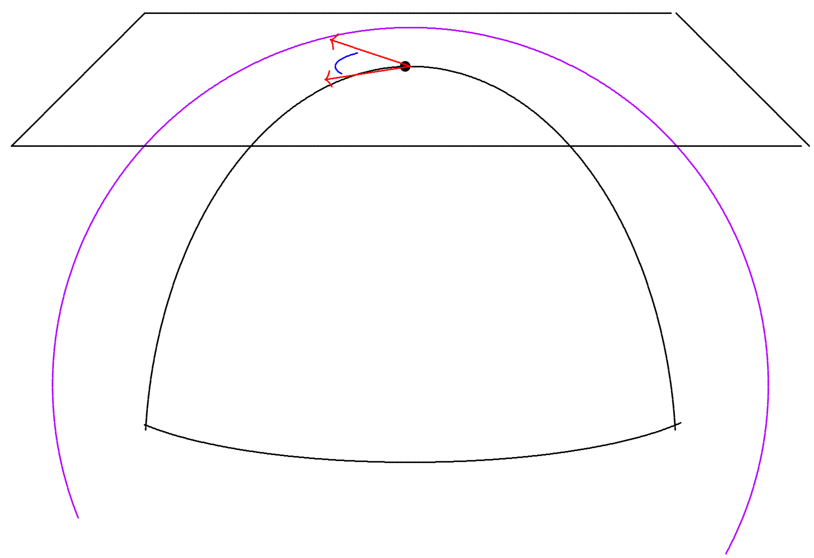
Ricci-flat Kähler metrics:

(M^{2m}, g): holonomy



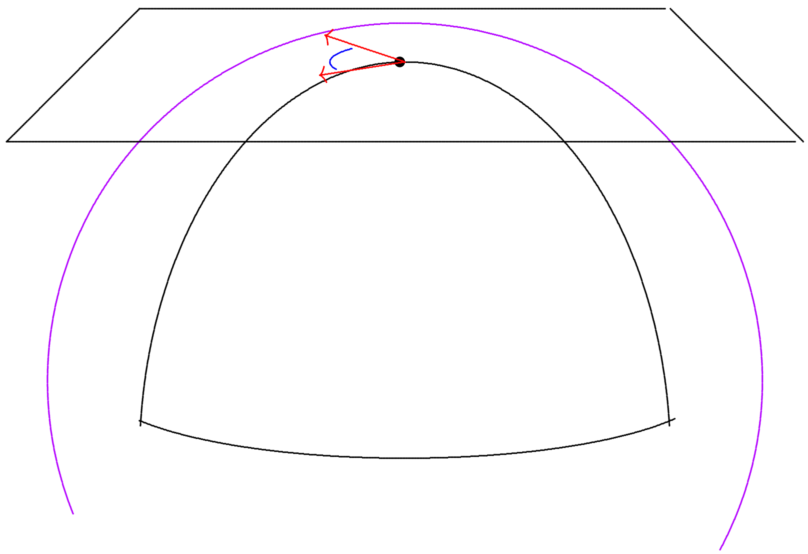
Ricci-flat Kähler metrics:

(M^{2m}, g): Ricci-flat Kähler $\Longleftarrow$ holonomy $\subset \mathbf{SU}(m)$



Ricci-flat Kähler metrics:

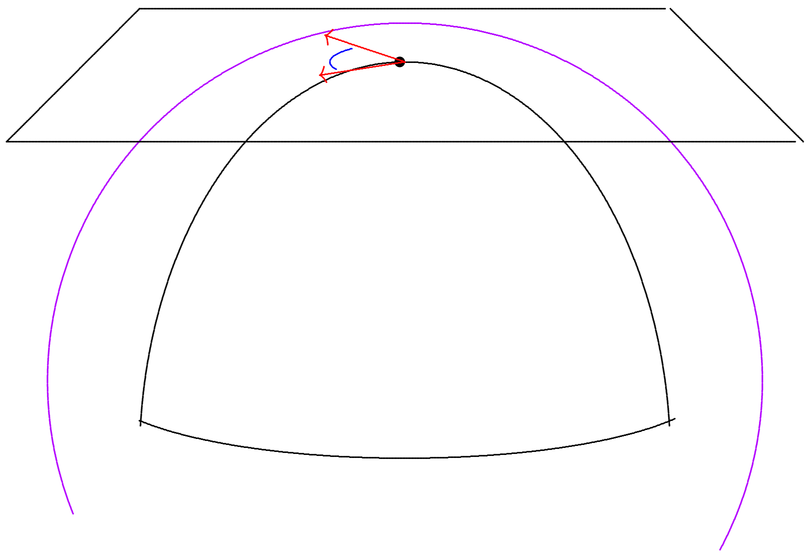
(M^{2m}, g): Ricci-flat Kähler $\Longleftarrow$ holonomy $\subset \mathbf{SU}(m)$



$\mathbf{SU}(m) \subset \mathbf{U}(m)$: $\{A \mid \det A = 1\}$

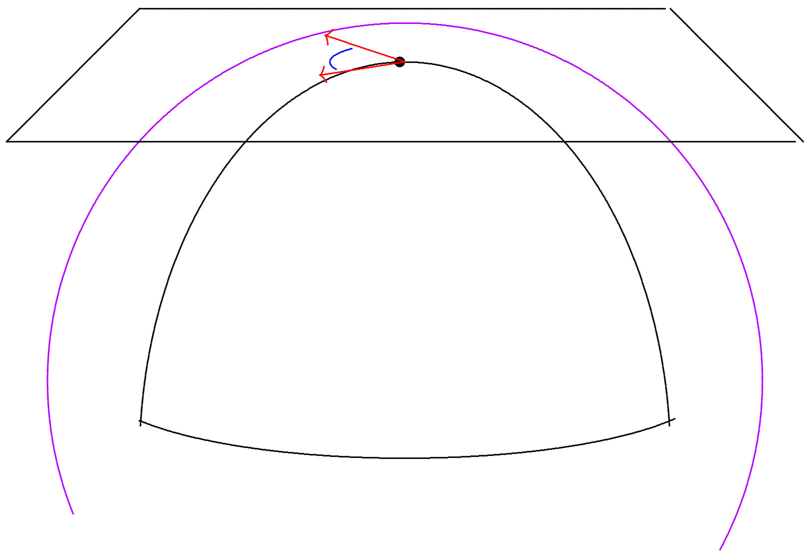
Ricci-flat Kähler metrics:

(M^{2m}, g): Ricci-flat Kähler $\Longleftarrow$ holonomy $\subset \mathbf{SU}(m)$



Ricci-flat Kähler metrics:

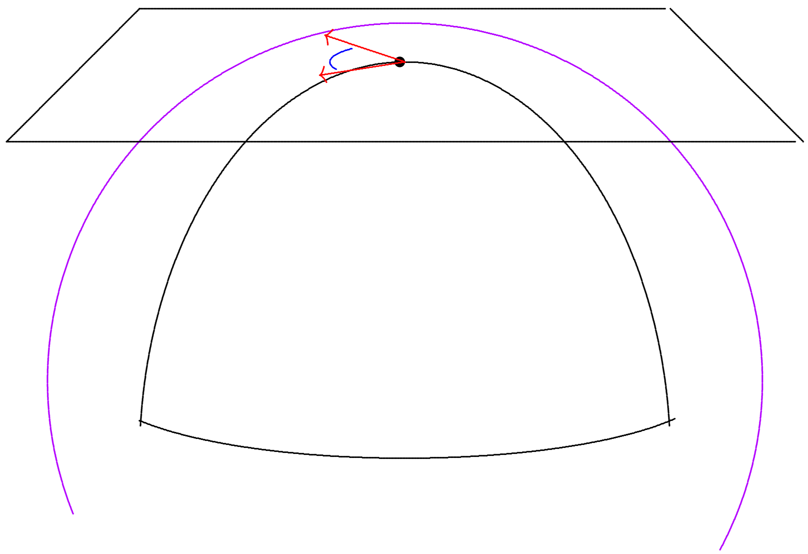
(M^{2m}, g): Ricci-flat Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{SU}(m)$



if M is simply connected.

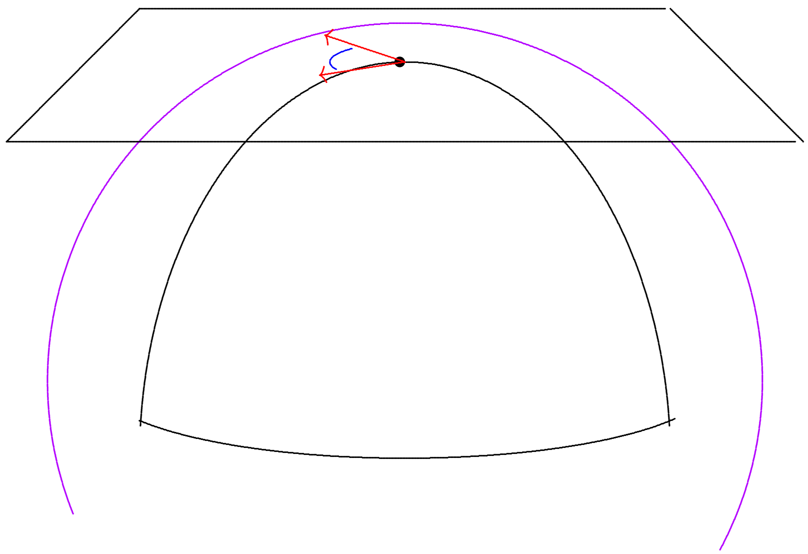
Calabi-Yau metrics:

(M^{2m}, g): Calabi-Yau $\Longleftrightarrow$ holonomy $\subset \mathbf{SU}(m)$



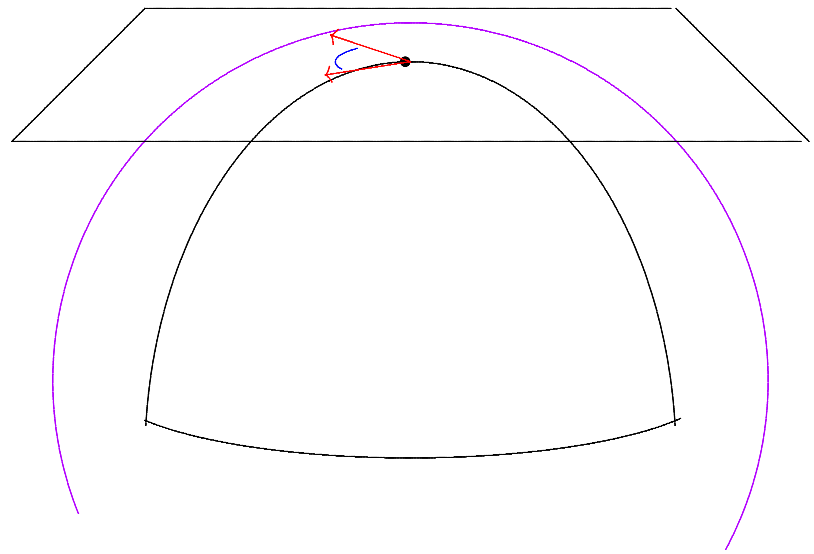
Hyper-Kähler metrics:

(M^{4k}, g) holonomy



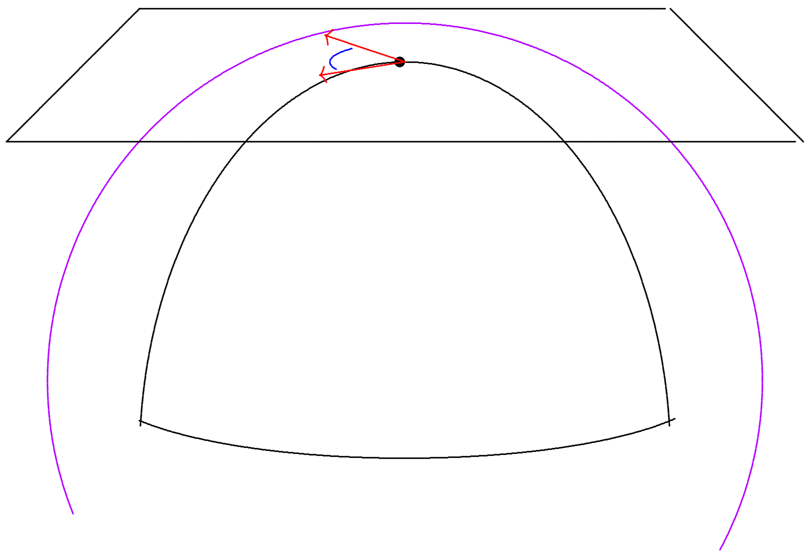
Hyper-Kähler metrics:

(M^{4k}, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(k)$



Hyper-Kähler metrics:

