Einstein Manifolds,

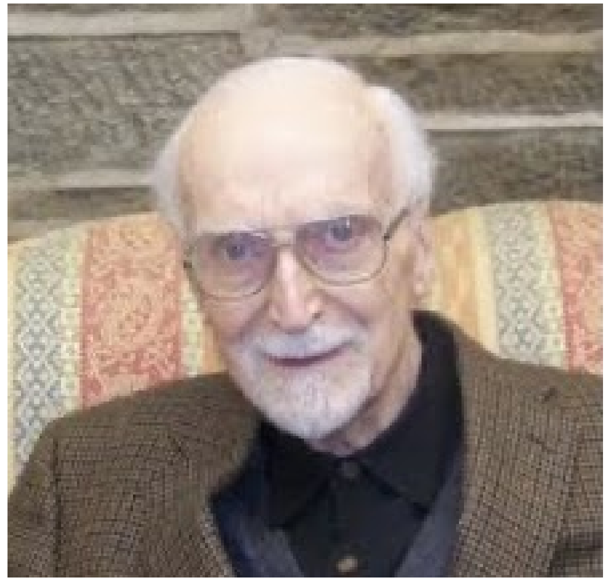
Extremal Kähler Metrics, &

Gravitational Instantons

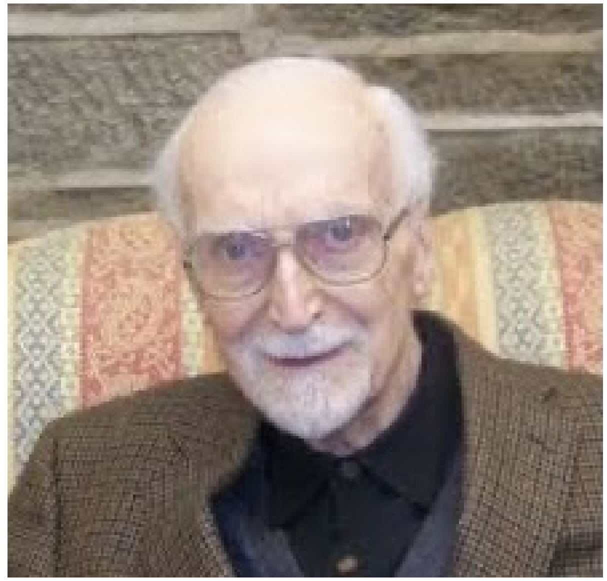
Claude LeBrun
Stony Brook University

38th Annual Geometry Festival
University of Pennsylvania
Philadelphia, April 20, 2024

In Memoriam



Eugenio Calabi



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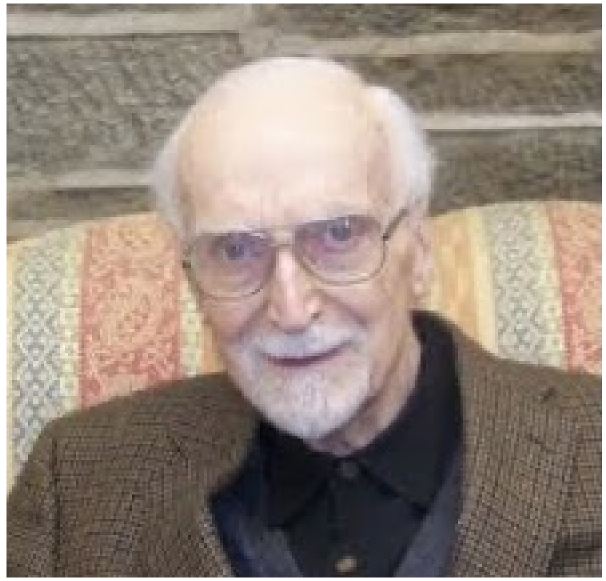

1923-2023

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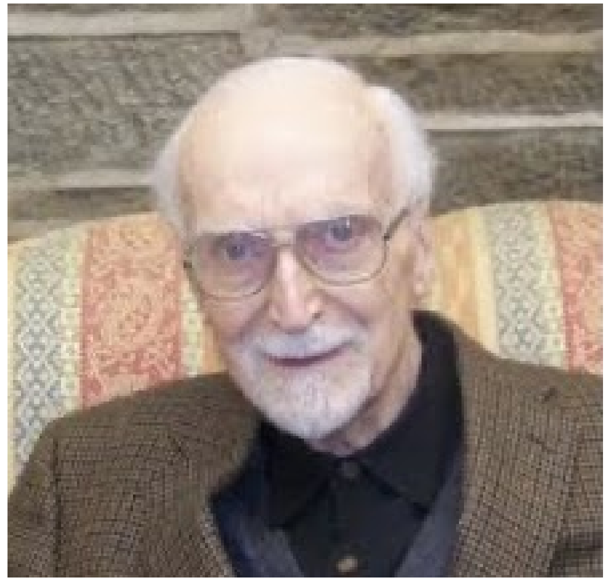

who pioneered many of the ideas central to this talk.

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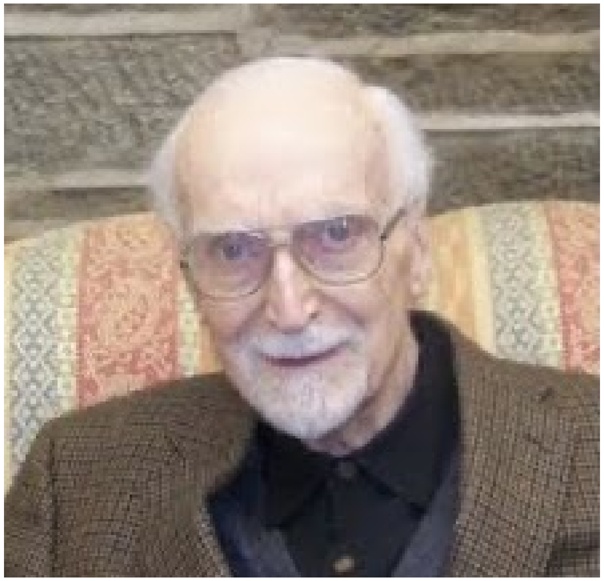

’l maestro di color chi sanno

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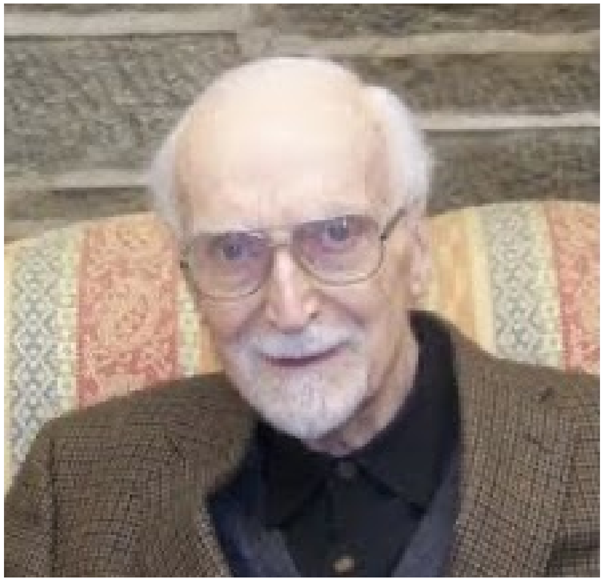

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—Dante Alighieri

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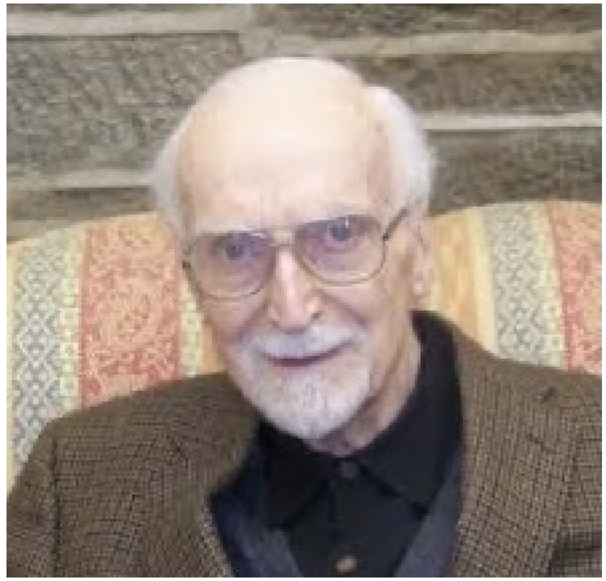

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interacting in ways he might have found surprising.

