MAT 531

Geometry/Topology II

Introduction to Smooth Manifolds

Claude LeBrun
Stony Brook University

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Let's not confuse one twin for the other!

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$$(B \circ A)^* = A^* \circ B^*$$

$$\mathbb{U} \xleftarrow{A^*} \mathbb{V}^* \xleftarrow{B^*} \mathbb{W}^*$$

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Or mirror twins!

Key Example. If M is a smooth n-manifold, and $p \in M$, the tangent space T_pM has a dual vector space

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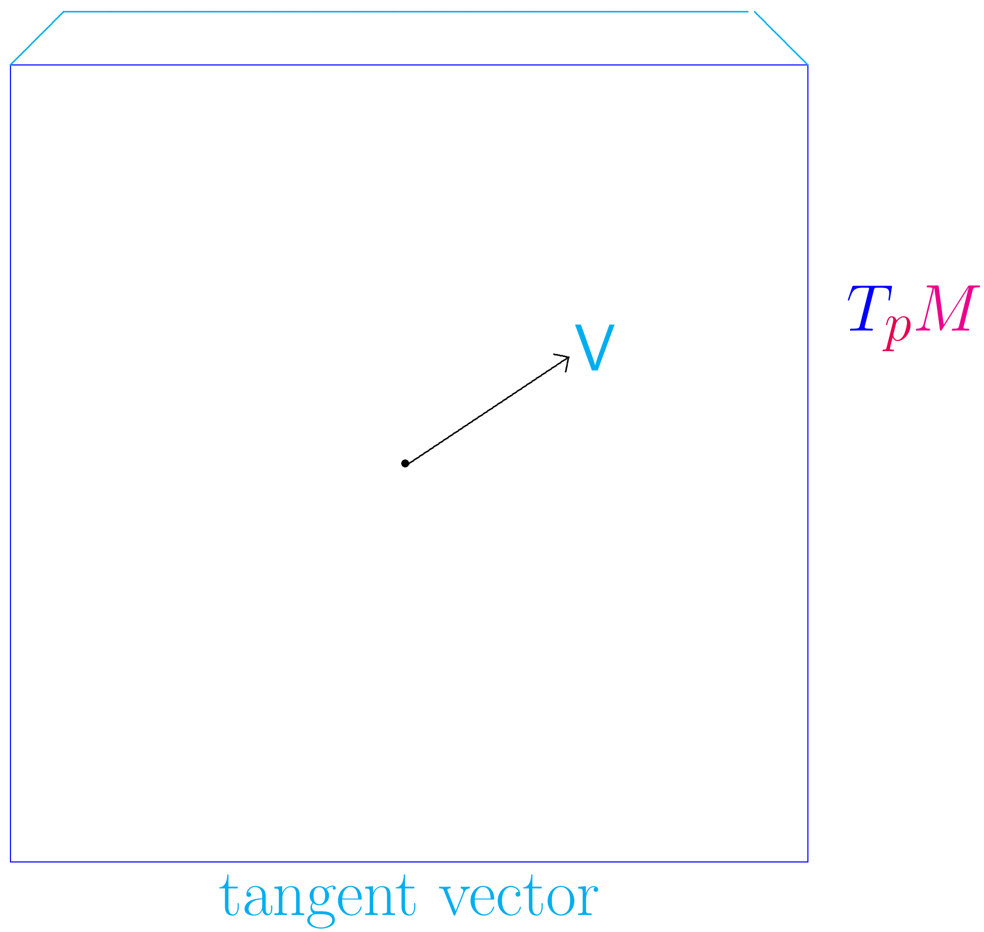
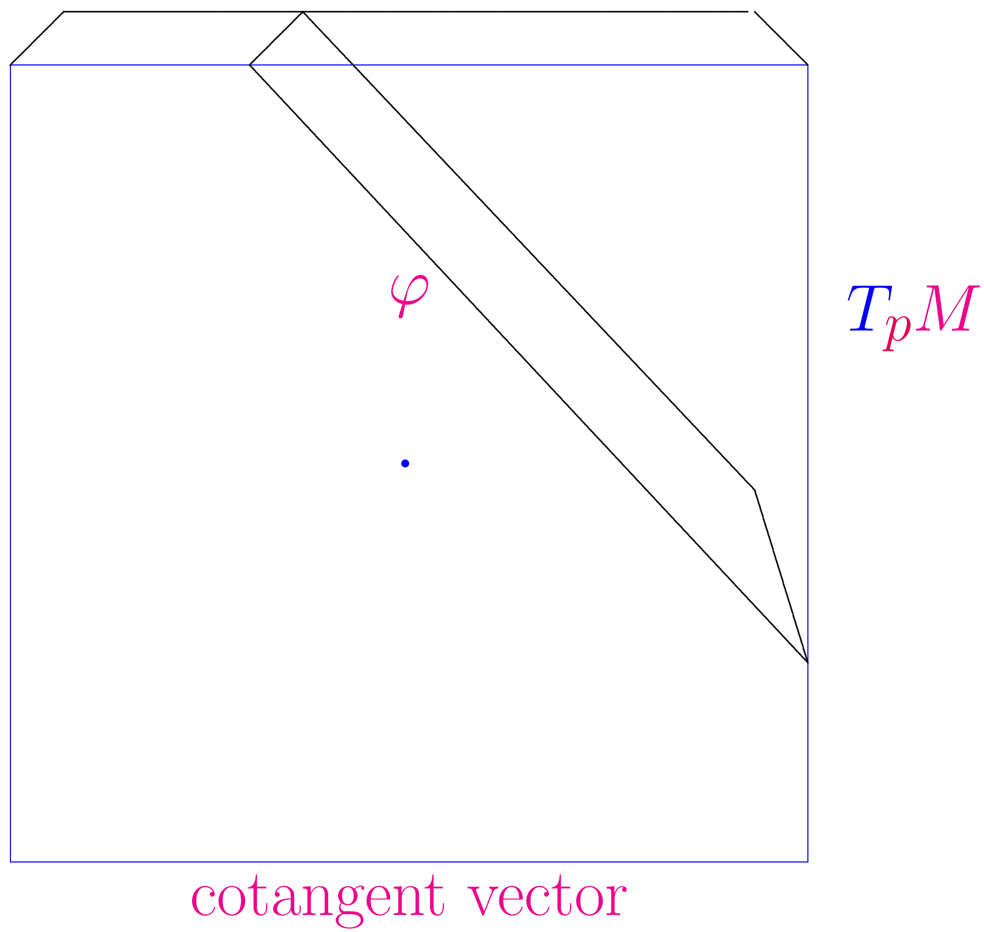

