MAT 531

Geometry/Topology II

Introduction to Smooth Manifolds

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Stony Brook University

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De Rham Cohomology:

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De Rham complex:

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$$\text{Image } d \subset \ker d$$

$$H^k(M) := \frac{\ker d : \Omega^k(M) \to \Omega^{k+1}(M)}{\text{Image } d : \Omega^{k-1}(M) \to \Omega^k(M)}$$

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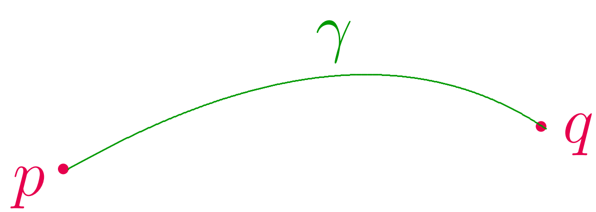

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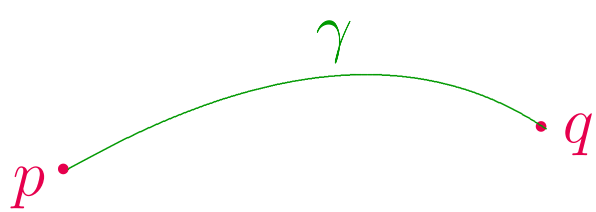

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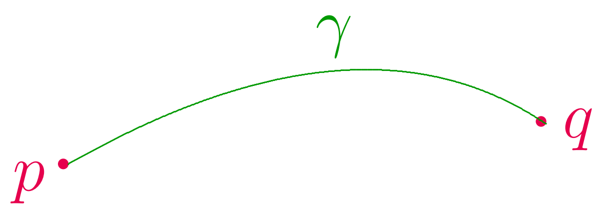

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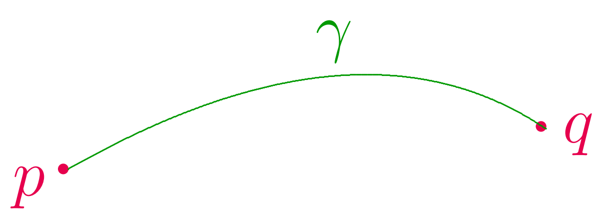

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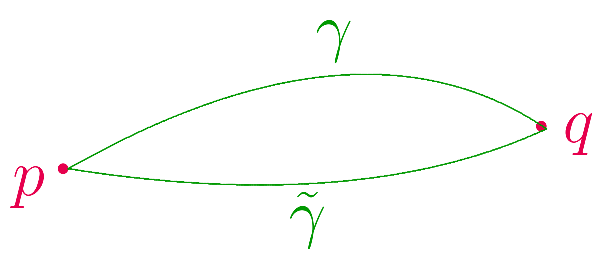

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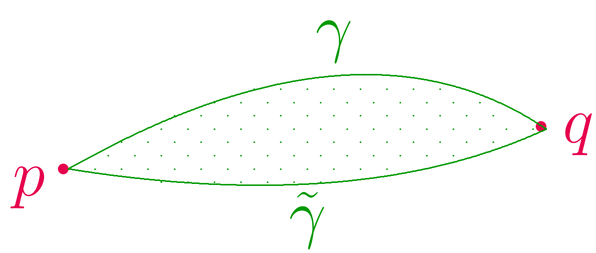

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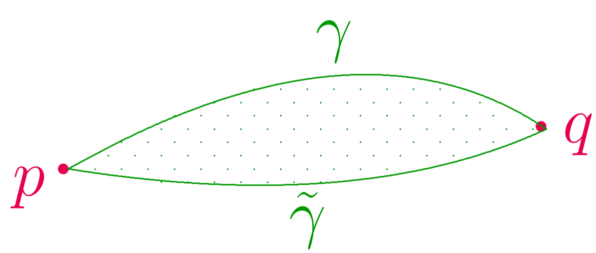

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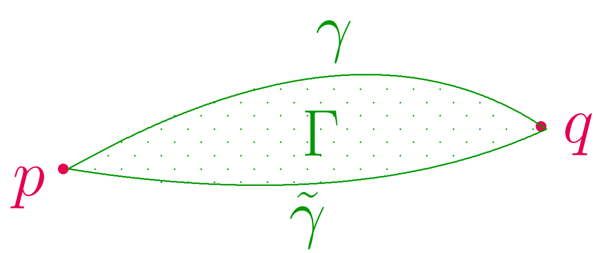

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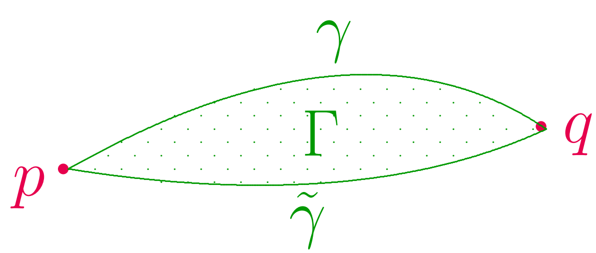