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for some constant $\lambda \in \mathbb{R}$.

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"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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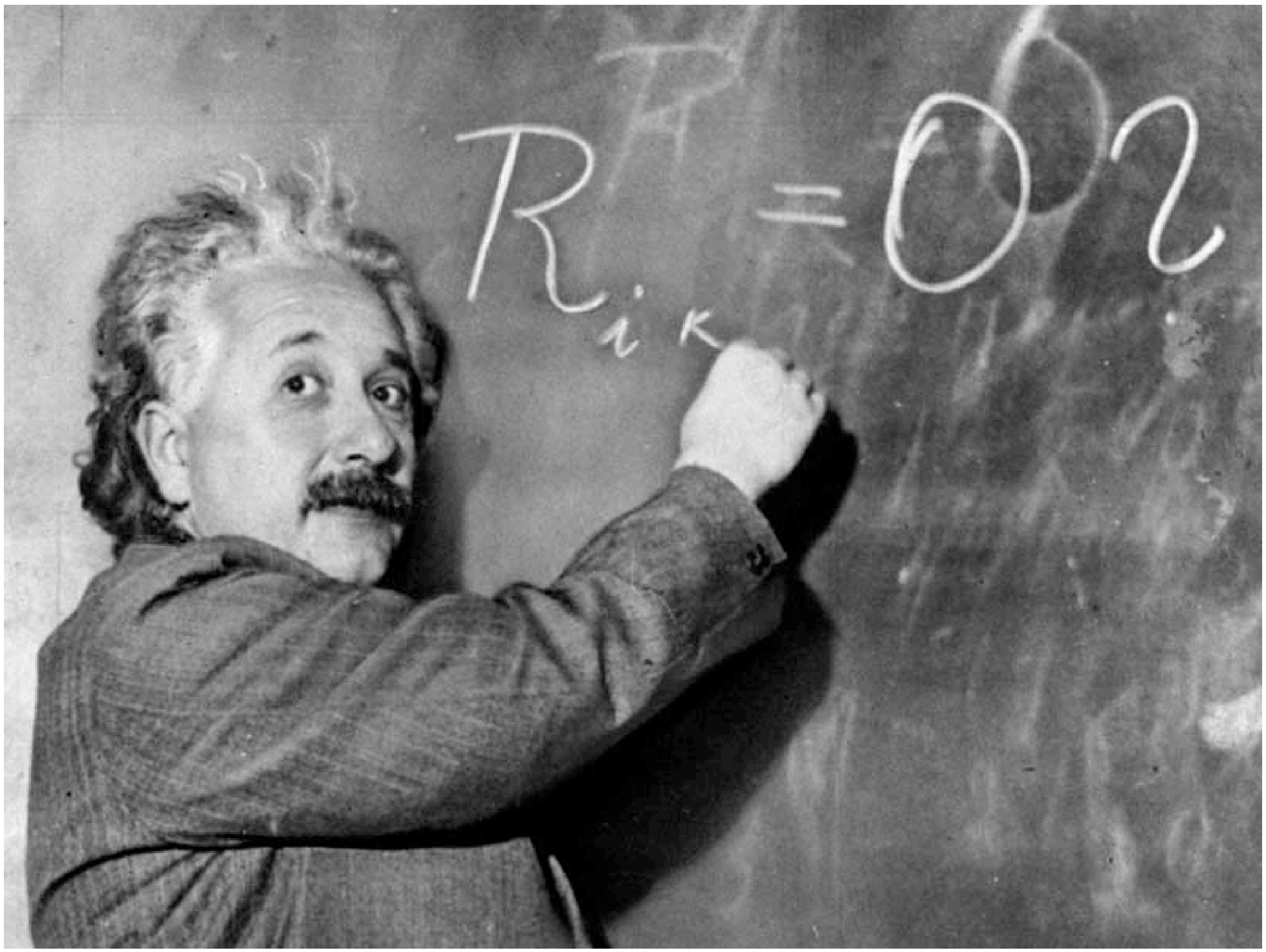

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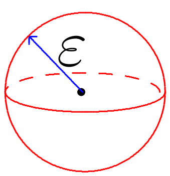
$$\frac{\mathrm{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s\,\frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



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Eugenio Calabi's most famous contributions to the subject concern the case when (M, g) is also a Kähler manifold.

His papers of 1954 essentially created the subject.

(M^n, g):

holonomy

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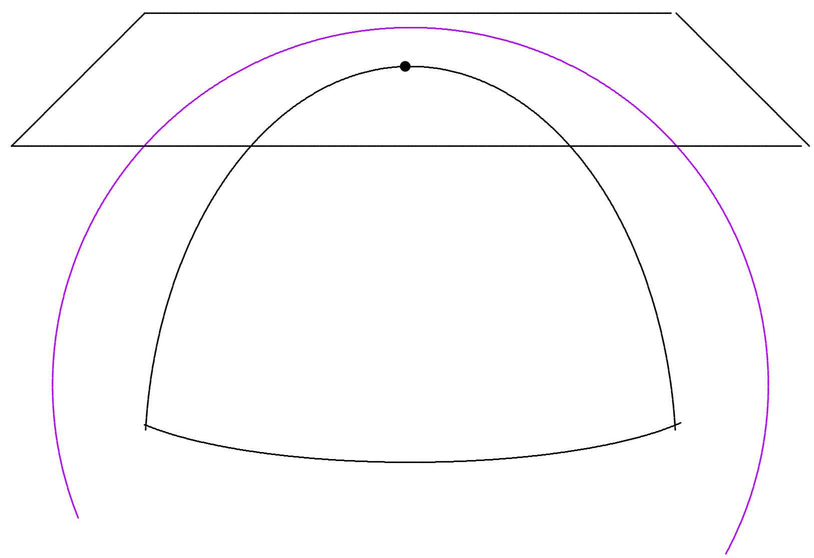

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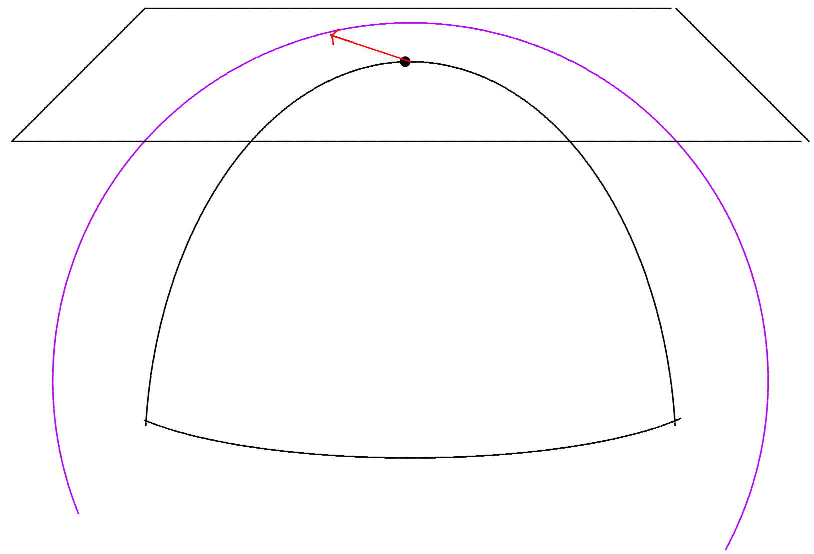