V
T_pM
tangent vector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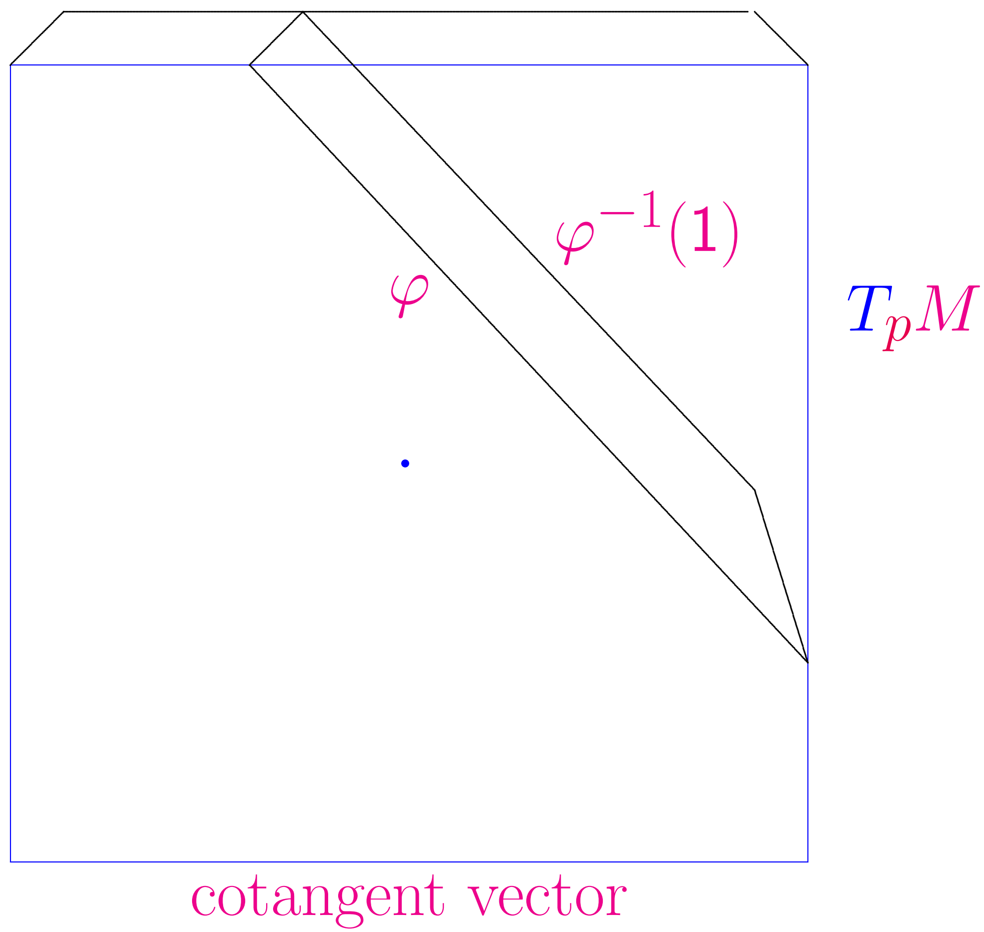