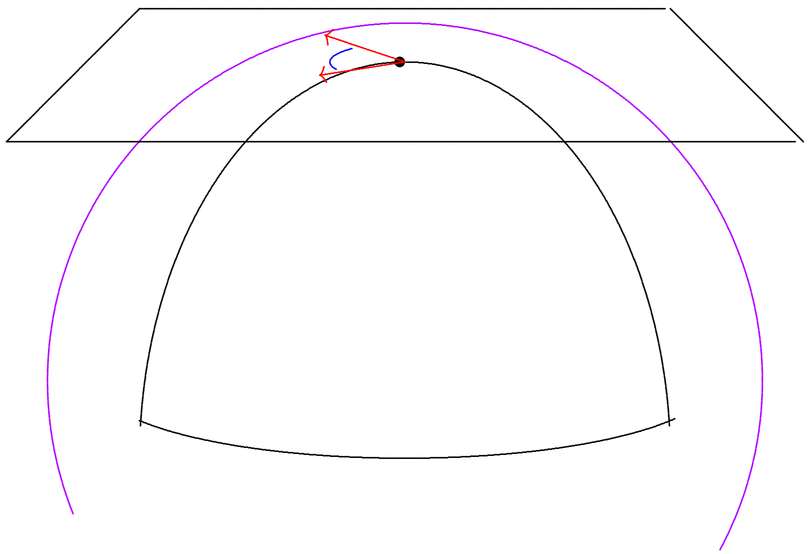
(M^{4k}, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(k)$



$$\mathbf{Sp}(k) := \mathbf{O}(4k) \cap \mathbf{GL}(\ell, \mathbb{H})$$

Hyper-Kähler metrics:

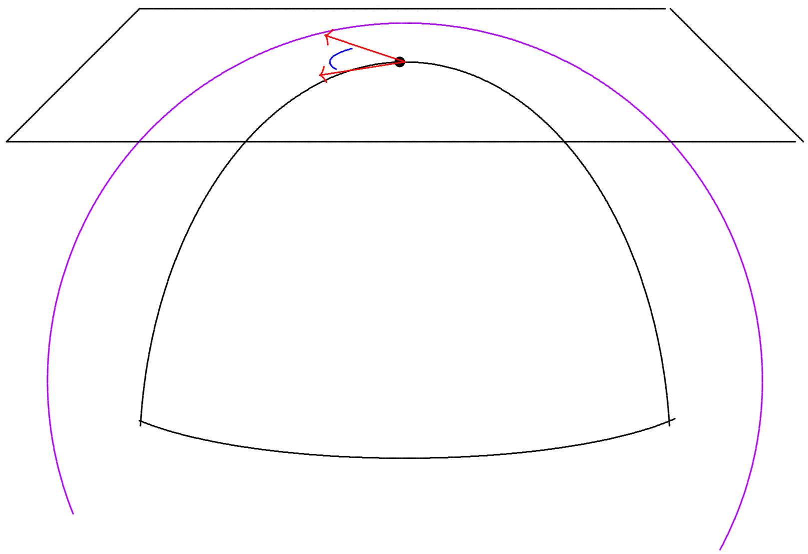
(M^{4k}, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(k)$



$\mathbf{Sp}(k) \subset \mathbf{SU}(2k)$

Hyper-Kähler metrics:

(M^{4k}, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(k)$

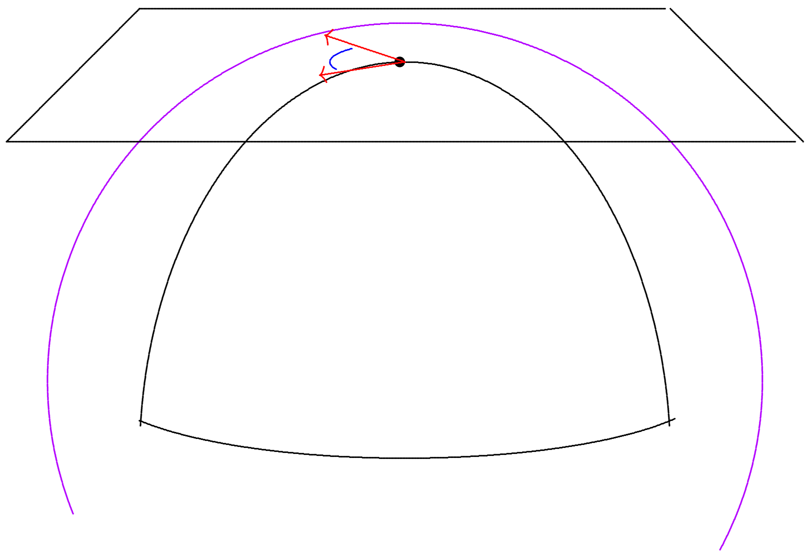


$\mathbf{Sp}(k) \subset \mathbf{SU}(2k)$

in many ways!

Hyper-Kähler metrics:

(M^{4k}, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(k)$

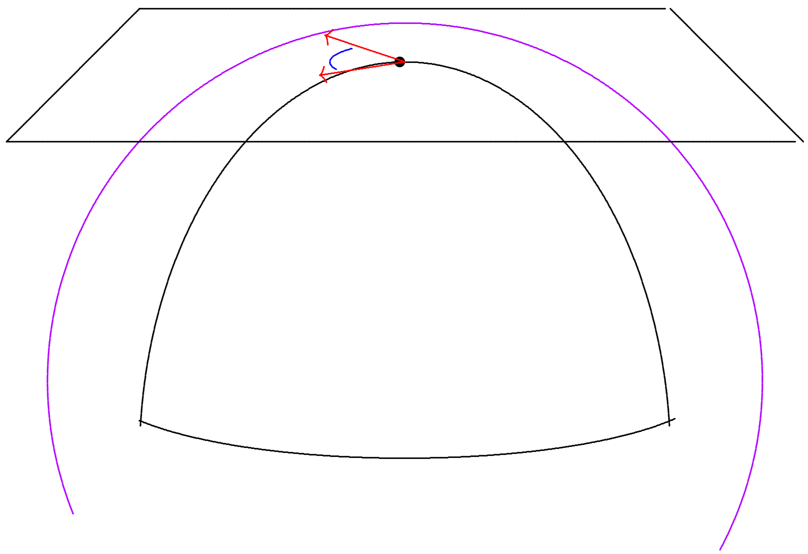


$$\mathbf{Sp}(k) \subset \mathbf{SU}(2k)$$

in many ways! (For example, permute i, j, $k\ldots$)

Hyper-Kähler metrics:

(M^{4k}, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(k)$



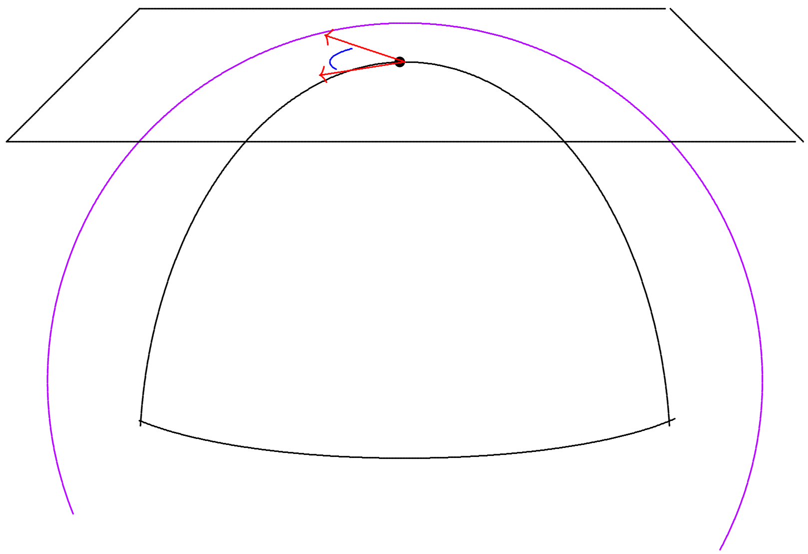
$$\mathbf{Sp}(k) \subset \mathbf{SU}(2k)$$

Ricci-flat and Kähler,

for many different complex structures!

Hyper-Kähler metrics:

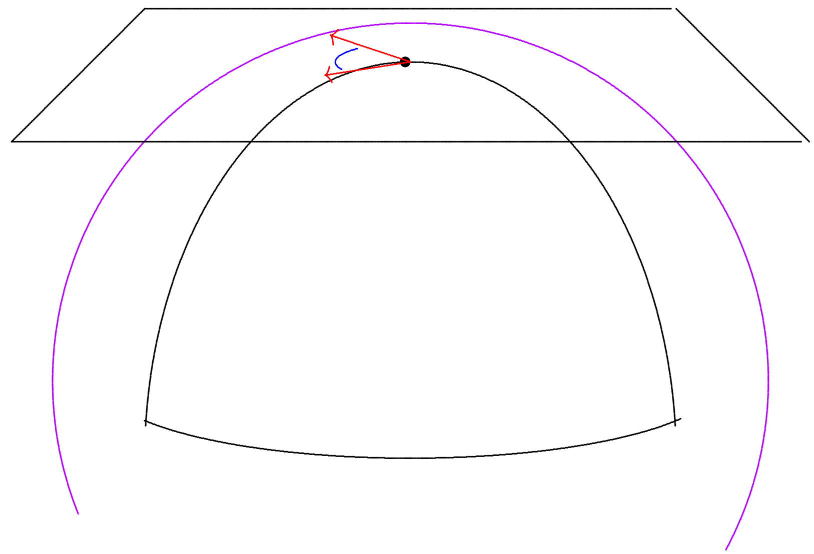
(M^{4k}, g) hyper-Kähler $\iff$ holonomy $\subset \mathbf{Sp}(k)$



$\mathbf{Sp}(k) \subset \mathbf{SU}(2k)$

Hyper-Kähler metrics:

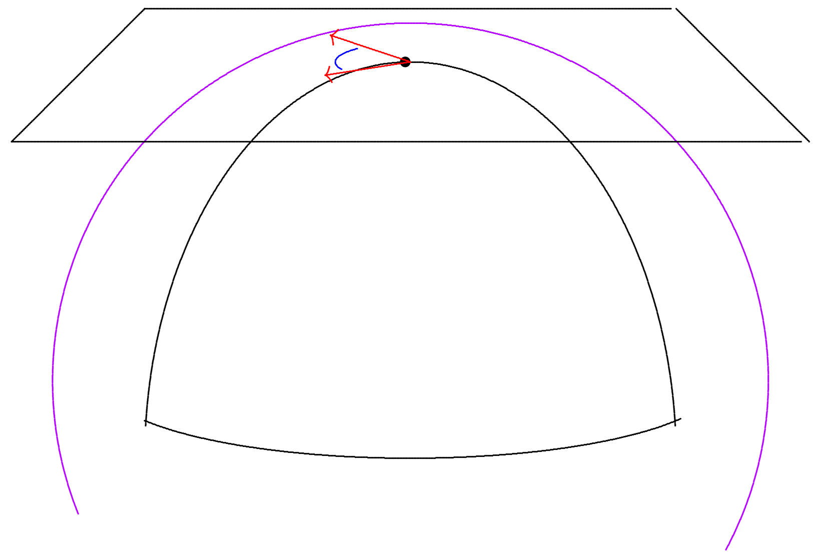
(M^4, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(1)$



$$\mathbf{Sp}(1) = \mathbf{SU}(2)$$

Hyper-Kähler metrics:

(M^4, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(1)$



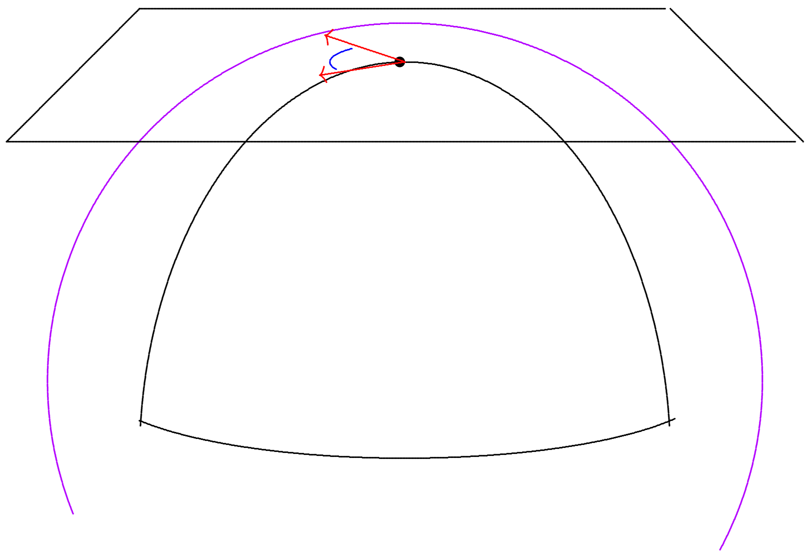
$$\mathbf{Sp}(1) = \mathbf{SU}(2)$$

For (M^4, g):

hyper-Kähler $\Longleftrightarrow$ Calabi-Yau.