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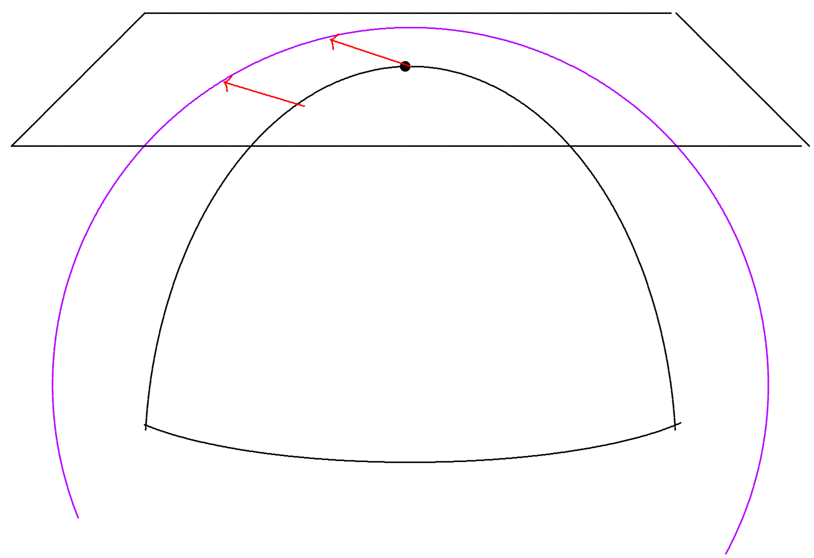

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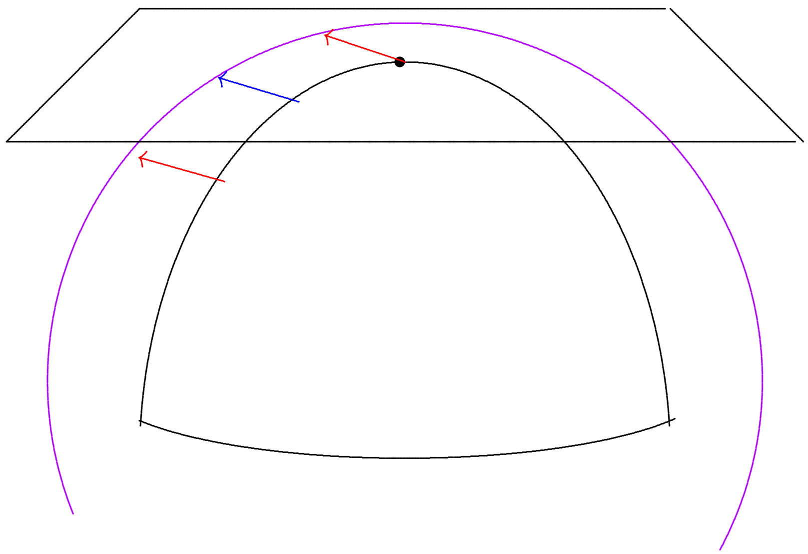

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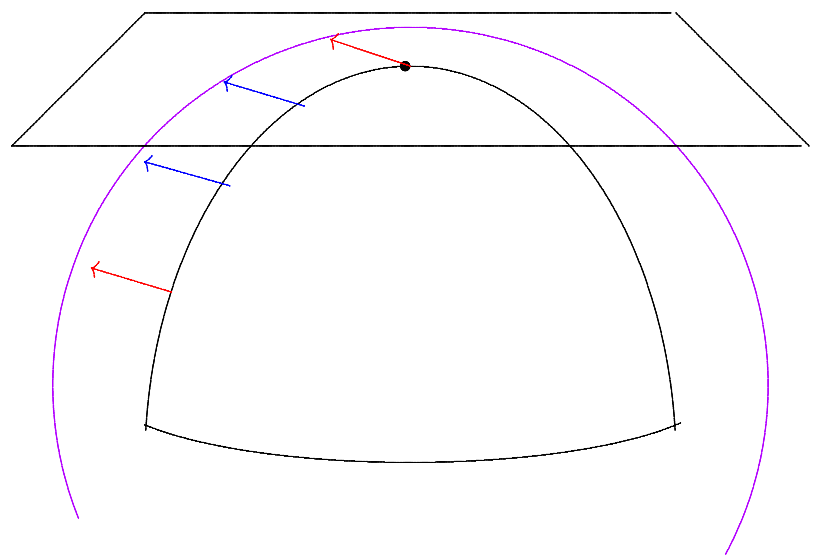

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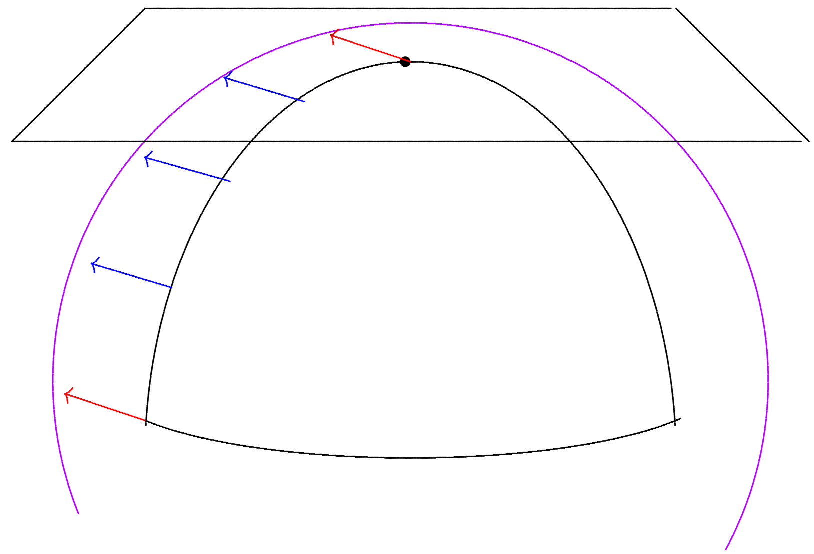

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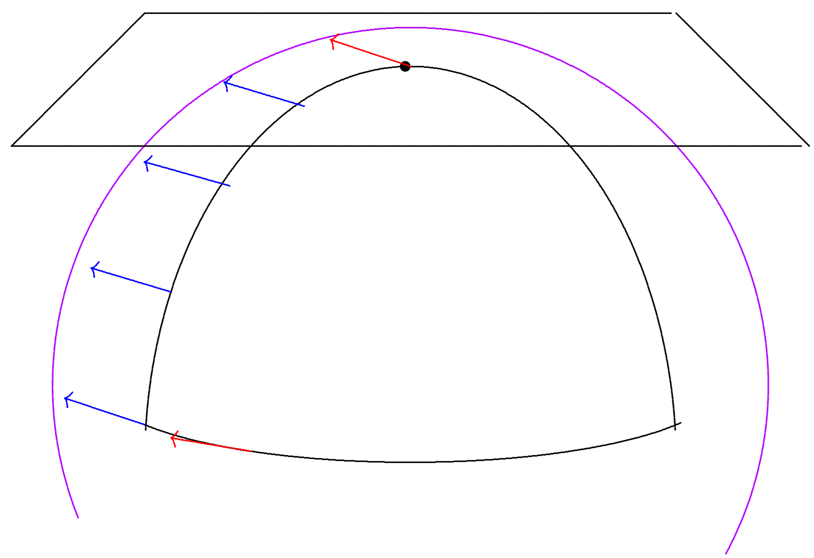