$\varphi^{-1}(1)$
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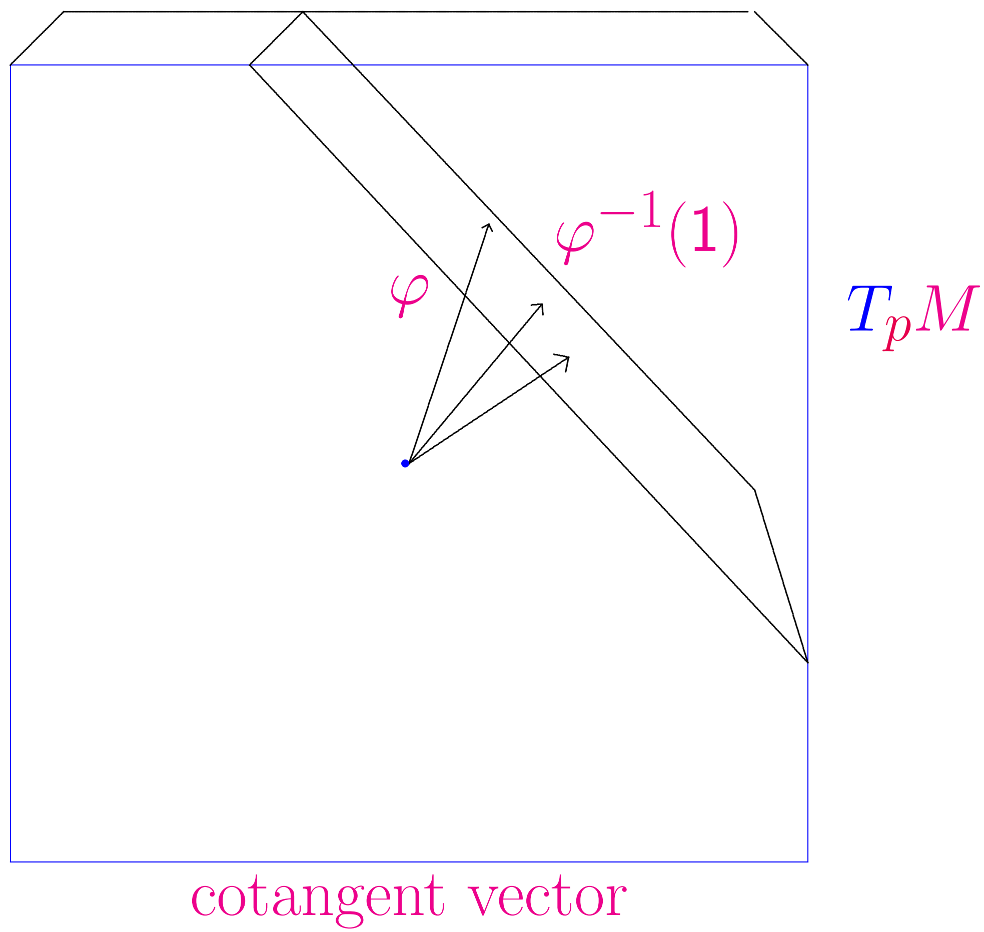