Hyper-Kähler metrics:

(M^4, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(1)$



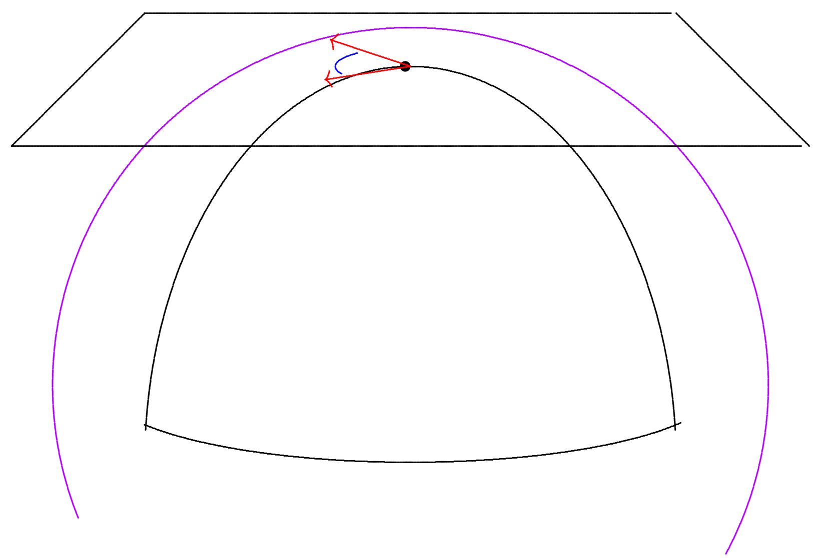
$$\mathbf{Sp}(1) = \mathbf{SU}(2)$$

When (M^4, g) simply connected:

hyper-Kähler $\Longleftrightarrow$ Ricci-flat Kähler.

Hyper-Kähler metrics:

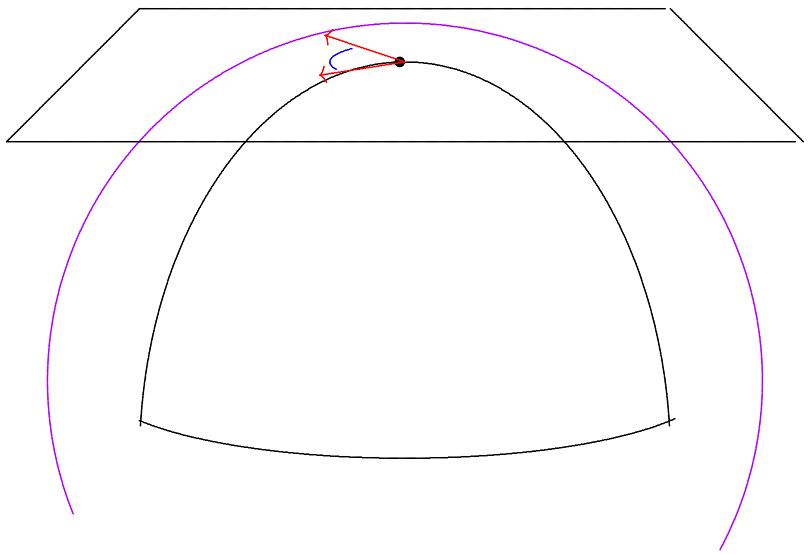
(M^4, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(1)$



$\mathbf{Sp}(1) = \mathbf{SU}(2)$

Hyper-Kähler metrics:

(M^4, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(1)$



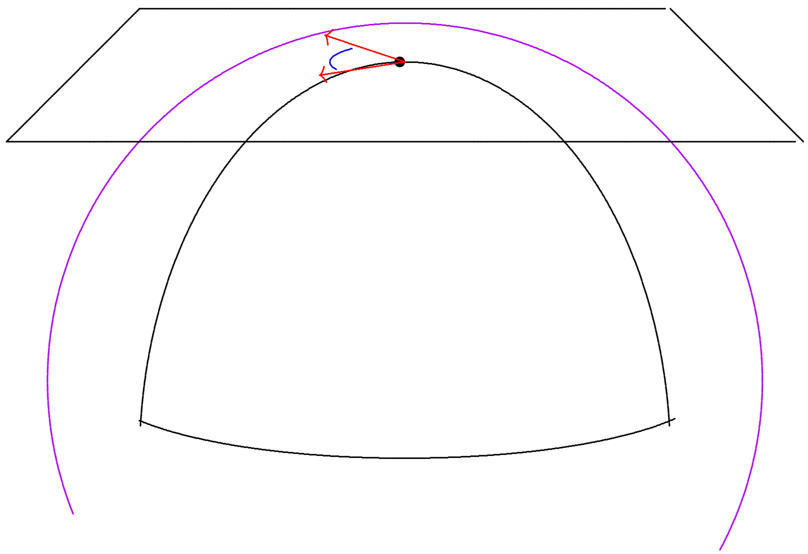
$$\mathbf{Sp}(1) = \mathbf{SU}(2)$$

Ricci-flat and Kähler,

for many different complex structures!

Hyper-Kähler metrics:

(M^{4k}, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(k)$



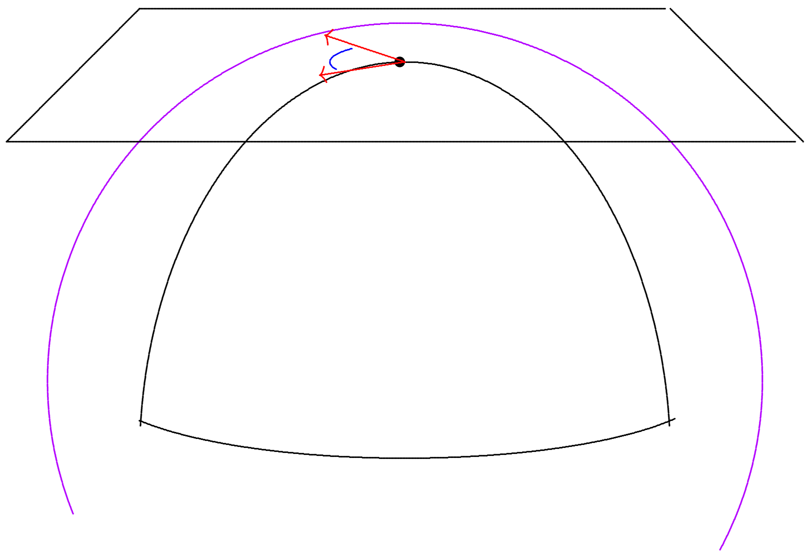
$\mathbf{Sp}(k) \subset \mathbf{SU}(2k)$

Ricci-flat and Kähler,

for many different complex structures!

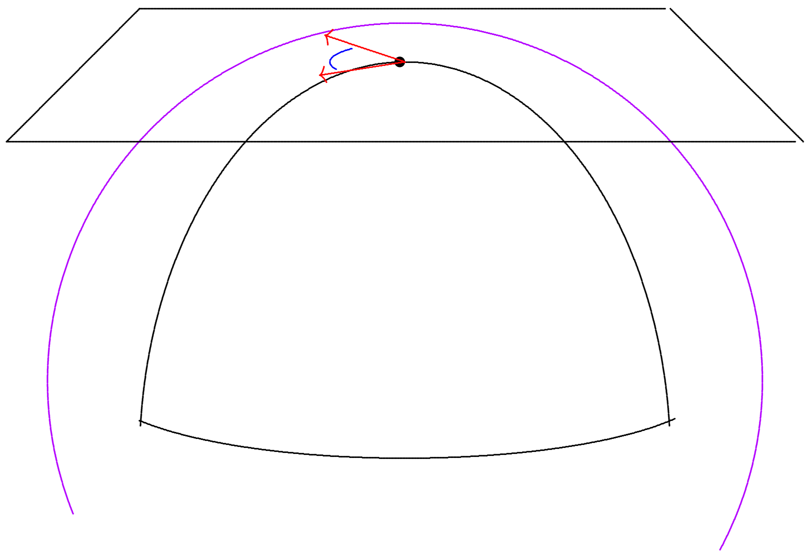
Kähler metrics:

(M^{2m}, g): holonomy



Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$



Kähler metrics:

(M^{2m}, g) Kähler $\iff$ holonomy $\subset \mathbf{U}(m)$

$\iff \exists$ almost complex-structure J with $\nabla J = 0$
and $g(J\cdot, J\cdot) = g$.

Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$

$\Longleftrightarrow$ $\exists$ almost complex-structure J with $\nabla J = 0$ and $g(J\cdot, J\cdot) = g$.

$\Longleftrightarrow$ (M^{2m}, g) is a complex manifold & $\exists$ J-invariant closed 2-form ω such that $g = \omega(\cdot, J\cdot)$.

Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$

$\Longleftrightarrow$ $\exists$ almost complex-structure J with $\nabla J = 0$ and $g(J\cdot, J\cdot) = g$.

$\Longleftrightarrow$ (M^{2m}, g) is a complex manifold & $\exists$ J-invariant closed 2-form ω such that $g = \omega(\cdot, J\cdot)$.

$$d\omega = 0$$

Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$

$\Longleftrightarrow$ $\exists$ almost complex-structure J with $\nabla J = 0$ and $g(J\cdot, J\cdot) = g$.

$\Longleftrightarrow$ (M^{2m}, g) is a complex manifold & $\exists$ J-invariant closed 2-form ω such that $g = \omega(\cdot, J\cdot)$.

$$d\omega = 0$$

$$[\omega] \in H^2(M)$$

"Kähler class"

Kähler metrics:

Kähler metrics:

Original definition:

Kähler metrics:

Original definition:

M can be made into a complex manifold,

Kähler metrics:

Original definition:

M can be made into a complex manifold, in such a manner that, locally,

Kähler metrics:

Original definition:

M can be made into a complex manifold, in such a manner that, locally,

$$g = \sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \left[dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j \right]$$

Kähler metrics:

Original definition:

M can be made into a complex manifold, in such a manner that, locally,

$$g = \sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \left[dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j \right]$$

for a locally defined function f.

Kähler metrics:

Original definition:

M can be made into a complex manifold, in such a manner that, locally,

$$g = \sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \left[dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j \right]$$

for a locally defined function f.

Associated Kähler form:

$$\omega = i \sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \, dz^j \wedge d\bar{z}^k$$

Kähler metrics:

Original definition:

M can be made into a complex manifold, in such a manner that, locally,

$$g = \sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \left[dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j \right]$$

for a locally defined function f.

Modern definition:

Kähler metrics:

Original definition:

M can be made into a complex manifold, in such a manner that, locally,

$$g = \sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \left[dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j \right]$$

for a locally defined function f.

Modern definition:

(M^{2m}, g) has holonomy $\subset \mathbf{U}(m)$.

Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$

$\Longleftrightarrow$ $\exists$ almost complex-structure J with $\nabla J = 0$ and $g(J\cdot, J\cdot) = g$.

$\Longleftrightarrow$ (M^{2m}, g) is a complex manifold & $\exists$ J-invariant closed 2-form ω such that $g = \omega(\cdot, J\cdot)$.

$$d\omega = 0$$

Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$

$\Longleftrightarrow$ $\exists$ almost complex-structure J with $\nabla J = 0$ and $g(J\cdot, J\cdot) = g$.

$\Longleftrightarrow$ (M^{2m}, g) is a complex manifold & $\exists$ J-invariant closed 2-form ω such that $g = \omega(\cdot, J\cdot)$.

$$d\omega = 0$$

ω is also defines a symplectic structure on M.

Symplectic 4-manifolds:

Symplectic 4-manifolds:

(M^4, ω) where ω closed non-degenerate 2-form:

Symplectic 4-manifolds:

(M^4, ω) where ω closed non-degenerate 2-form:

$$d\omega = 0, \qquad \lrcorner\,\omega : TM \xrightarrow{\cong} T^*M.$$

Symplectic 4-manifolds:

(M^4, ω) where ω closed non-degenerate 2-form:

$$\omega = dx \wedge dy + dz \wedge dt$$

Symplectic 4-manifolds:

(M^4, ω) where ω closed non-degenerate 2-form:

$$\omega = dx \wedge dy + dz \wedge dt$$

Induces preferred orientation:

Symplectic 4-manifolds:

(M^4, ω) where ω closed non-degenerate 2-form:

$$\omega = dx \wedge dy + dz \wedge dt$$

Induces preferred orientation:

$$\frac{\omega \wedge \omega}{2} = dx \wedge dy \wedge dz \wedge dt$$

Symplectic 4-manifolds:

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω.

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω.

On such manifolds, Seiberg-Witten theory mimics Kähler geometry when treating non-Kähler metrics.

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω.

On such manifolds, Seiberg-Witten theory mimics Kähler geometry when treating non-Kähler metrics.

Some Suggestive Questions.

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω.

On such manifolds, Seiberg-Witten theory mimics Kähler geometry when treating non-Kähler metrics.

Some Suggestive Questions. *If* (M^4, ω) *is a symplectic* 4*-manifold,*

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω.

On such manifolds, Seiberg-Witten theory mimics Kähler geometry when treating non-Kähler metrics.

Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold, when does M^4 admit an Einstein metric g*

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω.

On such manifolds, Seiberg-Witten theory mimics Kähler geometry when treating non-Kähler metrics.

Some Suggestive Questions. *If* (M^4, ω) *is a symplectic* 4*-manifold, when does* M^4 *admit an Einstein metric* g *(unrelated to* ω*)?*

Symplectic 4-manifolds:

A laboratory for exploring Einstein metrics.

Kähler geometry is a rich source of examples.

If M admits a Kähler metric, it of course admits a symplectic form ω.

On such manifolds, Seiberg-Witten theory mimics Kähler geometry when treating non-Kähler metrics.

Some Suggestive Questions. *If* (M^4, ω) *is a symplectic* 4*-manifold, when does* M^4 *admit an Einstein metric* g *(unrelated to* ω*)? What if we also require* $\lambda \geq 0$*?*

Theorem (L '09)**.**

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold*

Theorem (L '09). *Suppose that M is a smooth compact oriented* 4*-manifold which admits a symplectic structure* ω.