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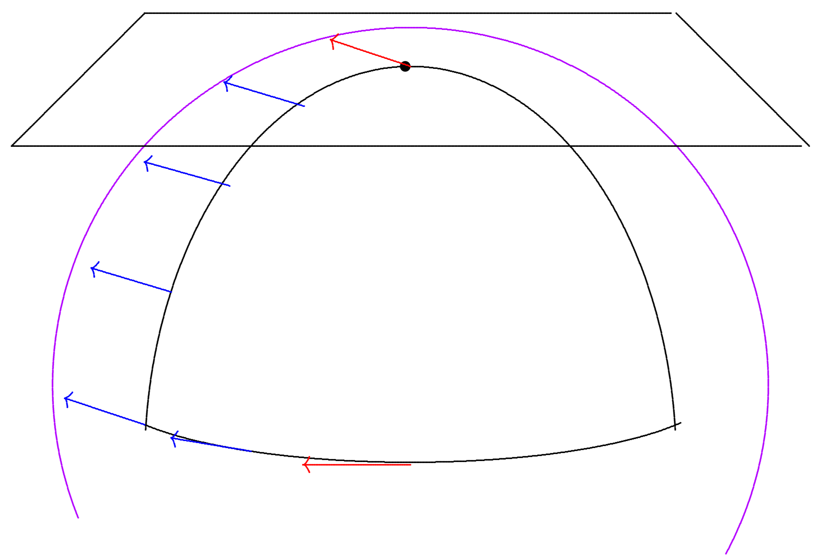

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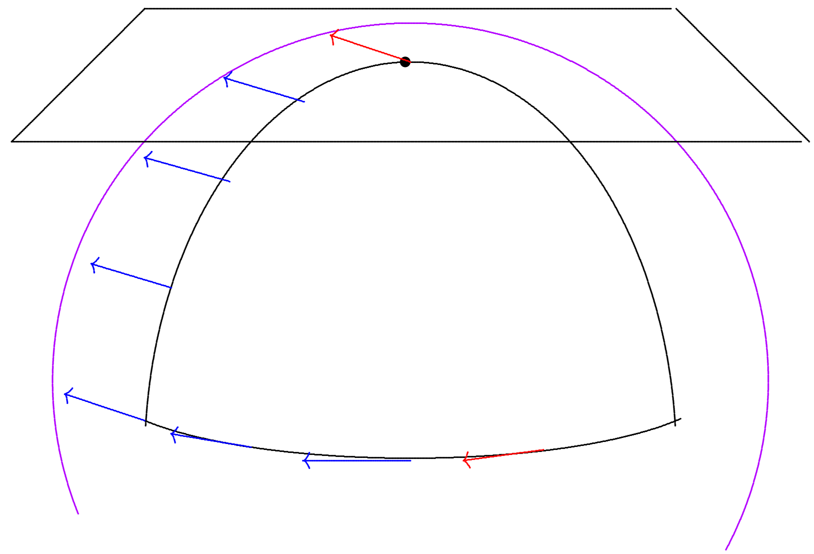

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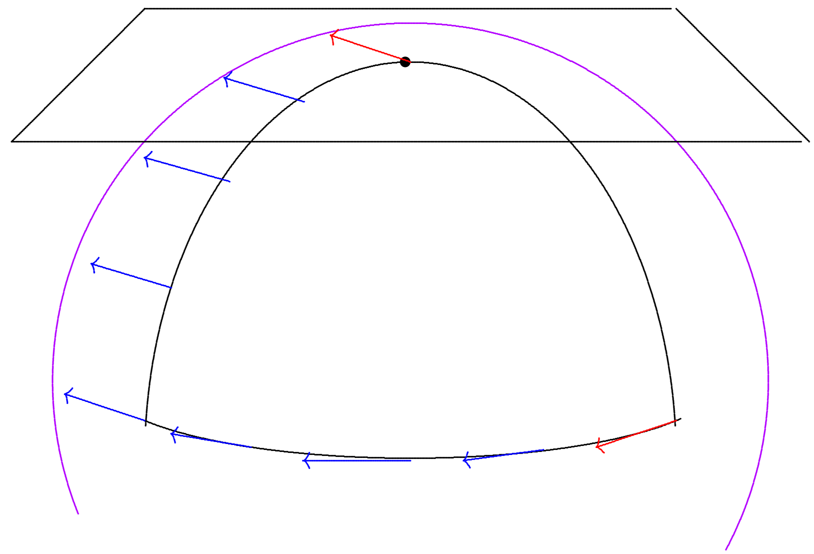

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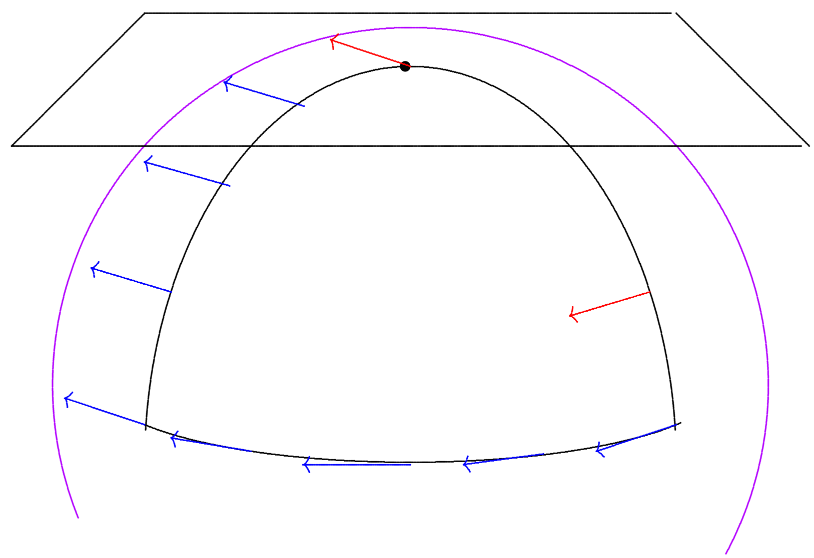

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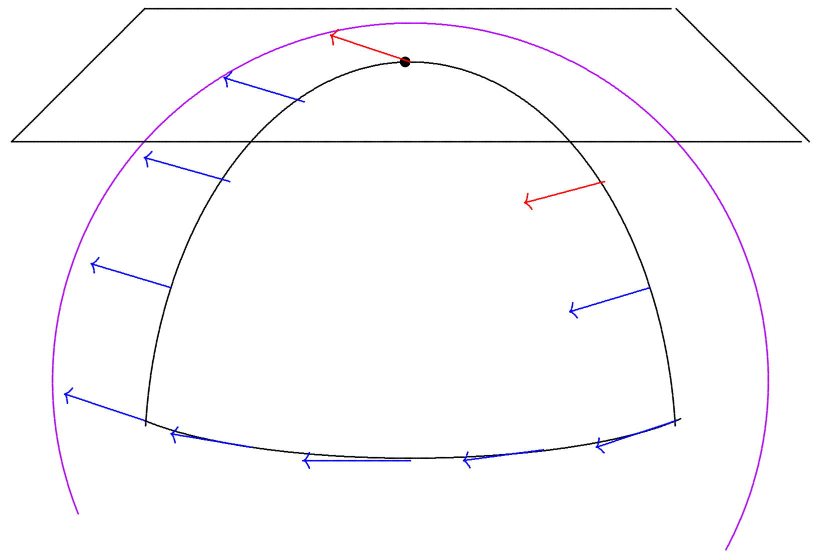