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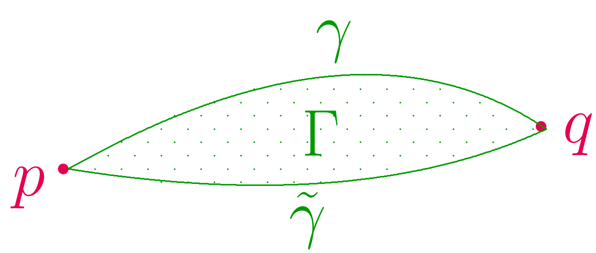

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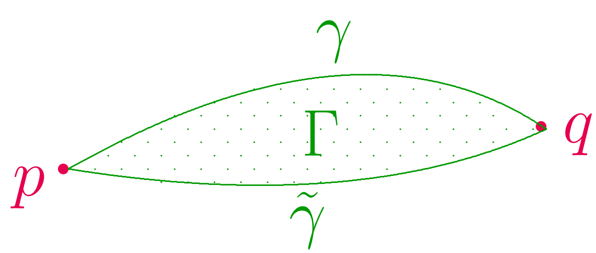

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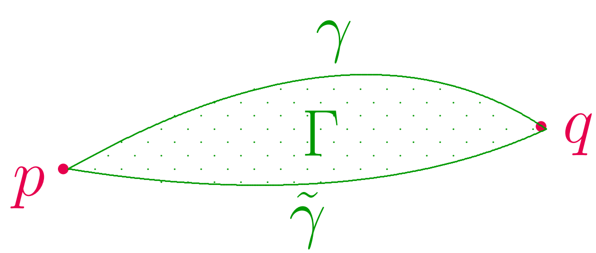

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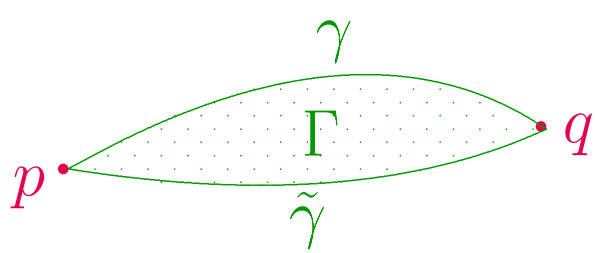

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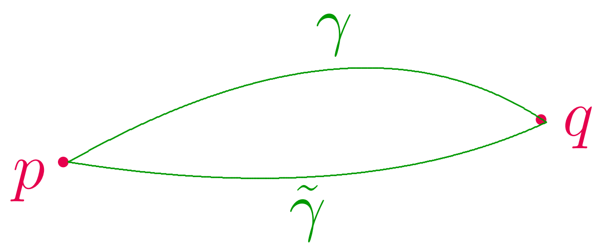

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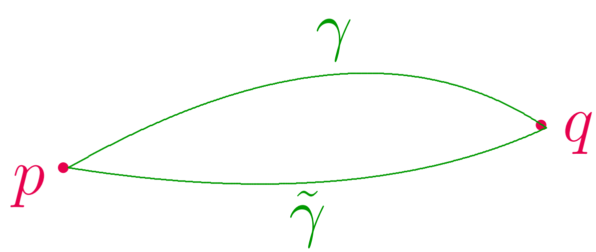

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for every n-form φ on N.

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$$\begin{aligned} \int_M G^*\varphi - \int_M F^*\varphi &= \int_{\partial(M\times[0,1])} \Phi^*\varphi \\ &= \int_{M\times[0,1]} \Phi^* d\varphi = 0. \end{aligned}$$

The homotopy invariance of the degree is actually symptomatic of a more general principle. . .

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For this, it suffices to prove. . .

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induces an isomorphism

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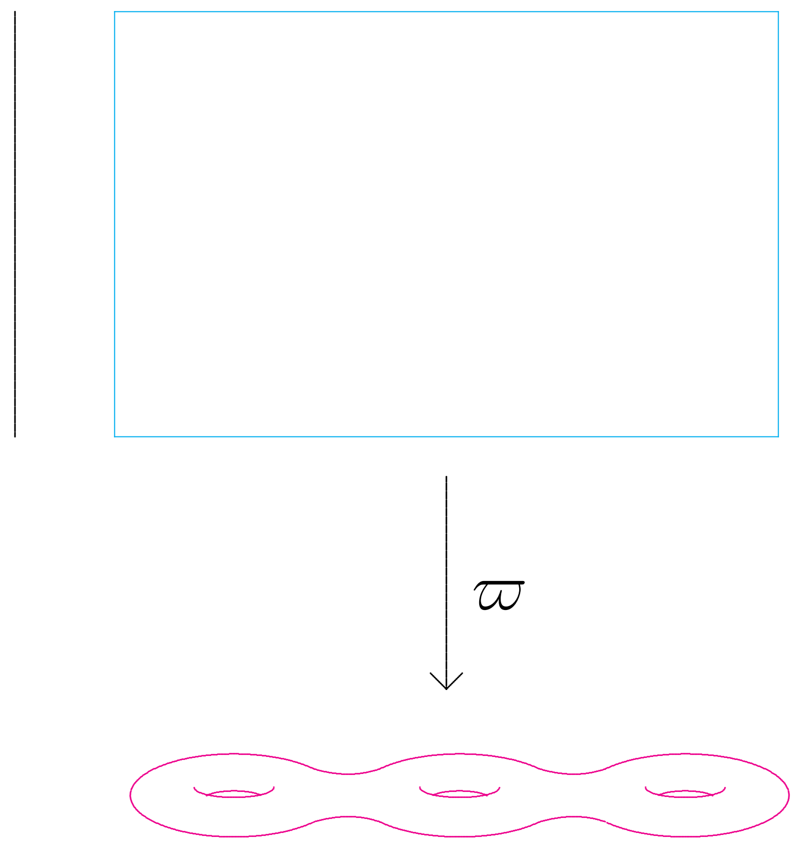