Theorem (L '09). *Suppose that M is a smooth compact oriented* 4*-manifold which admits a symplectic structure* ω. *Then M also admits an Einstein metric* g

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric g with $\lambda \geq 0$*

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if*

Theorem (L '09). *Suppose that* M *is a smooth compact oriented* 4*-manifold which admits a symplectic structure* ω. *Then* M *also admits an Einstein metric* g *with* $\lambda \geq 0$ *if and only if*

$$M \overset{diff}{\approx} \left\{ \right.$$

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if*

$$M \overset{diff}{\approx} \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8, \\ \\ \\ \end{cases}$$

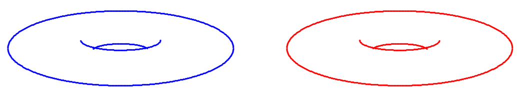
Conventions:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

Conventions:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

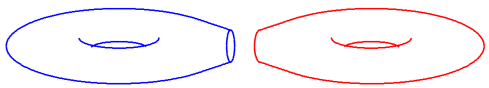
Connected sum #:



Conventions:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

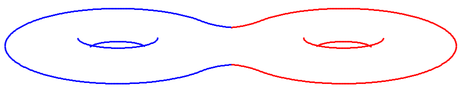
Connected sum #:



Conventions:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

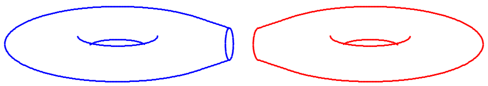
Connected sum $\#$:



Conventions:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

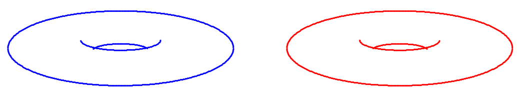
Connected sum #:



Conventions:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

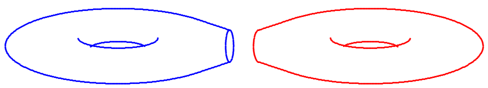
Connected sum $\#$:



Conventions:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

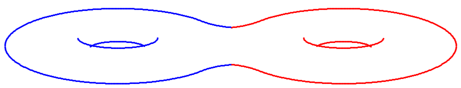
Connected sum #:



Conventions:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

Connected sum #:



Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if*

$$M \overset{diff}{\approx} \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}, \quad 0 \leq k \leq 8, \\ \\ \\ \\ \end{cases}$$

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if*

$$M \overset{diff}{\approx} \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}, & 0 \leq k \leq 8, \\ S^2 \times S^2, & \\ & \\ & \end{cases}$$

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if*

$$M \overset{diff}{\approx} \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}, & 0 \leq k \leq 8, \\ S^2 \times S^2, & \\ K3, & \\ & \\ & \\ & \end{cases}$$

$K3$ = Kummer-Kähler-Kodaira surface.

Simply connected complex surface with $c_1 = 0$.

$K3$ = Kummer-Kähler-Kodaira surface.

Simply connected complex surface with $c_1 = 0$.

Only one deformation type.

$K3$ = Kummer-Kähler-Kodaira surface.

Simply connected complex surface with $c_1 = 0$.

Only one diffeomorphism type.

$K3$ = Kummer-Kähler-Kodaira surface.

Simply connected complex surface with $c_1 = 0$.

$K3$ = Kummer-Kähler-Kodaira surface.

Simply connected complex surface with $c_1 = 0$.

Typical model: Smooth quartic in $\mathbb{CP}_3$.

$K3$ = Kummer-Kähler-Kodaira surface.

Simply connected complex surface with $c_1 = 0$.

Typical model: Smooth quartic in $\mathbb{CP}_3$.

$K3$ = Kummer-Kähler-Kodaira manifold.

Kummer construction:

Kummer: $T^4/\mathbb{Z}_2$: Singular quartic in $\mathbb{CP}_3$.

$K3$ = Kummer-Kähler-Kodaira manifold.

Kummer construction:

Kummer: $T^4/\mathbb{Z}_2$: Singular quartic in $\mathbb{CP}_3$.

T^4 = Picard torus of curve of genus 2.

$K3$ = Kummer-Kähler-Kodaira manifold.

Kummer construction:

Kummer: $T^4/\mathbb{Z}_2$: Singular quartic in $\mathbb{CP}_3$.

Remove singularities by deforming equation.

$K3$ = Kummer-Kähler-Kodaira manifold.

Kummer construction:

Kummer: $T^4/\mathbb{Z}_2$: Singular quartic in $\mathbb{CP}_3$.

Remove singularities by deforming equation.

$K3$ = Kummer-Kähler-Kodaira manifold.

Kummer construction:

Kummer: $T^4/\mathbb{Z}_2$: Singular quartic in $\mathbb{CP}_3$.

Remove singularities by deforming equation.

$K3$ = Kummer-Kähler-Kodaira manifold.

Kummer construction:

Kummer: $T^4/\mathbb{Z}_2$: Singular quartic in $\mathbb{CP}_3$.

Remove singularities by deforming equation.

$K3$ = Kummer-Kähler-Kodaira manifold.

Kummer construction:

Kummer: $T^4/\mathbb{Z}_2$: Singular quartic in $\mathbb{CP}_3$.

Remove singularities by deforming equation.

$K3$ = Kummer-Kähler-Kodaira manifold.

Kummer construction:

Kummer: $T^4/\mathbb{Z}_2$:   Singular quartic in $\mathbb{CP}_3$.

Generic quartic is a $K3$ surface.

$K3$ = Kummer-Kähler-Kodaira surface.

Simply connected complex surface with $c_1 = 0$.

Typical model: Smooth quartic in $\mathbb{CP}_3$.

$K3$ = Kummer-Kähler-Kodaira surface.

Simply connected complex surface with $c_1 = 0$.

Typical model: Smooth quartic in $\mathbb{CP}_3$.

Calabi/Yau: Admits Ricci-flat Kähler metrics.

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if*

$$M \overset{diff}{\approx} \begin{cases} \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ S^2 \times S^2, & \\ K3, & \\ & \\ & \end{cases}$$

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if*

$$M \overset{diff}{\approx} \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ S^2 \times S^2, & \\ K3, & \\ K3/\mathbb{Z}_2, & \\ & \\ & \end{cases}$$

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if*

$$M \overset{diff}{\approx} \begin{cases} \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ \\ \\ \end{cases}$$

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if*

$$M \overset{diff}{\approx} \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ \end{cases}$$

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if*

$$M \overset{\text{diff}}{\approx} \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{cases}$$

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if*

$$M \overset{diff}{\approx} \begin{cases} \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \textit{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{cases}$$

Existence: Yau, Tian, Page, Chen-L-Weber, et al.

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if*

$$M \overset{diff}{\approx} \begin{cases} \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{cases}$$

Existence: Yau, Tian, Page, Chen-L-Weber, et al.

No others: Hitchin-Thorpe, Seiberg-Witten, ...

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if*

$$M \overset{diff}{\approx} \begin{cases} \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{cases}$$

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if*

$$M \overset{diff}{\approx} \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{cases}$$

Extensive results in $\lambda < 0$ case, too.

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if*

$$M \overset{\text{diff}}{\approx} \begin{cases} \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \textit{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{cases}$$

Extensive results in $\lambda < 0$ case, too.

But that would be a topic for a different lecture!

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if*

$$M \overset{\text{diff}}{\approx} \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \textit{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{cases}$$

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if*

$$M \overset{diff}{\approx} \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \textit{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{cases}$$

Del Pezzo surfaces,
K3 surface, Enriques surface,
Abelian surface, Hyper-elliptic surfaces.

Del Pezzo surfaces:

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points,
in general position,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position,

Del Pezzo surfaces:

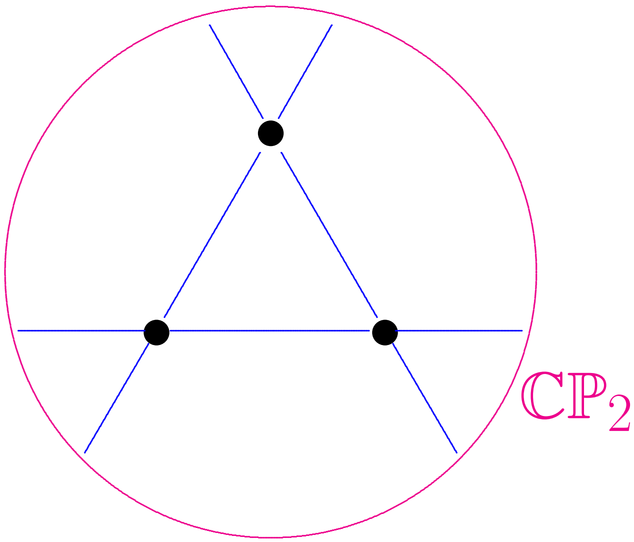
(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

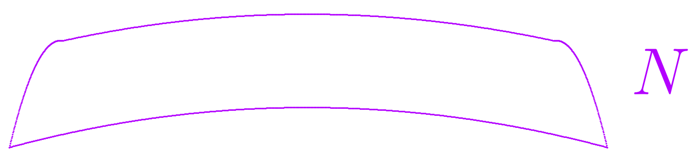
Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



Blowing up:

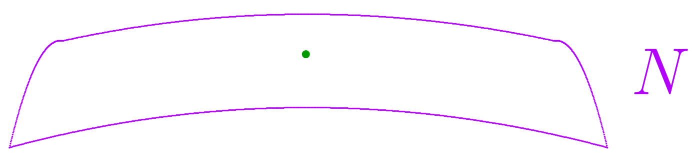
Blowing up:

If N is a complex surface,



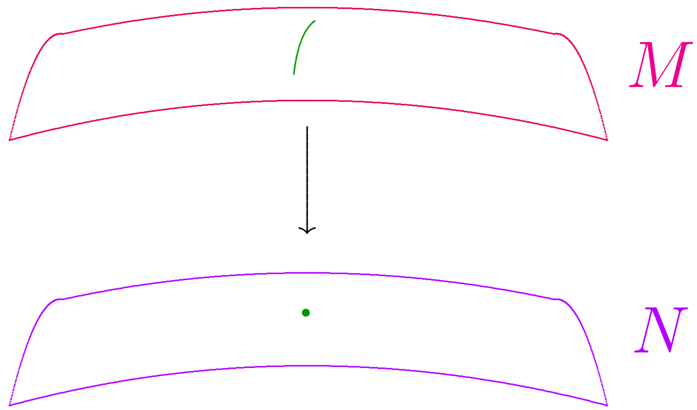
Blowing up:

If N is a complex surface, may replace $p \in N$



Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$

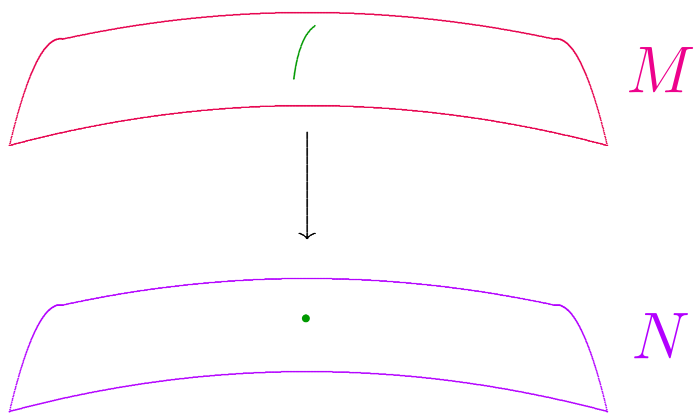


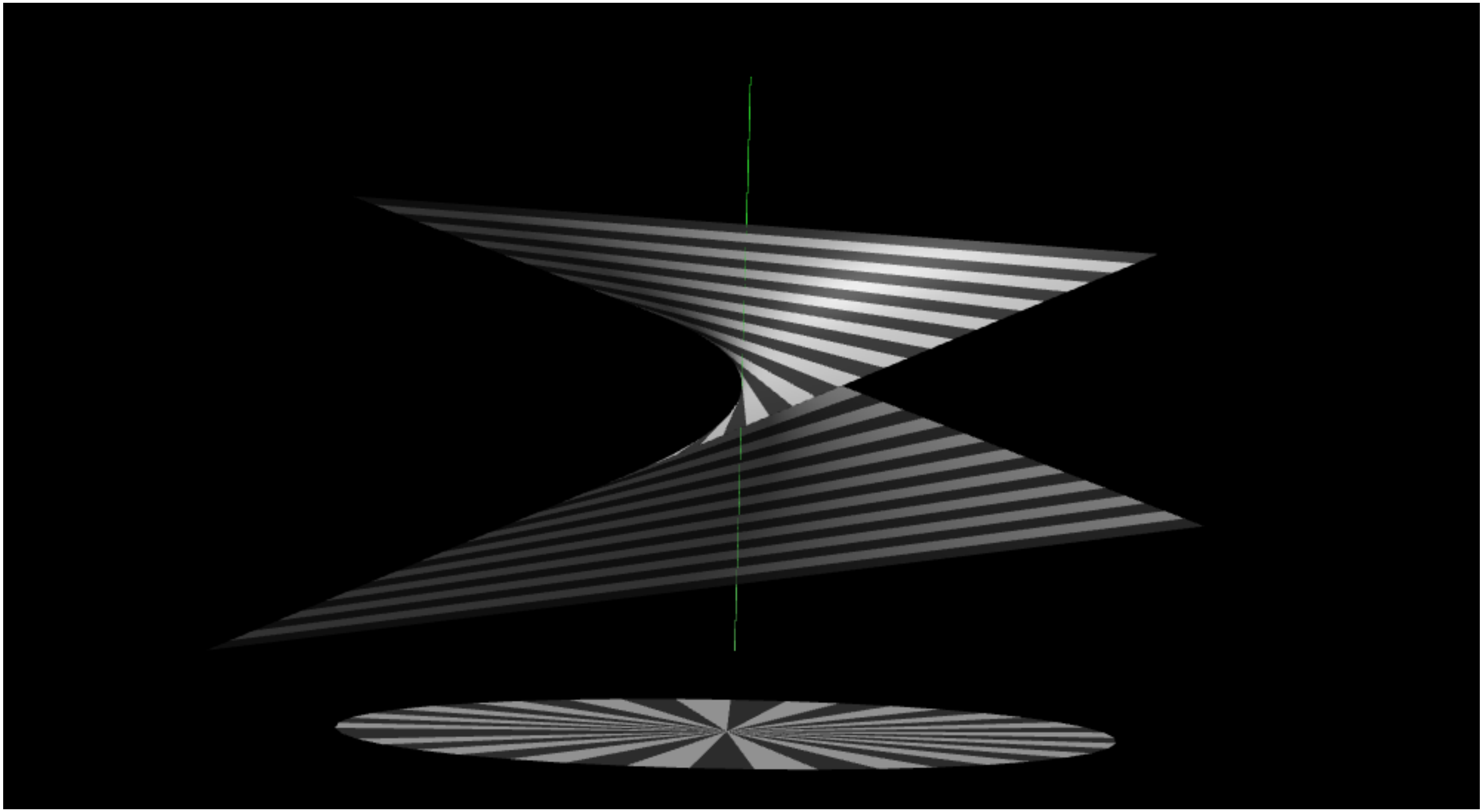
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