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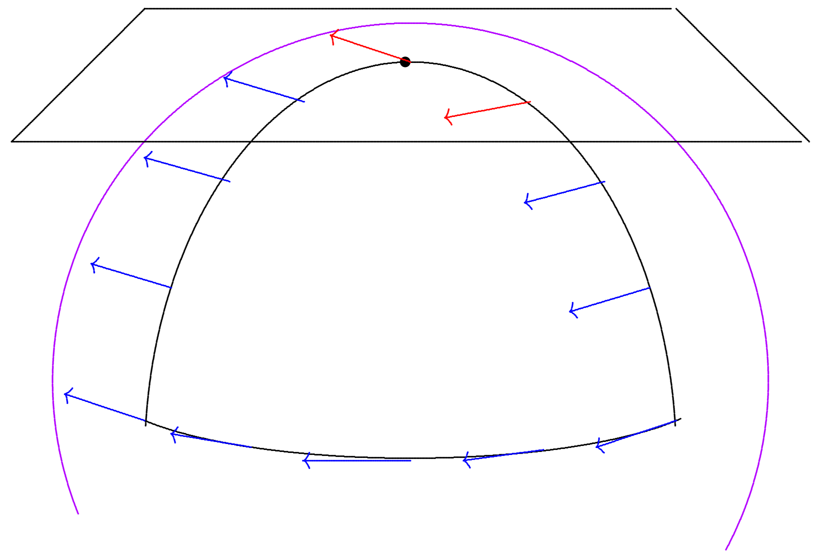

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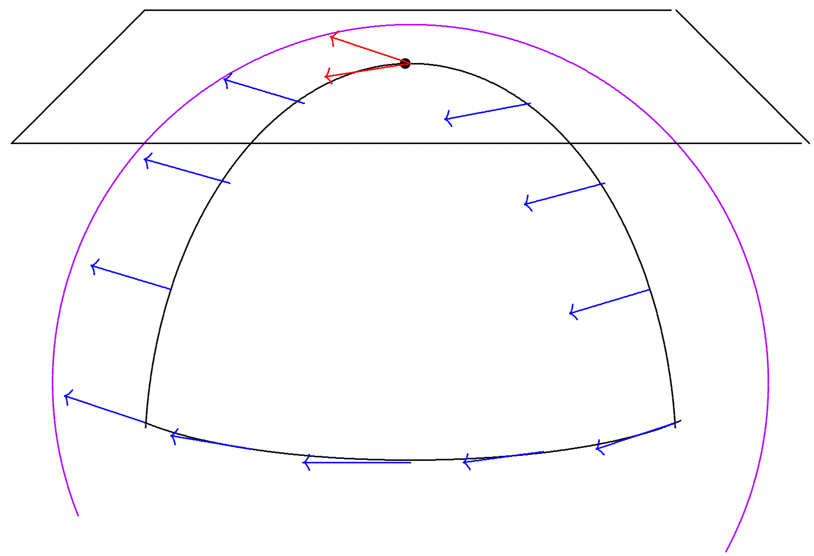

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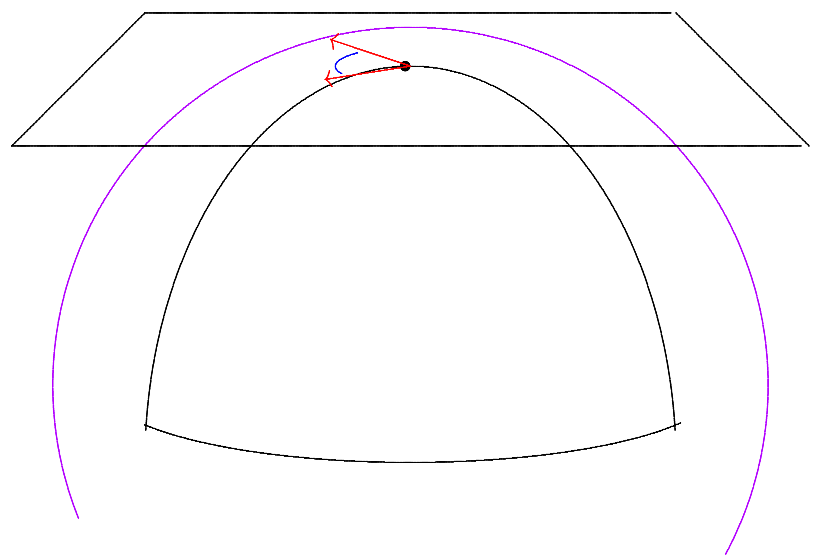
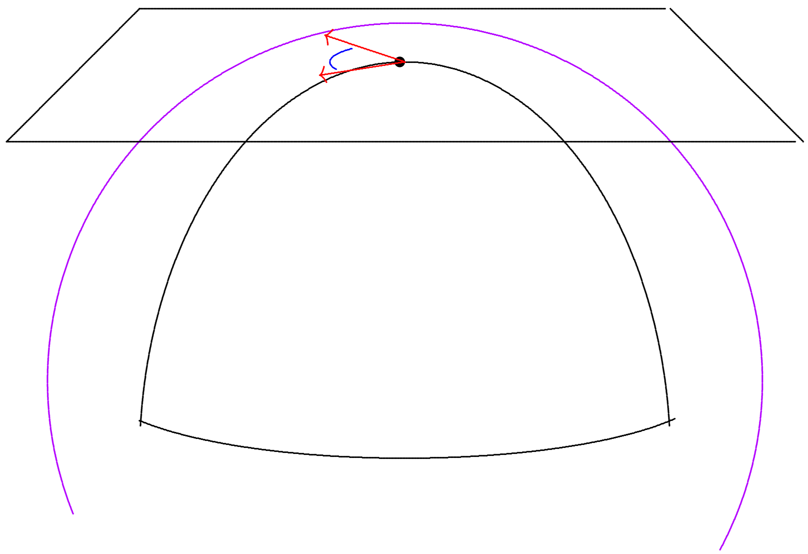
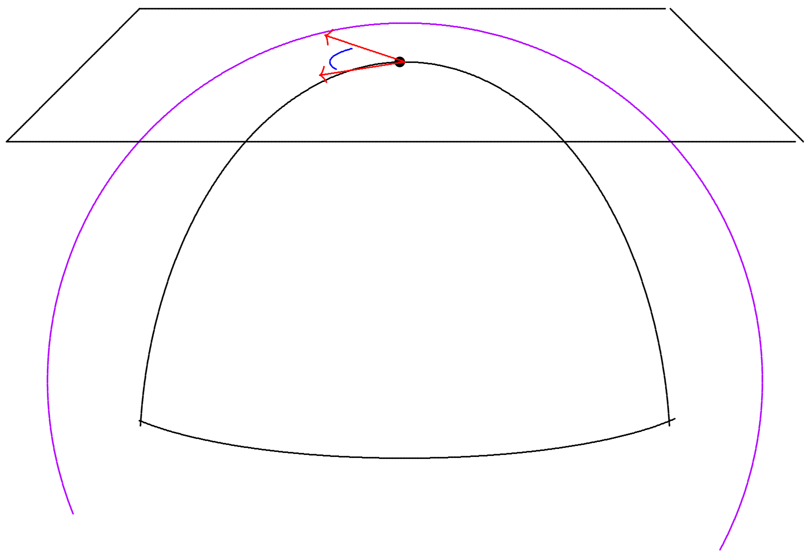