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Cartan's Magic Formula: If $\varphi \in \Omega^k(M)$, then, for any $\mathsf{V} \in \mathfrak{X}(M)$, then

$$\boxed{\mathcal{L}_{\mathsf{V}}\varphi = \mathsf{V} \lrcorner\, d\varphi + d(\mathsf{V} \lrcorner\, \varphi).}$$

When φ is closed:

$$\mathcal{L}_{\mathsf{V}}\varphi = d(\mathsf{V} \lrcorner\, \varphi).$$

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that gives F on $M \times \{0\}$ and G on $M \times \{1\}$.

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Exact: Kernel = Image at every stage.

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 & \downarrow d & & \downarrow d & & \downarrow d & \\
 & \vdots & & \vdots & & \vdots &
\end{array}
$$

Commutative diagram with exact rows!

Mayer-Vietoris sequence:

Suppose $M = \mathcal{U} \cup \mathcal{V}$.

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
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"Snake Lemma:" Induces long exact sequence. . .

$$0 \longrightarrow H^0(\mathcal{U} \cup \mathcal{V}) \longrightarrow H^0(\mathcal{U}) \oplus H^0(\mathcal{V}) \longrightarrow H^0(\mathcal{U} \cap \mathcal{V})$$

$$\xrightarrow{\delta} H^1(\mathcal{U} \cup \mathcal{V}) \longrightarrow H^1(\mathcal{U}) \oplus H^1(\mathcal{V}) \longrightarrow H^1(\mathcal{U} \cap \mathcal{V})$$

$$\xrightarrow{\delta} H^2(\mathcal{U} \cup \mathcal{V}) \longrightarrow H^2(\mathcal{U}) \oplus H^2(\mathcal{V}) \longrightarrow H^2(\mathcal{U} \cap \mathcal{V})$$

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Mayer-Vietoris long exact sequence therefore reads. . .

$$
0 \longrightarrow H^0(S^n) \longrightarrow H^0(\mathbb{R}^n) \oplus H^0(\mathbb{R}^n) \longrightarrow H^0(S^{n-1} \times \mathbb{R}) \dashrightarrow
$$

$$
H^{n-2}(S^n) \longrightarrow H^{n-2}(\mathbb{R}^n) \oplus H^{n-2}(\mathbb{R}^n) \longrightarrow H^{n-2}(S^{n-1} \times \mathbb{R}) \xrightarrow{\delta}
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for all $j \geq 2$, $n \geq 2$.

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Proposition.

$$H^k(S^n) = \begin{cases} \mathbb{R} & \textit{if } k = 0 \textit{ or } n, \\ 0 & \textit{otherwise.} \end{cases}$$

Finally, let's shore up our foundations. . .

De Rham Cohomology:

$$H^k(M) := \frac{\{\text{closed } k\text{-forms}\}}{\{\text{exact } k\text{-forms}\}}$$

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That is, an n-form is exact iff its integral $= 0$.

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Same conclusion if M compact manifold-with-boundary, where $\partial M \neq \varnothing$.

Compactly supported cohomology.

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Let $\Omega_c^k(M) = \{\text{compactly supported smooth } k\text{-forms}\}$.

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$$H_c^k(M) := \frac{\ker d : \Omega_c^k(M) \to \Omega_c^{k+1}(M)}{\text{Image } d : \Omega_c^{k-1}(M) \to \Omega_c^k(M)}$$

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Now proceed by induction...