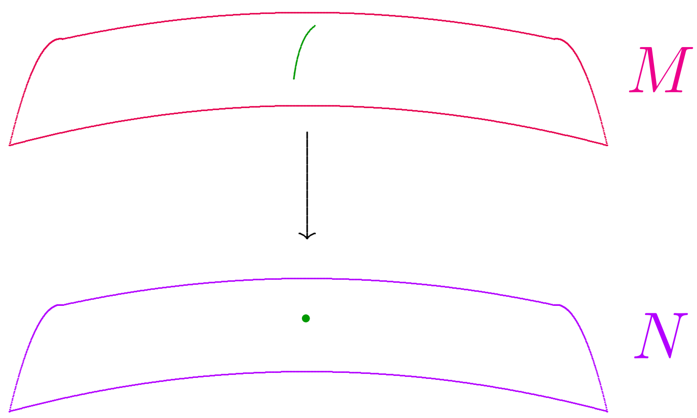


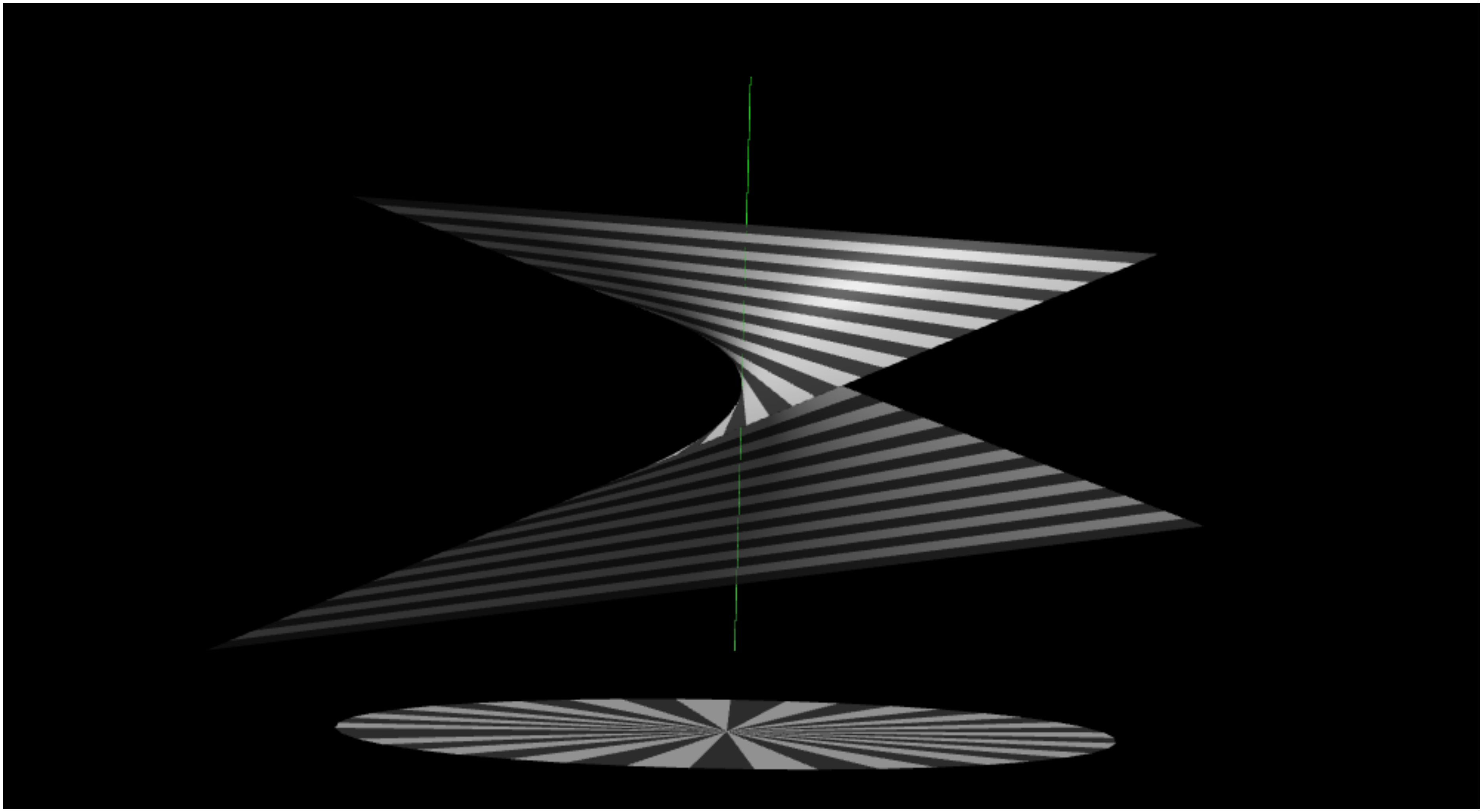
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}_2}$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



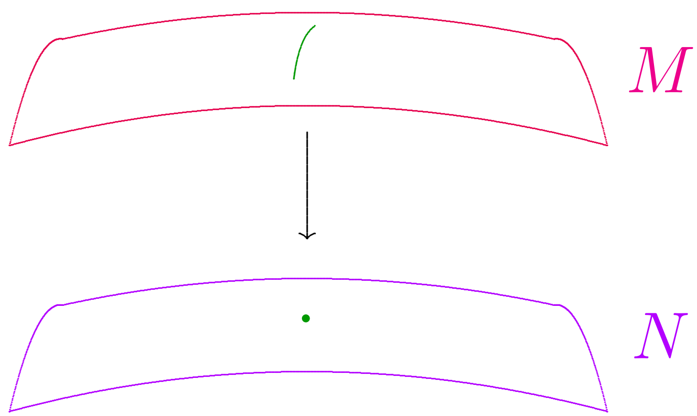


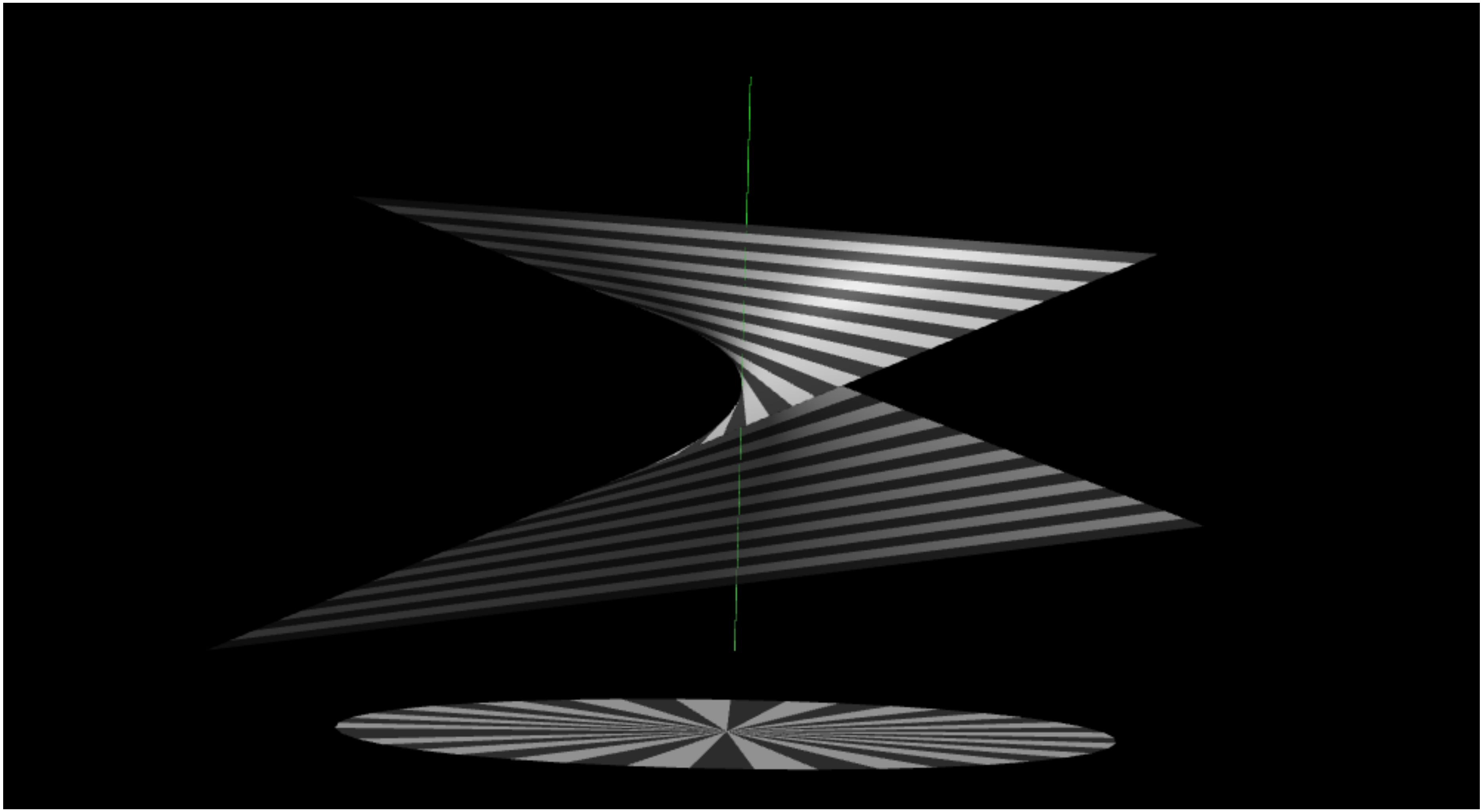
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



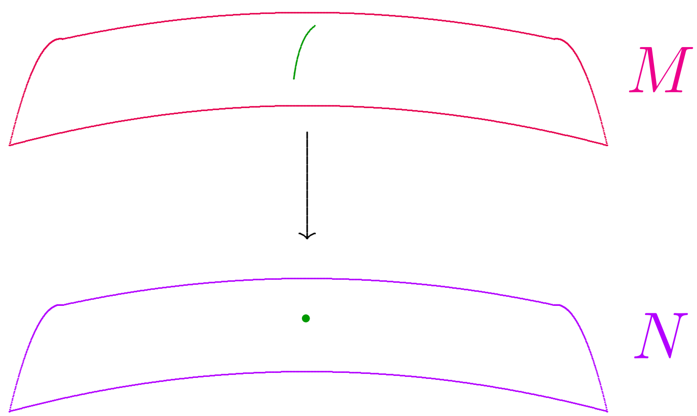


Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

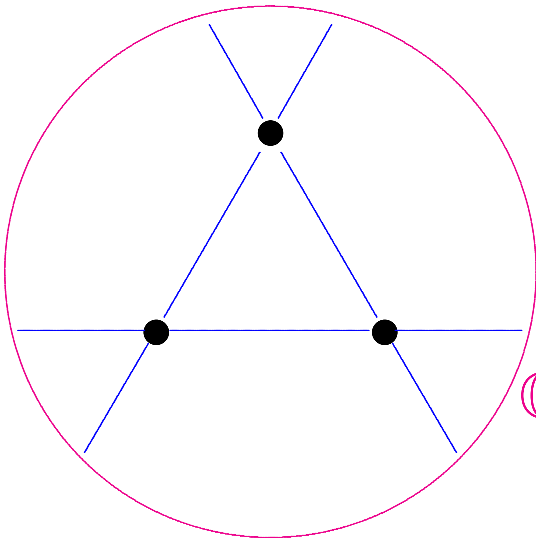
in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