(M^n, g): holonomy $\subset \mathbf{O}(n)$



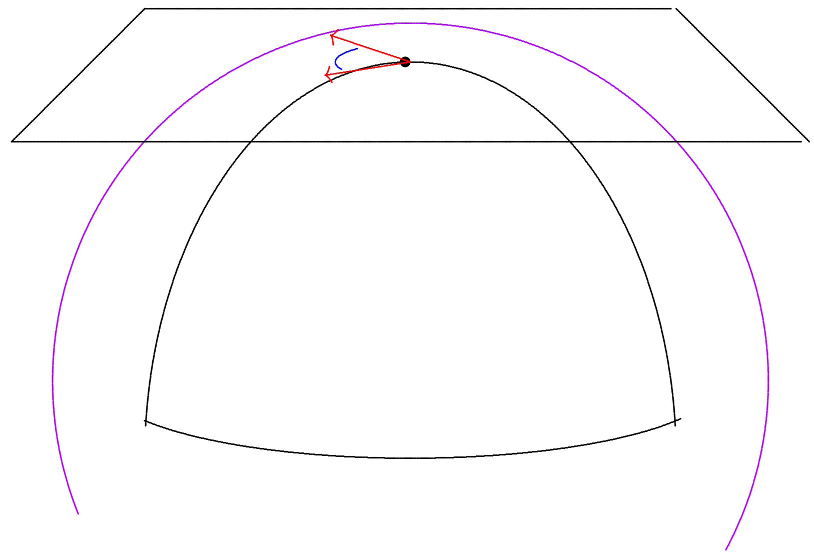
Kähler metrics:

(M^{2m}, g): holonomy



Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$



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$\Longleftrightarrow$ $\exists$ almost complex-structure J with $\nabla J = 0$ and $g(J\cdot, J\cdot) = g$.

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$\iff$ $\exists$ almost complex-structure J with $\nabla J = 0$ and $g(J\cdot, J\cdot) = g$.

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$$d\omega = 0$$

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"Kähler class"

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$$\omega = -i \sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \, dz^j \wedge d\bar{z}^k$$

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Kähler magic:

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Kähler magic:

$$r = -\sum_{j,k=1}^{m} \frac{\partial^2}{\partial z^j \partial \bar{z}^k} \log \det[g_{p\bar{q}}] \left[dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j\right]$$

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If we define the Ricci form by

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then $i\rho$ is curvature of canonical line bundle $\Lambda^{m,0}$.

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Calabi Conjecture (1954):

Completely describes the $\lambda = 0$ case.

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Calabi Conjecture (1954):
Moreover, when this happens, every Kähler class $[\omega] \in H^2(M, \mathbb{R})$ contains a unique J-compatible Kähler-Einstein metric.

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Calabi Conjecture (1954) proved by **Yau (1978).**

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So sometimes called the **Calabi-Yau Theorem**.

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Calabi Conjecture (1954) proved by **Yau (1978).**

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$\exists$ such a metric iff $-c_1(M, J)$ is a Kähler class.

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Product Einstein metric on $S^3 \times S^3$ is Hermitian with respect to Calabi-Eckmann complex structures, on a manifold than cannot admit Kähler metrics!

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Λ^+ self-dual 2-forms.
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$$|W_+|^2 = \frac{s^2}{24}$$

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and corresponds to harmonic primitive $(1,1)$-form

$$\psi := B(J\cdot,\cdot) = \frac{1}{12}\left[s\rho + 2i\partial\bar{\partial}s\right]_0$$

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So the critical points of restriction of $\mathcal{W}$ to $\{$Kähler metrics$\}$ also have $B = 0$!

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Up to constant, only Einstein conformal factor.

Complex Dimension 2 = Real Dimension 4...

Here, Calabi's extremal Kähler metrics related to Einstein metrics in a way that Gene never expected!

Proposition (L '97). *Let (M^4, J) be a complex surface, and suppose that g is an Einstein metric on M which is Hermitian with respect to J. Also suppose that g is not locally hyper-Kähler. Then g is globally conformal to a J-compatible extremal Kähler metric h.*

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Derdziński: If $W_+ \not\equiv 0$ and W_+ has repeated eigenvalue, Einstein metric must be conformally Kähler.

The Compact Case:

Theorem (CLW '08, L '12). *Let* (M, J) *be a compact complex surface. Then* (M, J) *admits a Hermitian, Einstein metric* $\Longleftrightarrow$ $b_1(M)$ *is even, and* $c_1(M, J)$ *has a sign:*

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What if we try to drop the Hermitian condition?

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Theorem (Hitchin et al.). *Suppose the compact complex surface* (M^4, J) *admits a Ricci-flat Kähler metric. Then any other Einstein metric on* M *is also Calabi-Yau (although usually adapted to some other complex structure* J'*). Consequently, the moduli space of Einstein metrics on* M *is connected.*

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Theorem (Wu '21/L '21)**.** *Let (M^4, g) be an oriented, simply-connected Einstein manifold with $\det(W_+) > 0$. Then (M, g) is a Hermitian, Einstein manifold with $\lambda > 0$.*

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In particular, the $\lambda > 0$ Einstein metrics we've discussed are isolated in the C^2 topology.

Finally, let me say something about gravitational instantons, Finally, let me say something about gravitational instantons, another subject to which Gene made seminal contributions.

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But Gene discovered this example independently!

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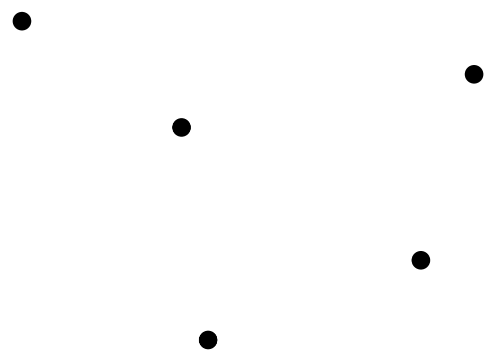

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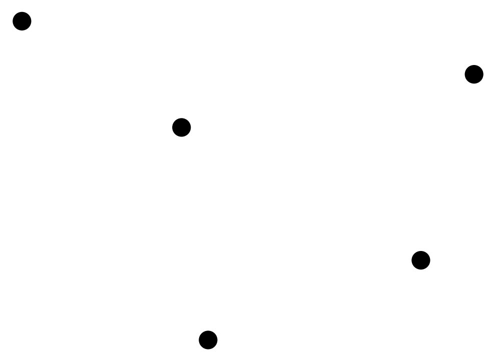

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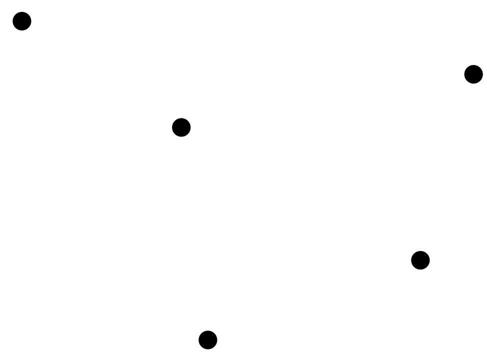

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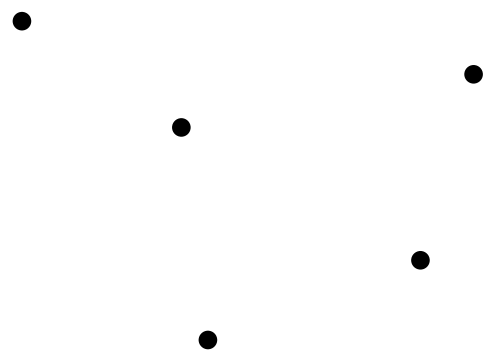