Proof

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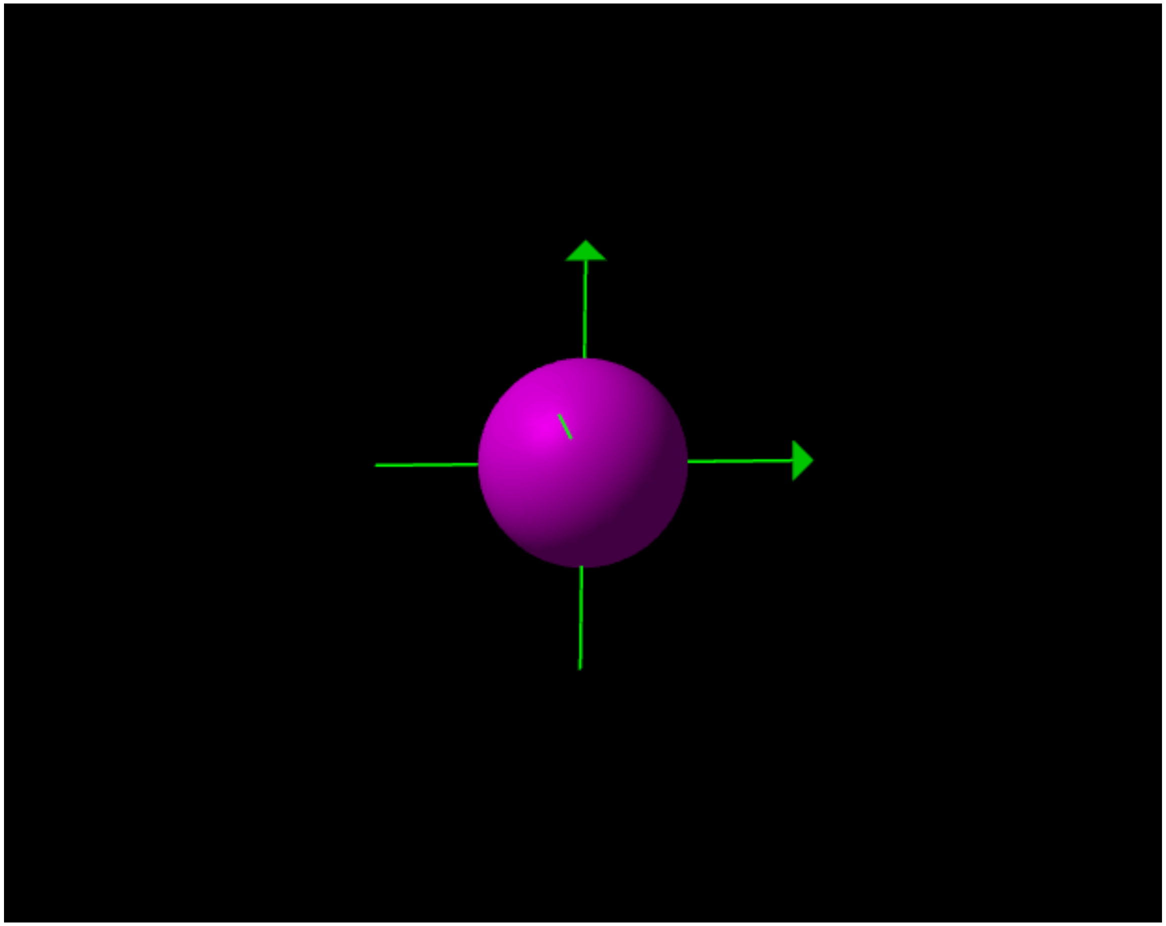

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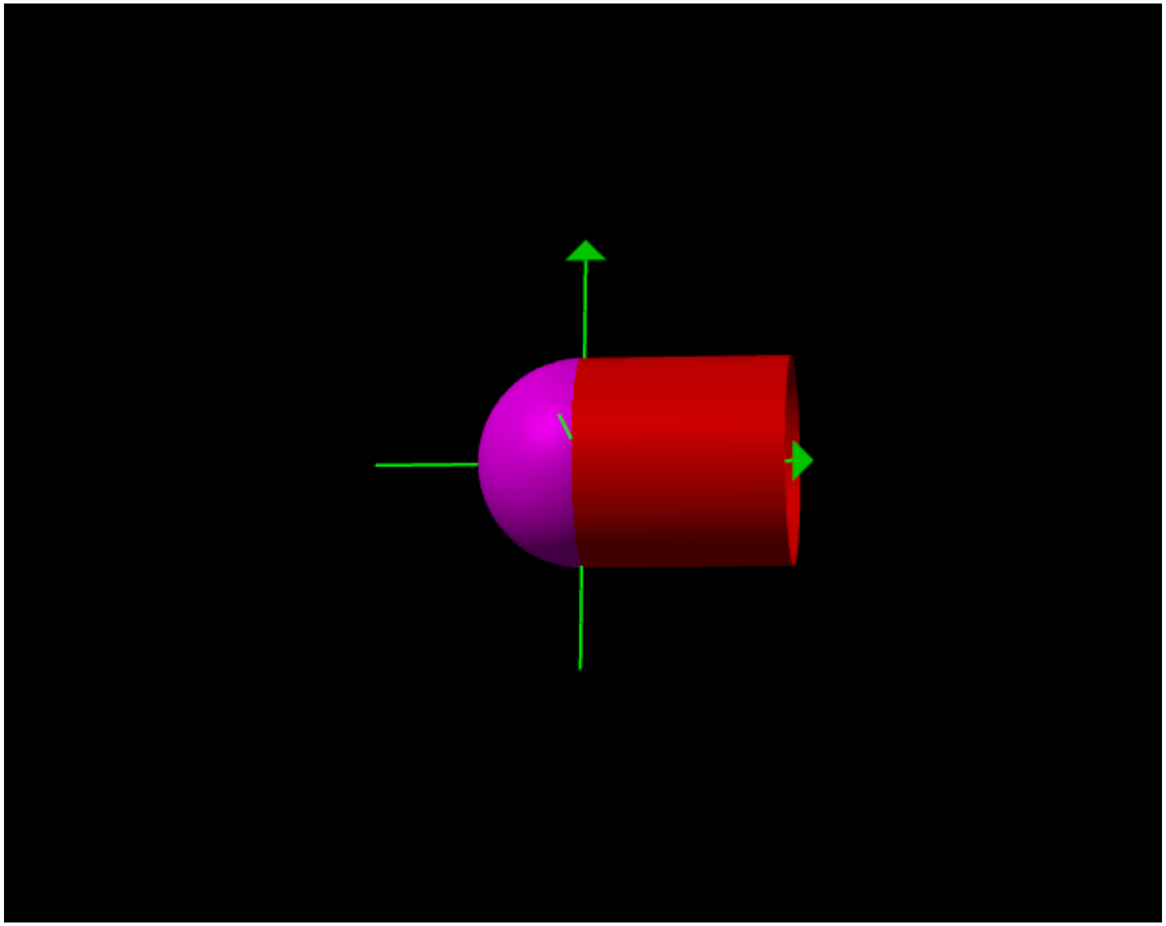