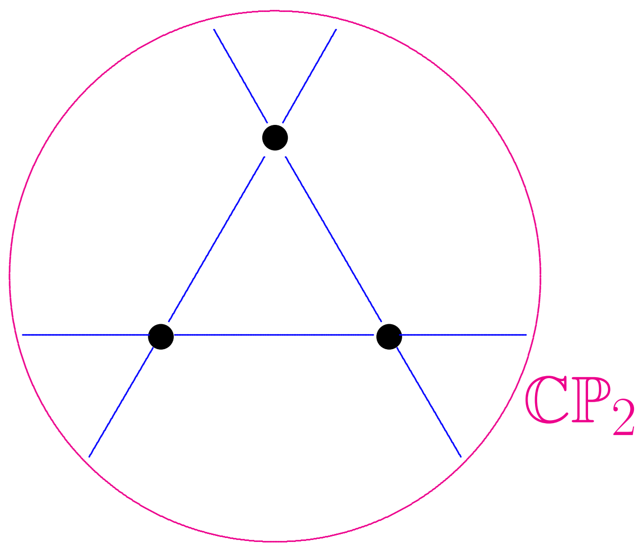
Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

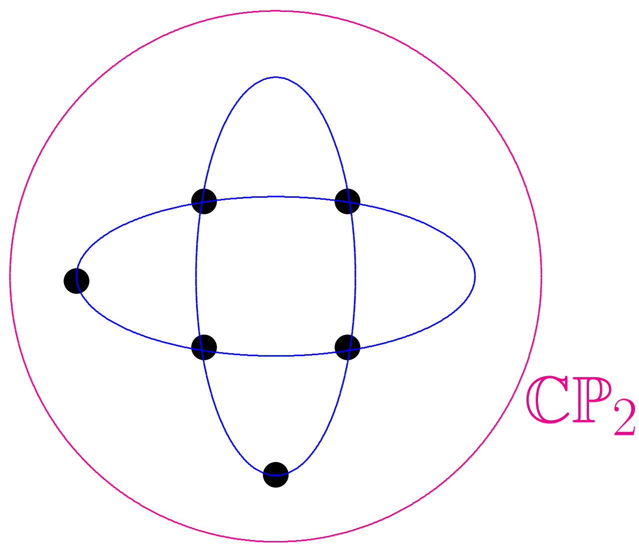


No 3 on a line,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

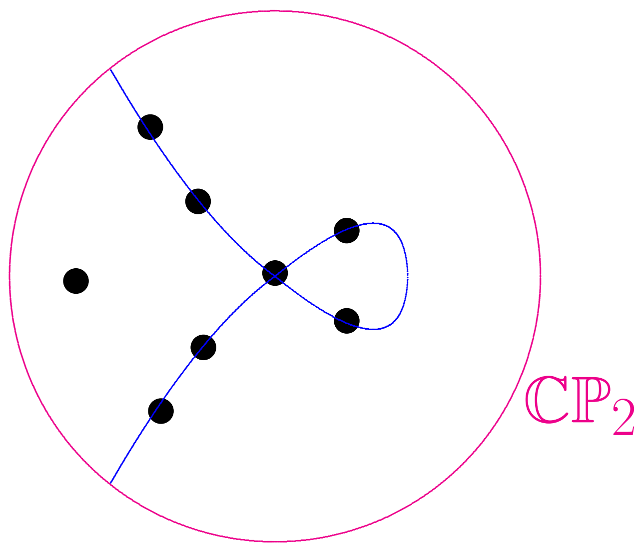


No 3 on a line, no 6 on conic,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



No 3 on a line, no 6 on conic, no 8 on nodal cubic.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo* (M^4, J) *admits a* J*-compatible*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo* (M^4, J) *admits a* J*-compatible conformally Kähler,*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric,*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo* (M^4, J) *admits a* J*-compatible conformally Kähler, Einstein metric, and this metric is unique*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is unique up to complex automorphisms and constant rescalings.*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Conformally Kähler:

$$g = u^2 h$$

$\exists$ some Kähler metric h & some smooth function u.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Conformally Kähler:

$$g = u^2 h$$

where Kähler metric h is extremal & $u = s_h^{-1}$.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Conformally Kähler:

$$g = u^2 h$$

$\exists$ some Kähler metric h & some smooth function u.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Conformally Kähler:

$$g = u^2 h$$

where Kähler metric h is extremal & $u = s_h^{-1}$.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page

1978

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński,

1983

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo* (M^4, J) *admits a* J*-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu,

1986

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu, Tian-Yau,

1987

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu, Tian-Yau, Tian,

1990

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu, Tian-Yau, Tian, Odaka-Spotti-Sun,

2016

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu, Tian-Yau, Tian, Odaka-Spotti-Sun, Chen-L-Weber.

2008

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Uniqueness: Bando-Mabuchi '87

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Uniqueness: Bando-Mabuchi '87, L '12.

One fundamental open problem:

One fundamental open problem:

Understand all Einstein metrics on del Pezzos.

One fundamental open problem:

Understand all Einstein metrics on del Pezzos.

Is Einstein moduli space connected?

One fundamental open problem:

Understand all Einstein metrics on del Pezzos.

Is Einstein moduli space connected?

Progress to date:

One fundamental open problem:

Understand all Einstein metrics on del Pezzos.

Is Einstein moduli space connected?

Progress to date:

Open characterizations of known Einstein metrics.

One fundamental open problem:

Understand all Einstein metrics on del Pezzos.

Is Einstein moduli space connected?

Progress to date:

Open characterizations of known Einstein metrics.

Peng Wu proposed one beautiful characterization, in terms of an open condition on

$$W_+ : \Lambda^+ \to \Lambda^+.$$

One fundamental open problem:

Understand all Einstein metrics on del Pezzos.

Is Einstein moduli space connected?

Progress to date:

Open characterizations of known Einstein metrics.

Peng Wu proposed one beautiful characterization, in terms of an open condition on

$$W_+ : \Lambda^+ \to \Lambda^+.$$

Wu's criterion:

$$\det(W_+) > 0.$$

Theorem (Wu/L '21)**.**

Theorem (Wu/L '21). *Let* (M, g) *be a compact oriented Einstein* 4*-manifold,*

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold,*

Theorem (Wu/L '21). *Let (M, g) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M.

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M*. Then* M *is diffeomorphic to a del Pezzo surface,*

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M. *Then* M *is diffeomorphic to a del Pezzo surface, and* g *is one of the conformally Kähler Einstein metrics we've discussed.*

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M*. Then* M *is diffeomorphic to a del Pezzo surface, and* g *is one of the conformally Kähler Einstein metrics we've discussed.*

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M*. Then* M *is diffeomorphic to a del Pezzo surface, and* g *is one of the conformally Kähler Einstein metrics we've discussed.*

Simplified version of my argument:

Theorem (Wu/L '21). *Let (M, g) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M. Then M is diffeomorphic to a del Pezzo surface, and g is one of the conformally Kähler Einstein metrics we've discussed.

Simplified version of my argument:

$$\det(W^+) > 0 \quad \Longrightarrow \quad W^+ \sim \begin{bmatrix} + & & \\ & - & \\ & & - \end{bmatrix}$$

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M*. Then* M *is diffeomorphic to a del Pezzo surface, and* g *is one of the conformally Kähler Einstein metrics we've discussed.*

Simplified version of my argument:

$$\det(W^+) > 0 \quad \Longrightarrow \quad W^+ \sim \begin{bmatrix} + & & \\ & - & \\ & & - \end{bmatrix}$$

Top eigenspace $L \subset \Lambda^+$ of W_+ is a line bundle.

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M*. Then* M *is diffeomorphic to a del Pezzo surface, and* g *is one of the conformally Kähler Einstein metrics we've discussed.*

Simplified version of my argument:
Set $h = \alpha^{2/3} g$, where α top eigenvalue of W_{+g},

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M*. Then* M *is diffeomorphic to a del Pezzo surface, and* g *is one of the conformally Kähler Einstein metrics we've discussed.*

Simplified version of my argument:
Set $h = \alpha^{2/3} g$, where α top eigenvalue of W_{+g}, and choose top eigenform $\omega \in \Lambda^+$ with $|\omega|_h \equiv 1$.

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M. *Then* M *is diffeomorphic to a del Pezzo surface, and* g *is one of the conformally Kähler Einstein metrics we've discussed.*

Simplified version of my argument:
Set $h = \alpha^{2/3} g$, where α top eigenvalue of W_{+g}, and choose top eigenform $\omega \in \Lambda^+$ with $|\omega|_h \equiv 1$.
A Weitzenböck argument

Theorem (Wu/L '21). *Let (M, g) be a simply-connected compact oriented Einstein 4-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M. Then M is diffeomorphic to a del Pezzo surface, and g is one of the conformally Kähler Einstein metrics we've discussed.

Simplified version of my argument:

Set $h = \alpha^{2/3} g$, where α top eigenvalue of W_{+g}, and choose top eigenform $\omega \in \Lambda^+$ with $|\omega|_h \equiv 1$. A Weitzenböck argument then shows that

$$0 \geq |\nabla \omega|^2 + 3\langle \omega, (d + d^*)^2 \omega \rangle$$

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M*. Then* M *is diffeomorphic to a del Pezzo surface, and* g *is one of the conformally Kähler Einstein metrics we've discussed.*

Simplified version of my argument:
Set $h = \alpha^{2/3} g$, where α top eigenvalue of W_{+g}, and choose top eigenform $\omega \in \Lambda^+$ with $|\omega|_h \equiv 1$. A Weitzenböck argument then shows that

$$0 \geq |\nabla \omega|^2 + 3\langle \omega, (d + d^*)^2 \omega \rangle$$

at every point, with respect to h.

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M*. Then* M *is diffeomorphic to a del Pezzo surface, and* g *is one of the conformally Kähler Einstein metrics we've discussed.*

Simplified version of my argument:
Set $h = \alpha^{2/3} g$, where α top eigenvalue of W_{+g}, and choose top eigenform $\omega \in \Lambda^+$ with $|\omega|_h \equiv 1$. A Weitzenböck argument then shows that

$$0 \geq |\nabla \omega|^2 + 3\langle \omega, (d + d^*)^2 \omega \rangle$$

at every point, with respect to h. Now integrate!

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M. *Then* M *is diffeomorphic to a del Pezzo surface, and* g *is one of the conformally Kähler Einstein metrics we've discussed.*

Theorem (Wu/L '21). *Let (M, g) be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M. Then M is diffeomorphic to a del Pezzo surface, and g is one of the conformally Kähler Einstein metrics we've discussed.

Corollary. *Every simply-connected compact oriented Einstein (M^4, h) with $\det(W_+) > 0$ is diffeomorphic to a del Pezzo surface.*

Theorem (Wu/L '21). *Let* (M, g) *be a simply-connected compact oriented Einstein* 4*-manifold, and suppose that its self-dual Weyl curvature*

$$W_+ : \Lambda^+ \to \Lambda^+$$

satisfies

$$\det(W_+) > 0$$

at every point of M*. Then* M *is diffeomorphic to a del Pezzo surface, and* g *is one of the conformally Kähler Einstein metrics we've discussed.*

Corollary. *Every simply-connected compact oriented Einstein* (M^4, h) *with* $\det(W_+) > 0$ *is diffeomorphic to a del Pezzo surface. Conversely, every del Pezzo* M^4 *carries Einstein* h *with* $\det(W_+) > 0$*, and these sweep out exactly one connected component of moduli space* $\mathscr{E}(M)$*.*

We've focused on compact Einstein manifolds.

We’ve focused on compact Einstein manifolds.

But non-compact, complete solutions are often key to proving theorems about compact ones.