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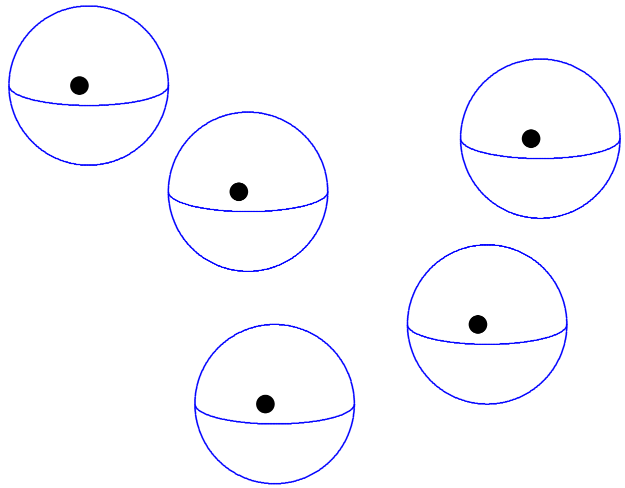

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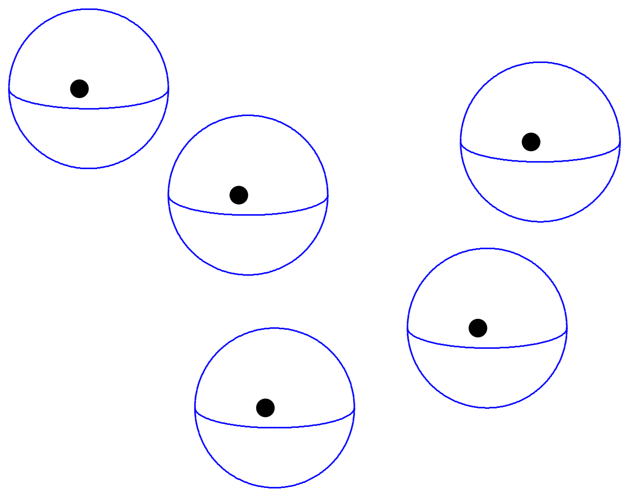

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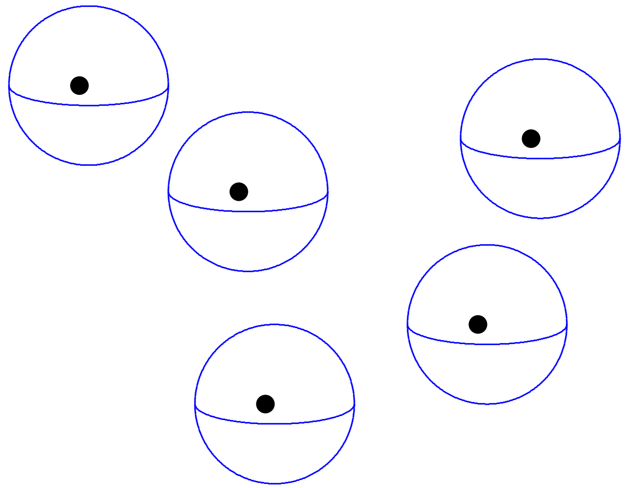

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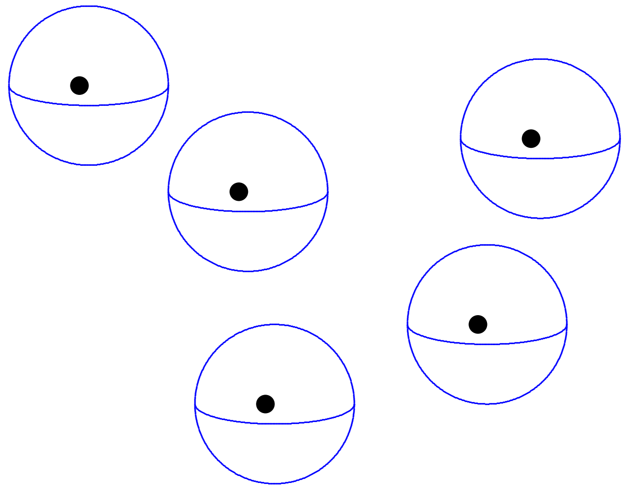

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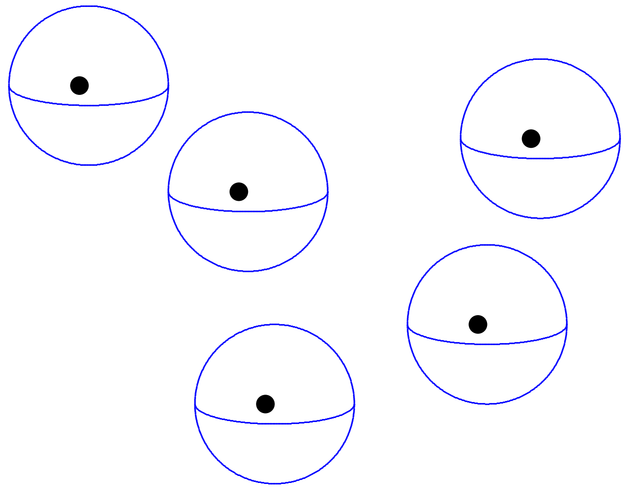

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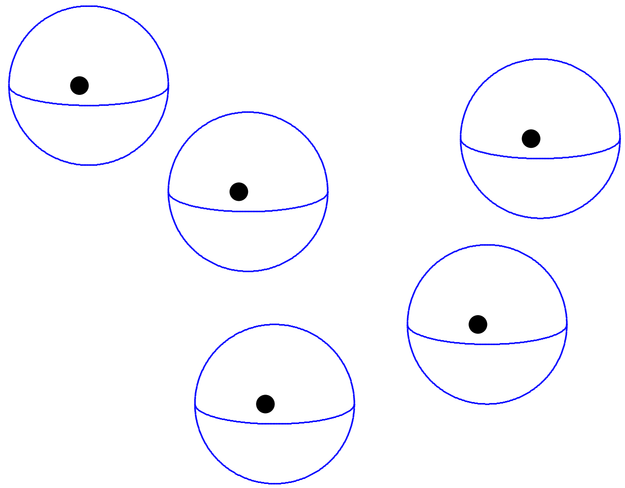

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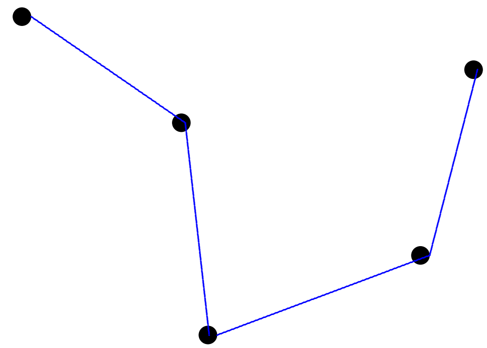

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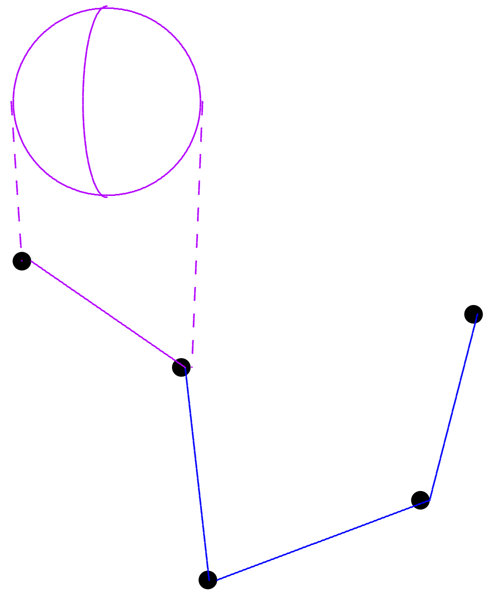

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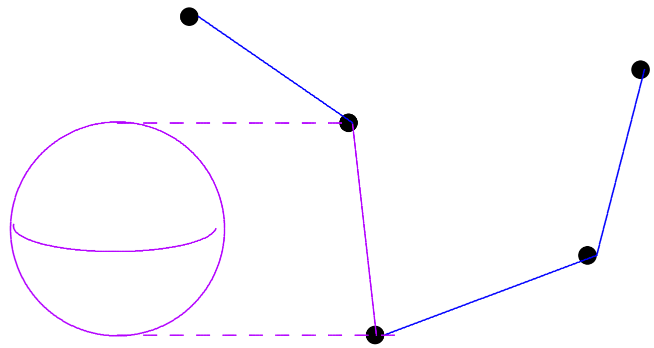