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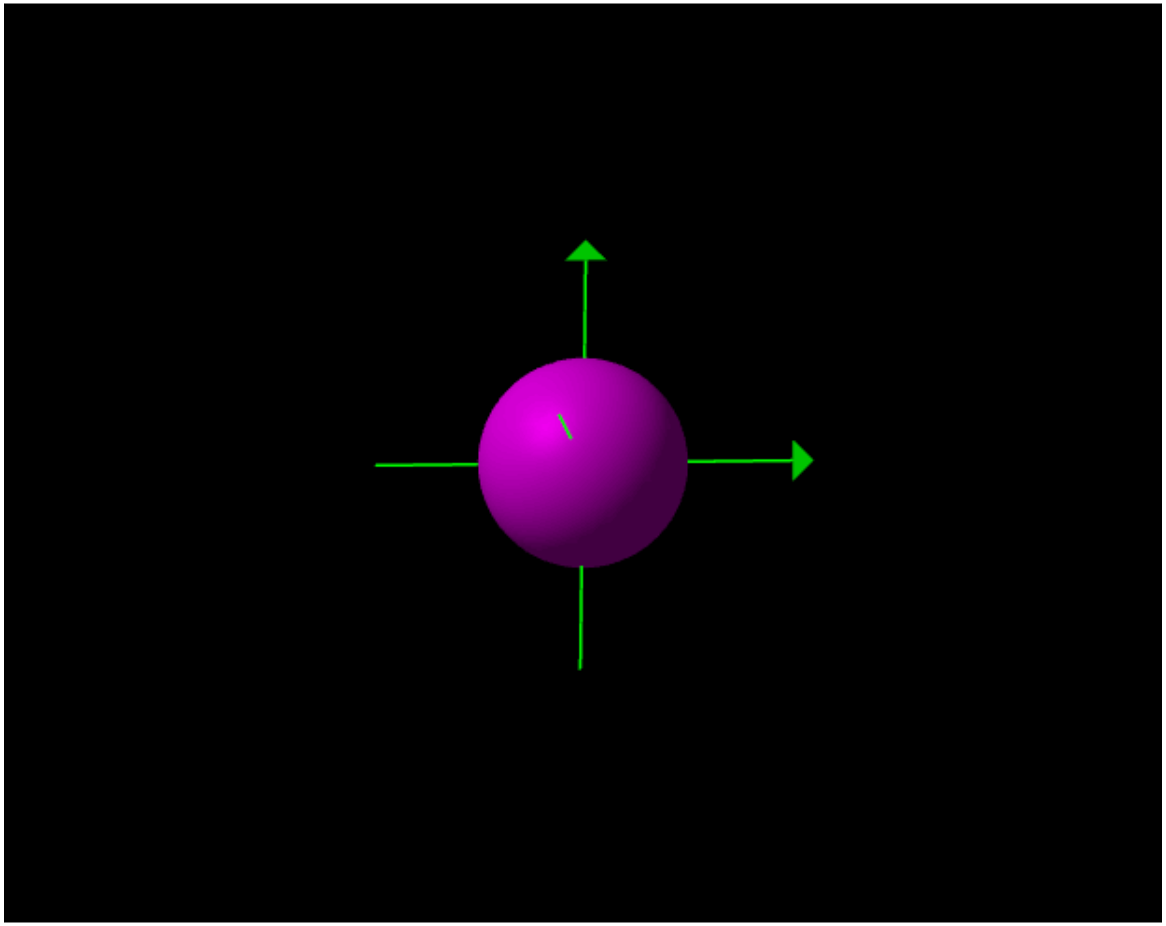