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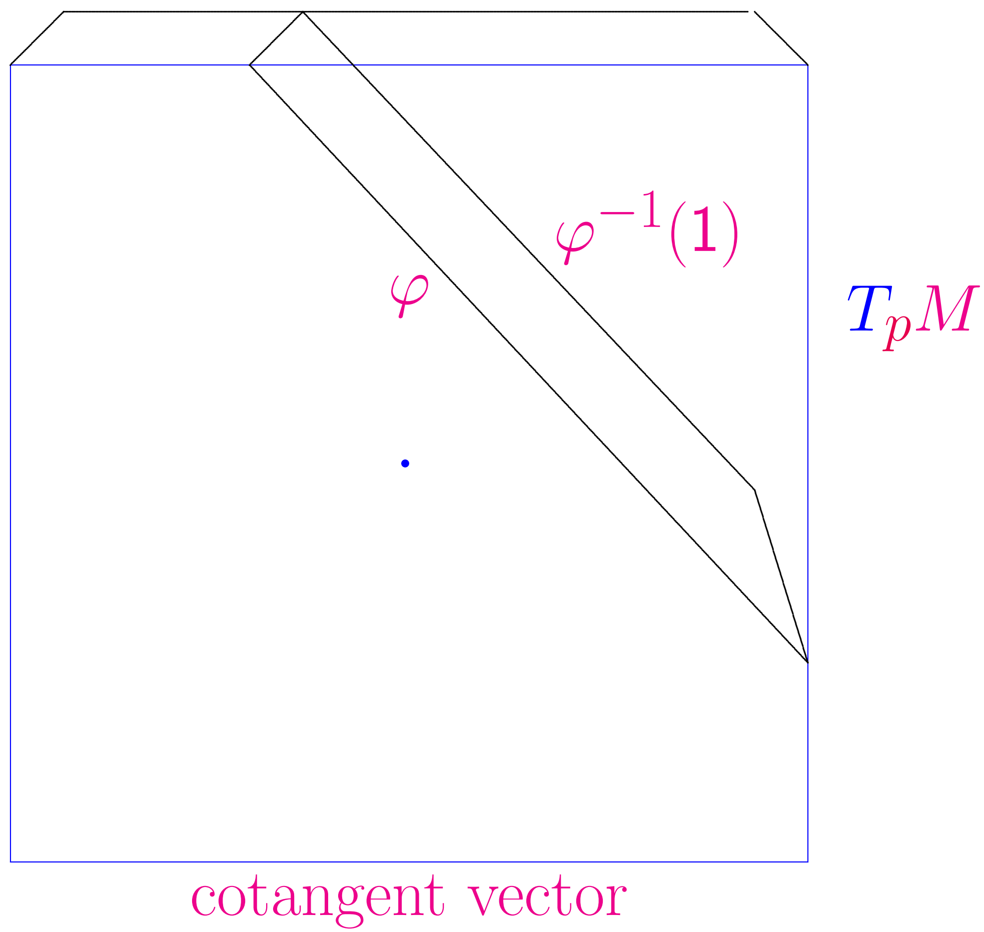