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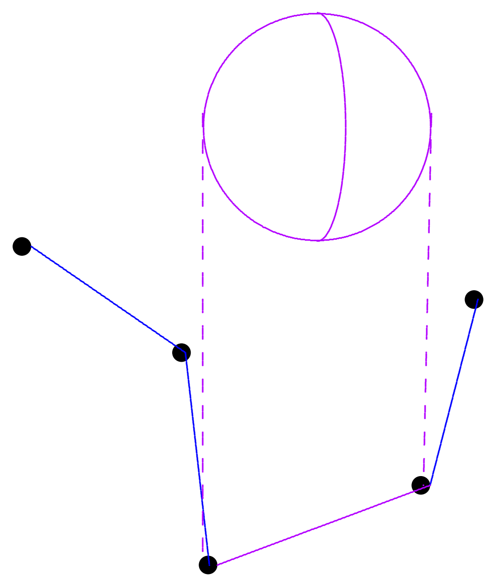

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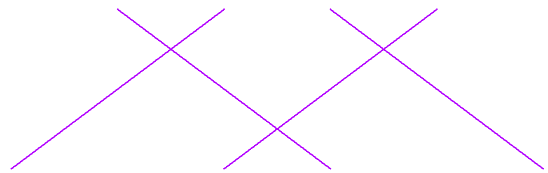

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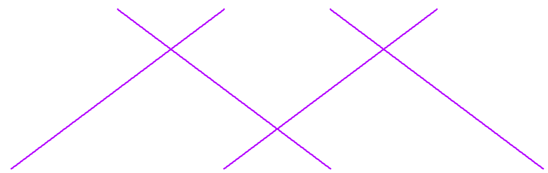

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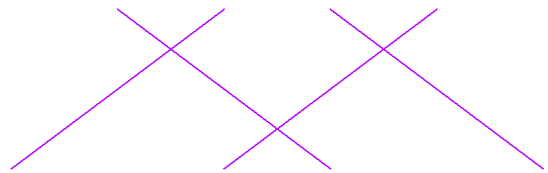

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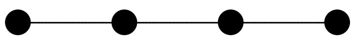

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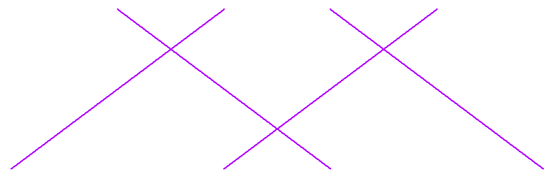

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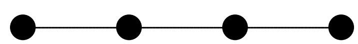

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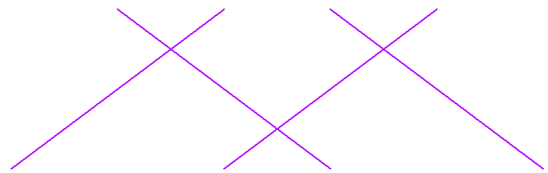

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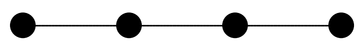

Diffeotype:

Plumb together k copies of T^*S^2
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This last property distinguishes the ALF spaces from other classes of gravitational instantons:

ALG, ALH, ALG*, ALH*, . . .

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Non-Kähler, but conformally Kähler!

Hawking also explored non-hyper-Kähler examples. . .

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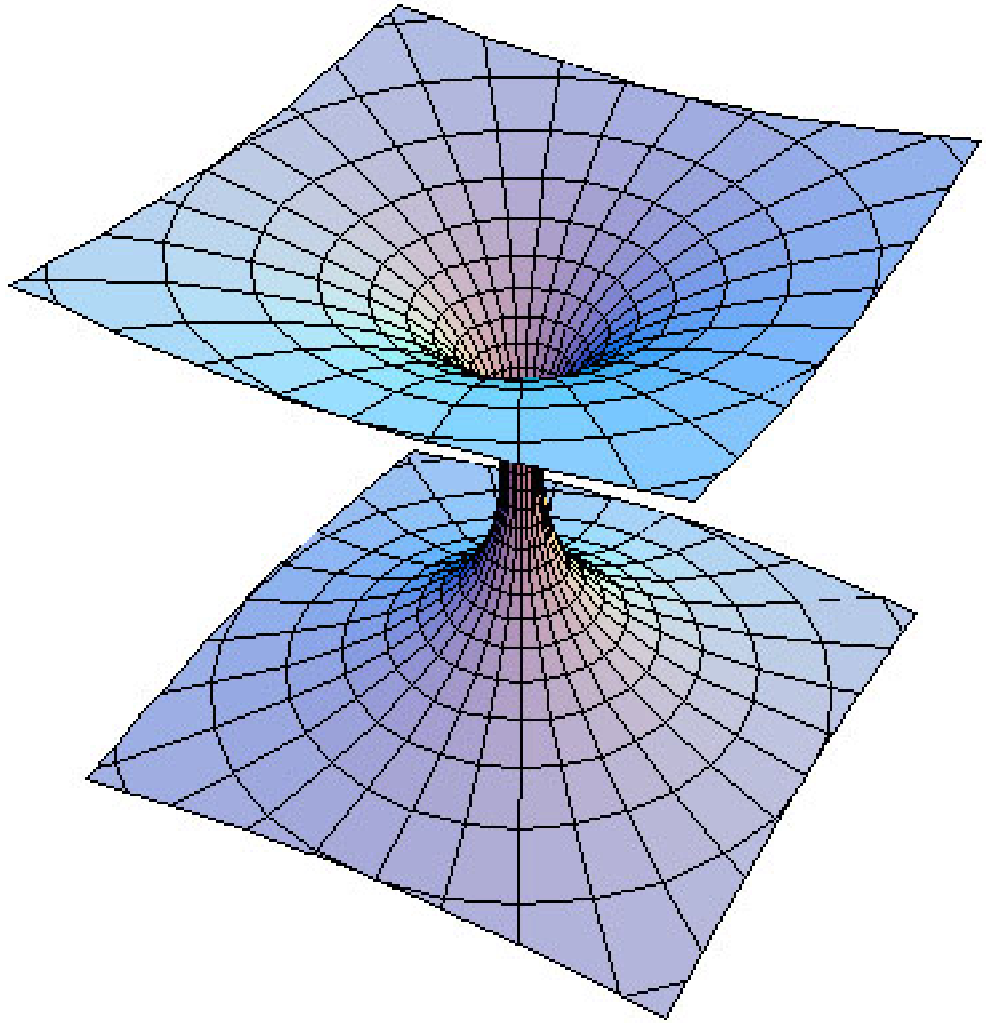

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Hitchin, Kronheimer, Cherkis-Hitchin, Minerbe, Hein, Chen-Chen, Hein-Sun-Viaclovsky-Zhang. . .

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Y. Chen & E. Teo, 2011

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Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric*, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT* metric*;*
- *the Taub-bolt* metric*;*
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*

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Notational warning:
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Joint work with

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This is suggestive, but not quite definitive.

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Assuming this, our results then imply:

Theorem B. *Let (M, g_0) be any toric Hermitian ALF gravitational instanton. Then any Ricci-flat metric g on M which is sufficiently C^3_1 close to g_0 must be another one of the gravitational instantons classified by Biquard-Gauduchon.*

Thanks for inviting me!

It's an honor to help celebrate the lasting influence of Gene's ideas!

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“Quintessentially Science Fiction”

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