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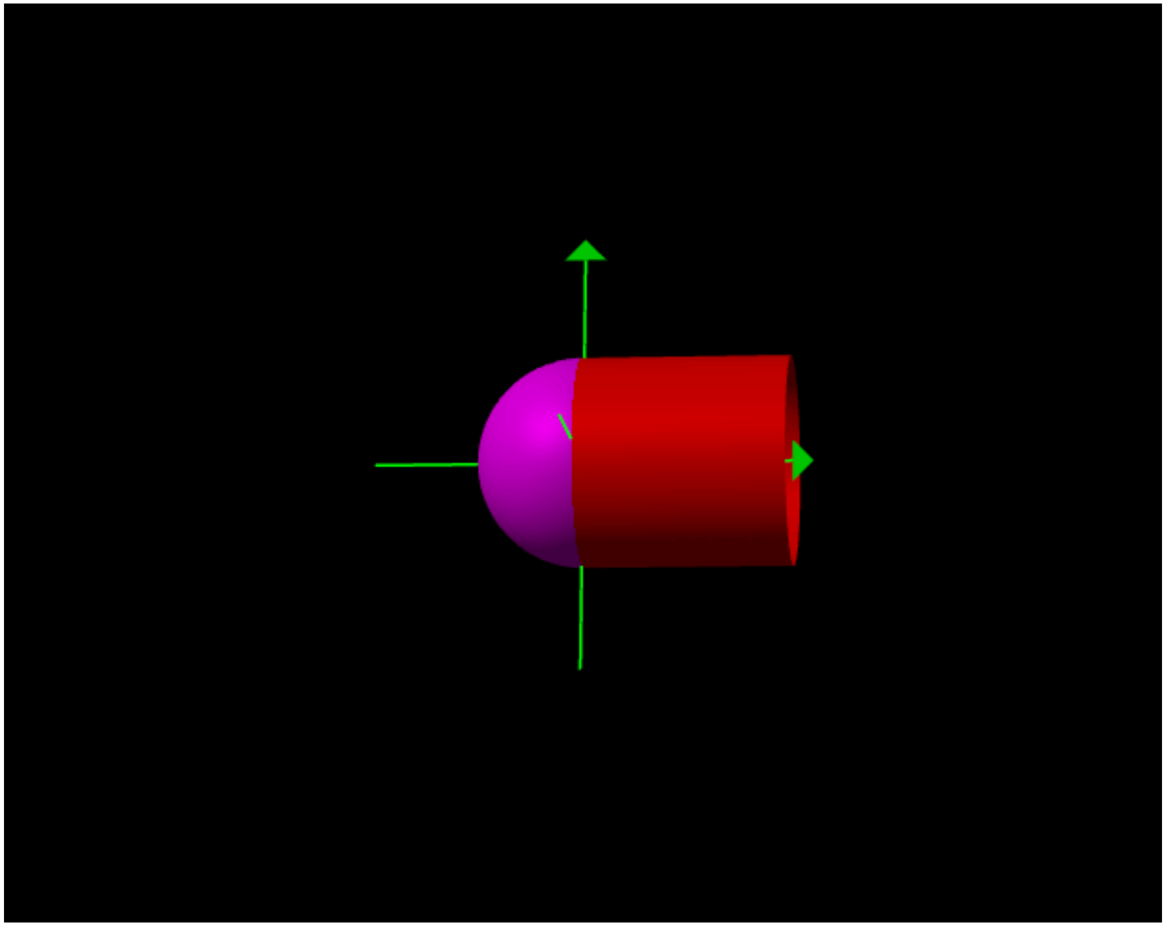

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Notation:

$$\begin{aligned} \pi : \mathbb{R}^n &\longrightarrow \mathbb{R}^{n-1} \\ (x^1, x^2, \ldots, x^n) &\longmapsto (x^2, \ldots, x^n) \end{aligned}$$