Definition. *A gravitational instanton is a*

Definition. *A gravitational instanton is a complete,*

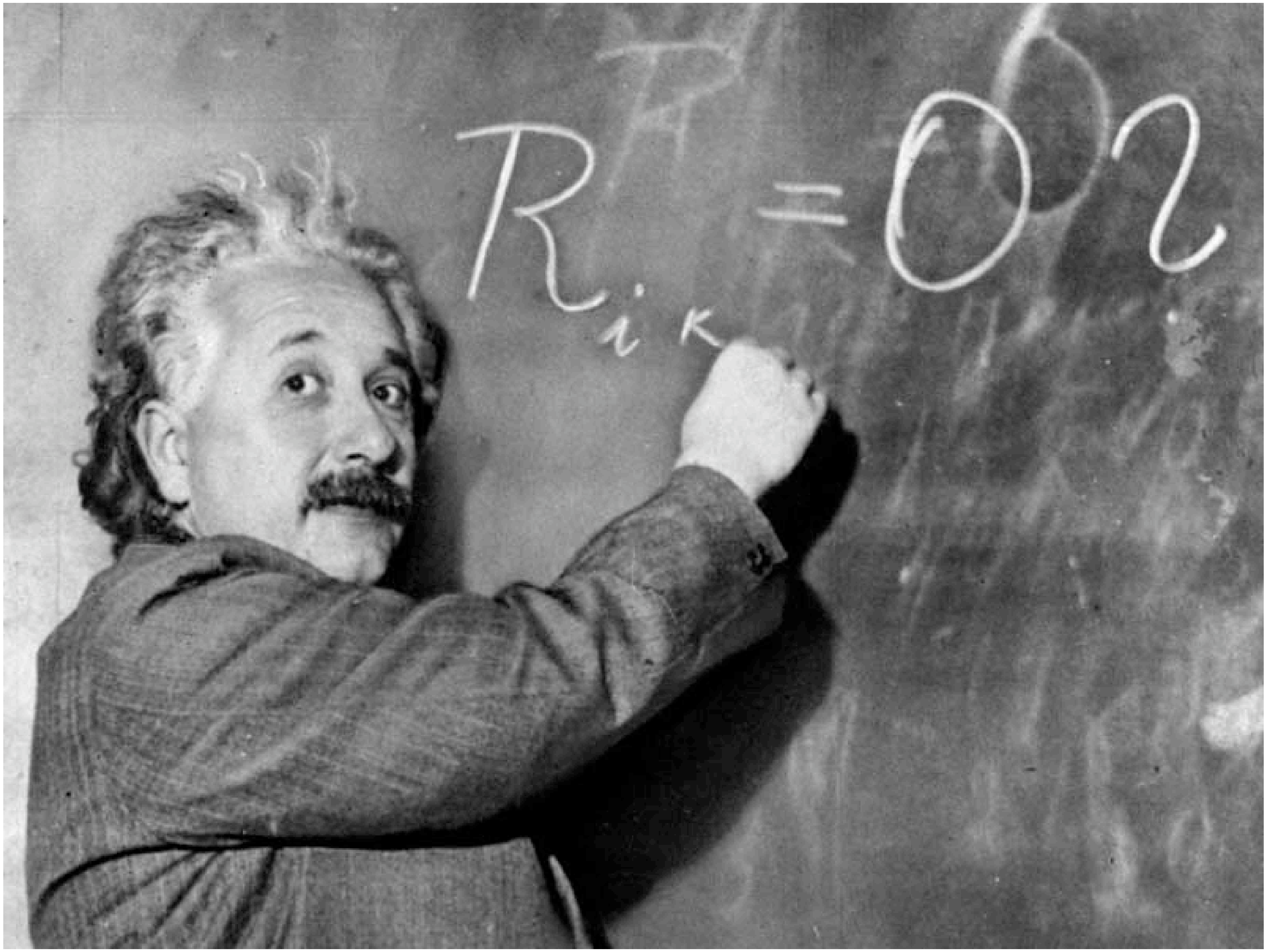
Definition. *A gravitational instanton is a complete, non-compact,*

Definition. *A gravitational instanton is a complete, non-compact, non-flat,*

Definition. *A gravitational instanton is a complete, non-compact, non-flat, Ricci-flat*

Definition. *A gravitational instanton is a complete, non-compact, non-flat, Ricci-flat Riemannian* 4*-manifold.*

Definition. *A gravitational instanton is a complete, non-compact, non-flat, Ricci-flat Riemannian 4-manifold.*



Definition. *A gravitational instanton is a complete, non-compact, non-flat, Ricci-flat Riemannian* 4*-manifold.*

Definition. *A gravitational instanton is a complete, non-compact, non-flat, Ricci-flat Riemannian* 4*-manifold.*

Terminology due to Gibbons & Hawking, late '70s

Definition. *A gravitational instanton is a complete, non-compact, non-flat, Ricci-flat Riemannian* 4*-manifold.*

Terminology due to Gibbons & Hawking, late '70s

Example.

Example. Riemannian Schwarzschild metric:

Example. Riemannian Schwarzschild metric:

$$g = \left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2 + \varrho^2 g_{S^2}$$

Example. Riemannian Schwarzschild metric:

$$g = \left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2 + \varrho^2 g_{S^2}$$

Conformal to

$$h = \frac{1}{\varrho^2}\left[\left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2\right] + g_{S^2}$$

Example. Riemannian Schwarzschild metric:

$$g = \left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2 + \varrho^2 g_{S^2}$$

Conformal to

$$h = \frac{1}{\varrho^2}\left[\left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2\right] + g_{S^2}$$

Kähler!

Example. Riemannian Schwarzschild metric:

$$g = \left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2 + \varrho^2 g_{S^2}$$

Conformal to

$$h = \frac{1}{\varrho^2}\left[\left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2\right] + g_{S^2}$$

Kähler! In fact, extremal Kähler!

Example. Riemannian Schwarzschild metric:

$$g = \left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2 + \varrho^2 g_{S^2}$$

Conformal to

$$h = \frac{1}{\varrho^2}\left[\left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2\right] + g_{S^2}$$

Kähler! In fact, extremal Kähler!

$$\bar{\partial}\nabla^{1,0} s = 0$$

Example. Riemannian Schwarzschild metric:

$$g = \left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2 + \varrho^2 g_{S^2}$$

Conformal to

$$h = \frac{1}{\varrho^2}\left[\left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2\right] + g_{S^2}$$

Kähler! In fact, extremal Kähler!

$$\bar{\partial}\nabla^{1,0} s = 0$$

Example. Riemannian Schwarzschild metric:

$$g = \left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2 + \varrho^2 g_{S^2}$$

Conformal to

$$h = \frac{1}{\varrho^2} \left[\left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2 \right] + g_{S^2}$$

Hawking: set $t = 4m\theta$ and $\varrho = 2m + \frac{r^2}{8m}$.
This makes g into a Ricci-flat metric on $\mathbb{R}^2 \times S^2$.

Example. Riemannian Schwarzschild metric:

$$g = \left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2 + \varrho^2 g_{S^2}$$

Conformal to

$$h = \frac{1}{\varrho^2}\left[\left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2\right] + g_{S^2}$$

Hawking: set $t = 4m\theta$ and $\varrho = 2m + \frac{r^2}{8m}$.
This makes g into a Ricci-flat metric on $\mathbb{R}^2 \times S^2$.

$$g = dr^2 + r^2 d\theta^2 + 4m^2 g_{S^2} + O(r^2)$$

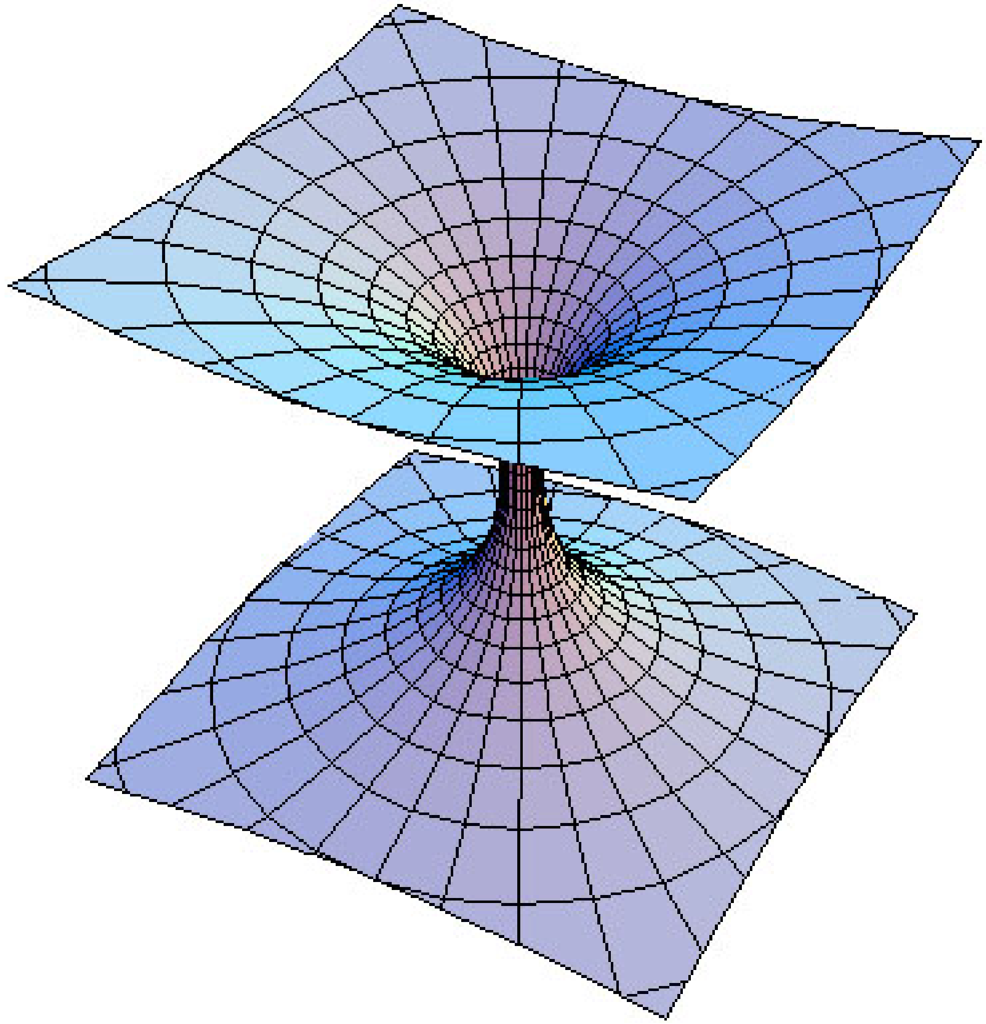
Example. Riemannian Schwarzschild metric:

$$g = \left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2 + \varrho^2 g_{S^2}$$

Conformal to

$$h = \frac{1}{\varrho^2} \left[\left(1 - \frac{2m}{\varrho}\right)^{-1} d\varrho^2 + \left(1 - \frac{2m}{\varrho}\right) dt^2 \right] + g_{S^2}$$

Hawking: set $t = 4m\theta$ and $\varrho = 2m + \frac{r^2}{8m}$.
This makes g into a Ricci-flat metric on $\mathbb{R}^2 \times S^2$.
Makes h into extremal Kähler metric on $\mathbb{C} \times \mathbb{CP}_1$.



$\mathbb{R} \times S^2 \subset \mathbb{R}^2 \times S^2$

Example. Taub-NUT metric:

Example. Taub-NUT metric:

$$g = \frac{r+1}{4r} dr^2 + r(1+r)[\sigma_1{}^2 + \sigma_2{}^2] + \frac{r}{r+1}\sigma_3{}^2$$

Example. Taub-NUT metric:

$$g = \frac{r+1}{4r} dr^2 + r(1+r)[{\sigma_1}^2 + {\sigma_2}^2] + \frac{r}{r+1} {\sigma_3}^2$$

for left-invariant coframe $\{\sigma_j\}$ on $S^3 = \mathbf{SU}(2)$.

Example. Taub-NUT metric:

$$g = \frac{r+1}{4r} dr^2 + r(1+r)[{\sigma_1}^2 + {\sigma_2}^2] + \frac{r}{r+1} {\sigma_3}^2$$

for left-invariant coframe $\{\sigma_j\}$ on $S^3 = \mathbf{SU}(2)$.

Complete, non-flat, Ricci-flat metric on $\mathbb{R}^4$.

Example. Taub-NUT metric:

$$g = \frac{r+1}{4r} dr^2 + r(1+r)[{\sigma_1}^2 + {\sigma_2}^2] + \frac{r}{r+1} {\sigma_3}^2$$

for left-invariant coframe $\{\sigma_j\}$ on $S^3 = \mathbf{SU}(2)$.

Complete, non-flat, Ricci-flat metric on $\mathbb{R}^4$.

Hermitian, non-Kähler metric on $\mathbb{C}^2$.

Example. Taub-NUT metric:

$$g = \frac{r+1}{4r} dr^2 + r(1+r)[{\sigma_1}^2 + {\sigma_2}^2] + \frac{r}{r+1} {\sigma_3}^2$$

for left-invariant coframe $\{\sigma_j\}$ on $S^3 = \mathbf{SU}(2)$.

Complete, non-flat, Ricci-flat metric on $\mathbb{R}^4$.

Hermitian, non-Kähler metric on $\mathbb{C}^2$.

(But for opposite orientation, actually hyper-Kahler!)

Example. Taub-NUT metric:

$$g = \frac{r+1}{4r}dr^2 + r(1+r)[{\sigma_1}^2 + {\sigma_2}^2] + \frac{r}{r+1}{\sigma_3}^2$$

for left-invariant coframe $\{\sigma_j\}$ on $S^3 = \mathbf{SU}(2)$.

Complete, non-flat, Ricci-flat metric on $\mathbb{R}^4$.

Hermitian, non-Kähler metric on $\mathbb{C}^2$.

(But for opposite orientation, actually hyper-Kahler!)

These examples are both Hermitian and toric:

Example. Taub-NUT metric:

$$g = \frac{r+1}{4r} dr^2 + r(1+r)[{\sigma_1}^2 + {\sigma_2}^2] + \frac{r}{r+1}{\sigma_3}^2$$

for left-invariant coframe $\{\sigma_j\}$ on $S^3 = \mathbf{SU}(2)$.

Complete, non-flat, Ricci-flat metric on $\mathbb{R}^4$.

Hermitian, non-Kähler metric on $\mathbb{C}^2$.

(But for opposite orientation, actually hyper-Kahler!)

These examples are both Hermitian and toric:

$\mathbb{T}^2$ acts on them isometrically.

Finally, they are ALF:

Example. Taub-NUT metric:

$$g = \frac{r+1}{4r}dr^2 + r(1+r)[{\sigma_1}^2 + {\sigma_2}^2] + \frac{r}{r+1}{\sigma_3}^2$$

for left-invariant coframe $\{\sigma_j\}$ on $S^3 = \mathbf{SU}(2)$.

Complete, non-flat, Ricci-flat metric on $\mathbb{R}^4$.