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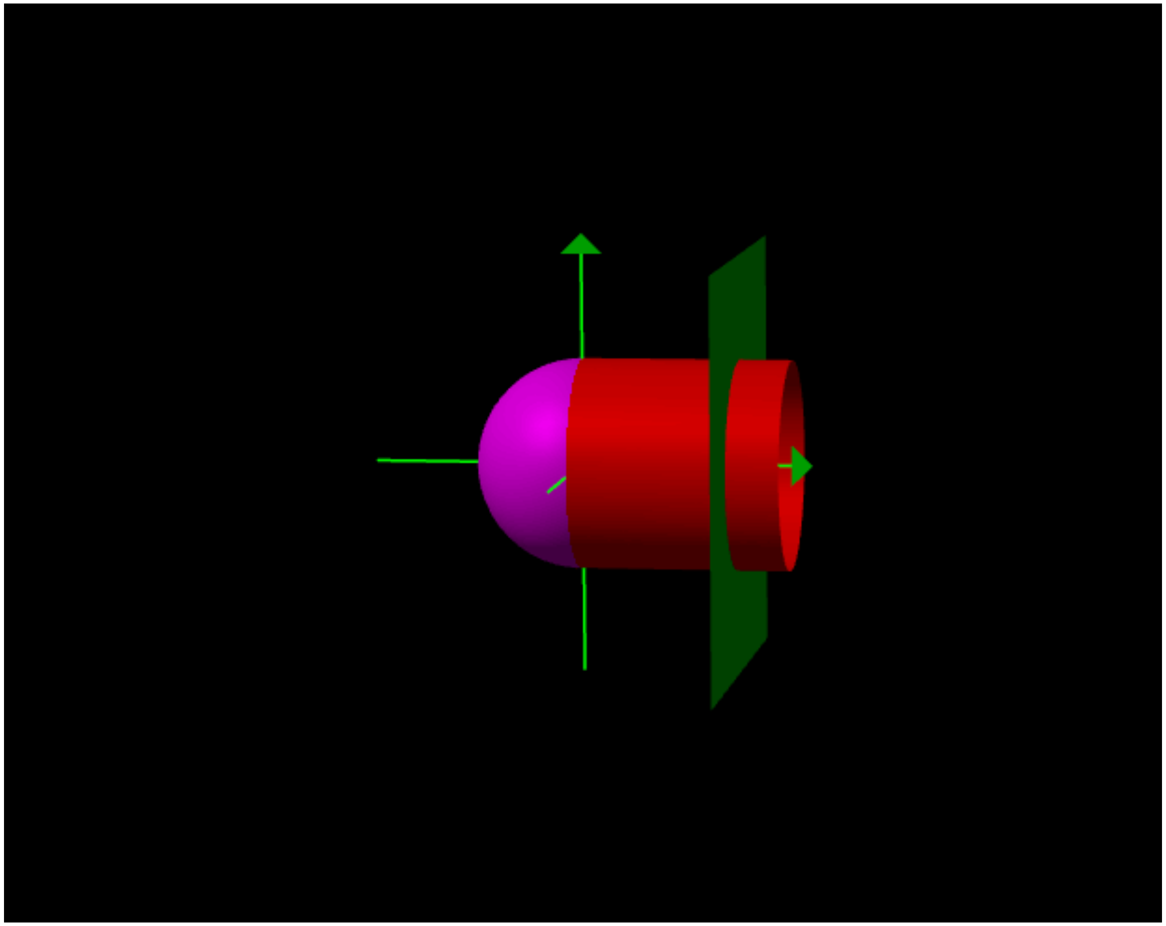
Reduction to $\mathbb{R}^{n-1}$:



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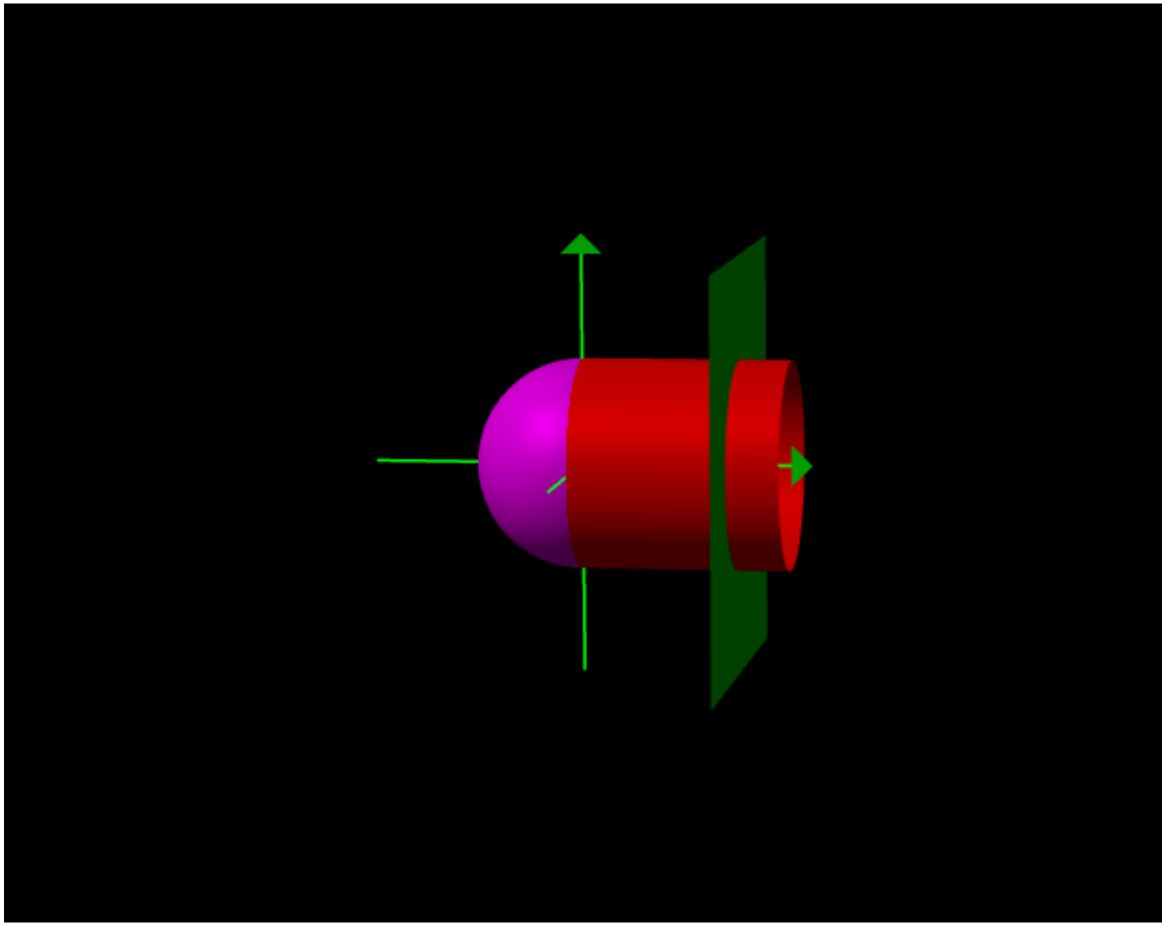

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Using this, we now prove a major generalization. . .

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Now recall an application of flow of a vector field. . .

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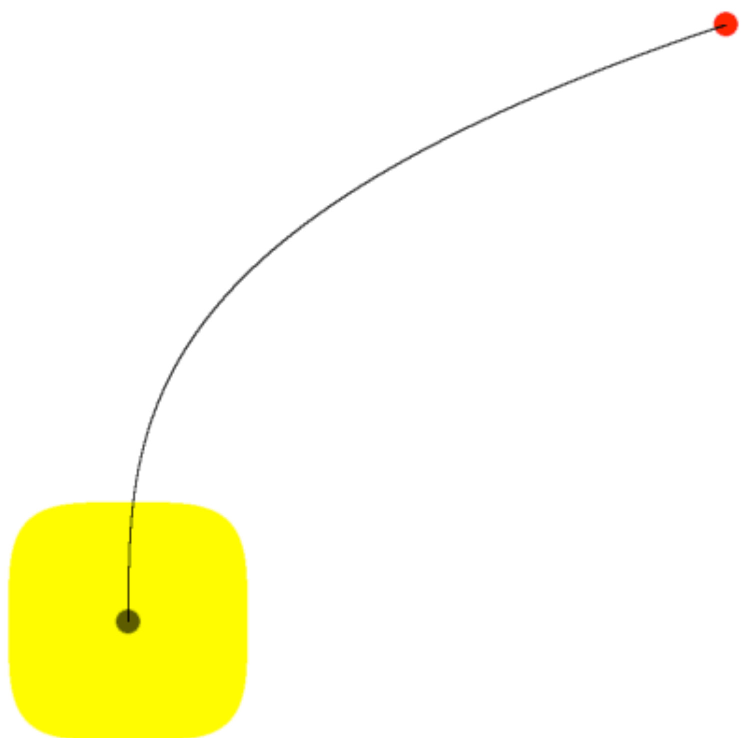

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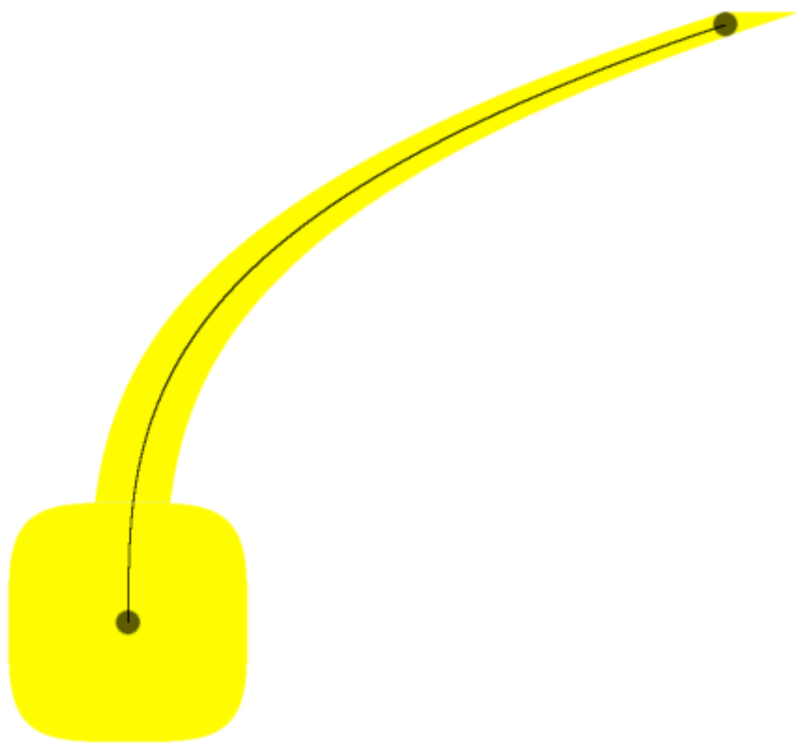

Proposition. *Let M^n be a connected, oriented n-manifold (without boundary), and let $\varphi \in \Omega^n_c(M)$ be a compactly supported n-form with*

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Then each n-forms φ_j is then compactly supported in $\mathscr{U}_j$, and

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This now implies...

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Specializing to the compact case, we thus have. . .

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By contrast…

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To see this, first recall…

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$$\Phi^2 = \text{identity}.$$

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Same trick used in non-compact case. . .

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Deduce non-orientable case by our double-cover trick.