Hermitian, non-Kähler metric on $\mathbb{C}^2$.

(But for opposite orientation, actually hyper-Kahler!)

These examples are both Hermitian and toric:

$\mathbb{T}^2$ acts on them isometrically.

Finally, they are ALF:

"Asymptotically Locally Flat"

Example. Taub-NUT metric:

$$g = \frac{r+1}{4r} dr^2 + r(1+r)[{\sigma_1}^2 + {\sigma_2}^2] + \frac{r}{r+1} {\sigma_3}^2$$

for left-invariant coframe $\{\sigma_j\}$ on $S^3 = \mathbf{SU}(2)$.

Complete, non-flat, Ricci-flat metric on $\mathbb{R}^4$.

Hermitian, non-Kähler metric on $\mathbb{C}^2$.

(But for opposite orientation, actually hyper-Kahler!)

These examples are both Hermitian and toric:

$\mathbb{T}^2$ acts on them isometrically.

Finally, they are ALF:

$$g = d\varrho^2 + \varrho^2 \gamma + \eta^2 + \mho$$

$$\mho = O(\varrho^{-1}), \quad \nabla \mho = O(\varrho^{-2}), \quad \ldots \quad \nabla^3 \mho = O(\varrho^{-4})$$

Example. Taub-NUT metric:

$$g = \frac{r+1}{4r} dr^2 + r(1+r)[{\sigma_1}^2 + {\sigma_2}^2] + \frac{r}{r+1} {\sigma_3}^2$$

for left-invariant coframe $\{\sigma_j\}$ on $S^3 = \mathbf{SU}(2)$.

Complete, non-flat, Ricci-flat metric on $\mathbb{R}^4$.

Hermitian, non-Kähler metric on $\mathbb{C}^2$.

(But for opposite orientation, actually hyper-Kahler!)

These examples are both Hermitian and toric:

$\mathbb{T}^2$ acts on them isometrically.

Finally, they are ALF:

$$\Longrightarrow \quad \mathrm{Vol}(B_\rho) \sim \mathrm{const} \cdot \rho^3$$

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat,*

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat*

Theorem (Biquard-Gauduchon '23)**.** *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is*

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric,*

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric,*

$\mathbb{T}^2$ acts effectively and isometrically

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric, ALF,*

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric, ALF,*

$$\Longrightarrow \quad \mathrm{Vol}(B_\rho) \sim \mathrm{const} \cdot \rho^3$$

Theorem (Biquard-Gauduchon '23)**.** *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric, ALF, and Hermitian*

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric, ALF, and Hermitian with respect to some integrable complex structure* J.

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric*, ALF, and Hermitian with respect to some integrable complex structure* J*.*

$$g(J\cdot, J\cdot) = g$$

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric*, ALF, and Hermitian with respect to some integrable complex structure* J*.*

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric, *ALF, and Hermitian with respect to some integrable complex structure* J. *Also assume that* (M, g, J) *is not Kähler.*

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric*, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT* metric*;*

Diffeomorphic to $\mathbb{R}^4$

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric, *ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT* metric*;*
- *the Taub-bolt* metric*;*

Diffeomorphic to $\mathbb{CP}_2 - \{pt\}$

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT metric;*
- *the Taub-bolt metric;*
- *a metric of the Kerr family;*

Diffeomorphic to $S^2 \times \mathbb{R}^2$

Theorem (Biquard-Gauduchon '23)**.** *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT metric;*
- *the Taub-bolt metric;*
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*

Y. Chen & E. Teo, 2011

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* *toric*, *ALF*, *and* *Hermitian* *with respect to some integrable complex structure* J. *Also assume that* (M, g, J) *is* *not Kähler*. *Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented)* *Taub-NUT* *metric;*
- *the* *Taub-bolt* *metric;*
- *a metric of the* *Kerr* *family; or*
- *a metric in the* *Chen-Teo* *family.*

Diffeomorphic to $\mathbb{CP}_2 - S^1$

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT metric;*
- *the Taub-bolt metric;*
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric*, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT metric;*
- *the Taub-bolt metric;*
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*

By the Riemannian Goldberg-Sachs Theorem,

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is toric, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT metric;*
- *the Taub-bolt metric;*
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*

By the Riemannian Goldberg-Sachs Theorem, the Hermitian assumption is

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric, *ALF, and Hermitian with respect to some integrable complex structure* J. *Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT metric;*
- *the Taub-bolt metric;*
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*

By the Riemannian Goldberg-Sachs Theorem, the Hermitian assumption is equivalent to assuming that the Einstein metric g is conformally Kähler.

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric, *ALF, and Hermitian with respect to some integrable complex structure* J. *Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT metric;*
- *the Taub-bolt metric;*
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*

This assertion is peculiar to dimension 4.

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric, *ALF, and Hermitian with respect to some integrable complex structure* J. *Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT metric;*
- *the Taub-bolt metric;*
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*

This assertion is peculiar to dimension 4.
It is false in all higher dimensions!

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric, *ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT* metric;
- *the Taub-bolt* metric;
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*

Derdziński:

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric, *ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT metric;*
- *the Taub-bolt metric;*
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*

Derdziński: the conformally related Kähler metric is also automatically extremal in the sense of Calabi!

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat 4-manifold that is toric, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT metric;*
- *the Taub-bolt metric;*
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*

Derdziński:

$$\bar{\partial}\nabla^{1,0}s_h = 0$$

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric*, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT metric;*
- *the Taub-bolt metric;*
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*

Derdziński:

$$J\nabla s_h \text{ Killing}$$

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric, *ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT metric;*
- *the Taub-bolt metric;*
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*

Derdziński:

$$g = \qquad s_h{}^{-2} h$$

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric, *ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT metric;*
- *the Taub-bolt metric;*
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*

Derdziński:

$$g = \text{const}\ {s_h}^{-2} h$$

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric, *ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT* metric;
- *the Taub-bolt* metric;
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric, *ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT metric;*
- *the Taub-bolt metric;*
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*

But are Hermitian, toric hypotheses really needed?

Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric, *ALF, and Hermitian with respect to some integrable complex structure* J. *Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT* metric;
- *the Taub-bolt metric;*
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*

Peng Wu's criterion still applicable this setting. . .

Theorem (Biquard-Gauduchon-L)**.**

Theorem (Biquard-Gauduchon-L). *Let* (M, g_0) *be any toric Hermitian ALF gravitational instanton.*

Theorem (Biquard-Gauduchon-L). *Let* (M, g_0) *be any toric Hermitian ALF gravitational instanton. Then any Ricci-flat metric* g *on* M *which is sufficiently* C^3_1 *close to* g_0

Theorem (Biquard-Gauduchon-L). *Let* (M, g_0) *be any toric Hermitian ALF gravitational instanton. Then any Ricci-flat metric* g *on* M *which is sufficiently* C^3_1 *close to* g_0

$$\|g - g_0\|_{C^3_1} < \varepsilon$$

Theorem (Biquard-Gauduchon-L). *Let* (M, g_0) *be any toric Hermitian ALF gravitational instanton. Then any Ricci-flat metric* g *on* M *which is sufficiently* C^3_1 *close to* g_0

$$\|g - g_0\|_{C^3_1} < \varepsilon$$

$$\|\mho\|_{C^3_1} := \sup_M \sum_{j=0}^{3} (1 + \mathsf{dist})^{j+1} |\nabla^j \mho|_{g_0}$$

Theorem (Biquard-Gauduchon-L). *Let* (M, g_0) *be any toric Hermitian ALF gravitational instanton. Then any Ricci-flat metric* g *on* M *which is sufficiently* C^3_1 *close to* g_0

$$\|g - g_0\|_{C^3_1} < \varepsilon$$

$$\|\mho\|_{C^3_1} := \sup_M \sum_{j=0}^{3} (1 + \mathsf{dist})^{j+1} |\nabla^j \mho|_{g_0}$$

$$|\mho|_{g_0} = O(\varrho^{-1}), \quad |\nabla \mho|_{g_0} = O(\varrho^{-2}), \quad \ldots$$

Theorem (Biquard-Gauduchon-L). *Let* (M, g_0) *be any toric Hermitian ALF gravitational instanton. Then any Ricci-flat metric* g *on* M *which is sufficiently* C^3_1 *close to* g_0

Theorem (Biquard-Gauduchon-L)**.** *Let* (M, g_0) *be any toric Hermitian ALF gravitational instanton. Then any Ricci-flat metric* g *on* M *which is sufficiently* C^3_1 *close to* g_0 *must be another one of the gravitational instantons*

Theorem (Biquard-Gauduchon-L). *Let* (M, g_0) *be any toric Hermitian ALF gravitational instanton. Then any Ricci-flat metric* g *on* M *which is sufficiently* C^3_1 *close to* g_0 *must be another one of the gravitational instantons classified by Biquard-Gauduchon.*

Theorem (Biquard-Gauduchon-L)**.** *Let* (M, g_0) *be any toric Hermitian ALF gravitational instanton. Then any Ricci-flat metric* g *on* M *which is sufficiently* C^3_1 *close to* g_0 *must be another one of the gravitational instantons classified by Biquard-Gauduchon.*

This improved statement of our result also depends on a result of Mingyang Li, `arXiv:2310.13197`.

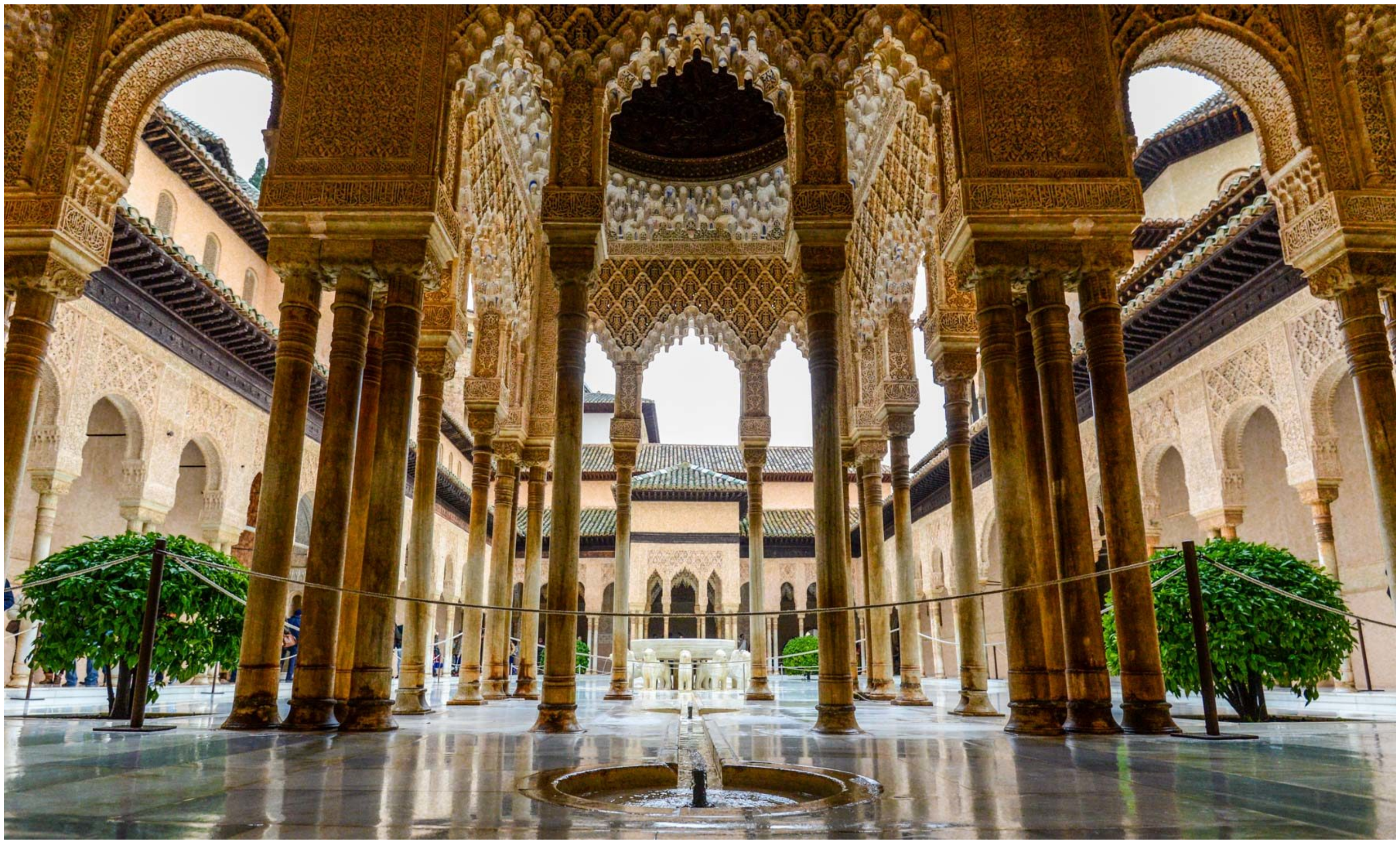
¡Muchísimas gracias por la invitación!

¡Muchísimas gracias por la invitación!

¡Estoy muy feliz de estar aquí!

¡Muchísimas gracias por la invitación!

¡Estoy muy feliz de estar aquí!



Many thanks for the invitation!

It's really a pleasure to be here!