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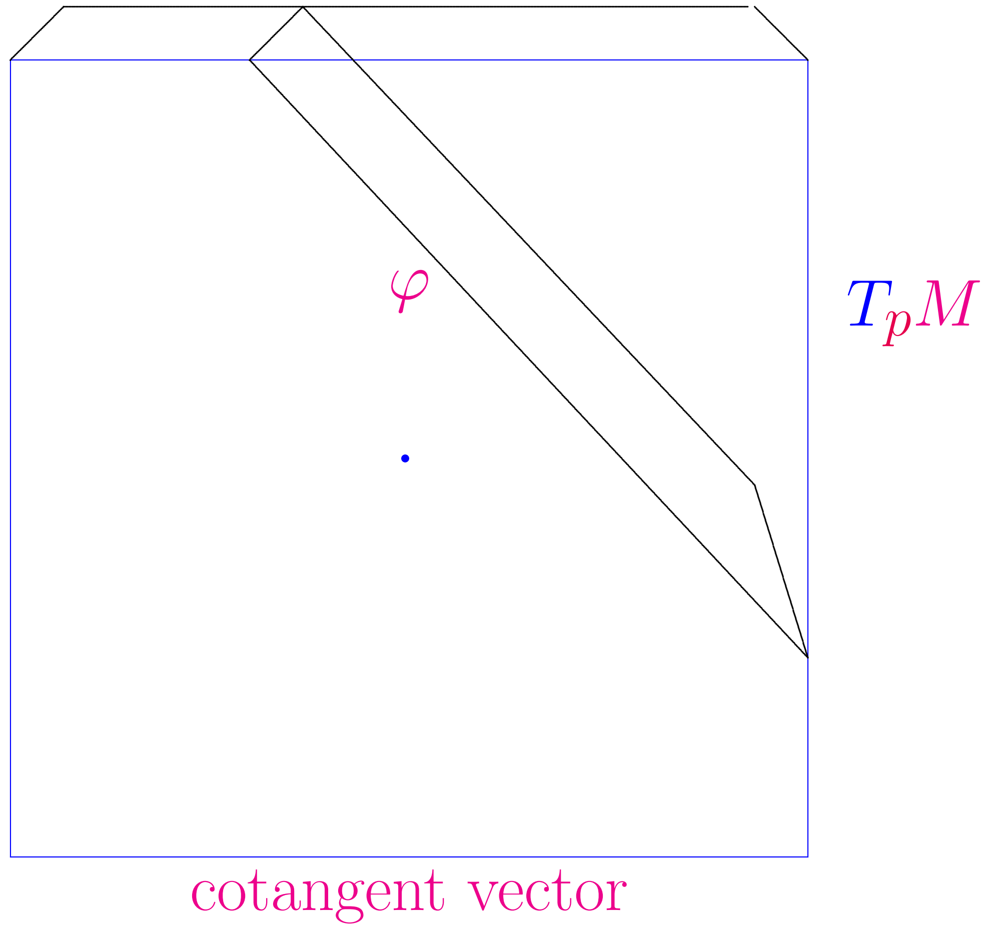

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Zero covector $\leftrightarrow \varnothing$: "hyperplane at infinity."

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Thus, for $v \in T_pM$, we have

$$(df)(v) := vf.$$

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$$dx^1, \ldots, dx^n \in T_p^* M$$

provide a basis for $T_p^* M$.

Exactly dual basis of coordinate basis for $T_p M$:

$$(dx^j)\left(\frac{\partial}{\partial x^k}\right) = \delta_k^j = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

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In particular, df is linear combination of dx^j:

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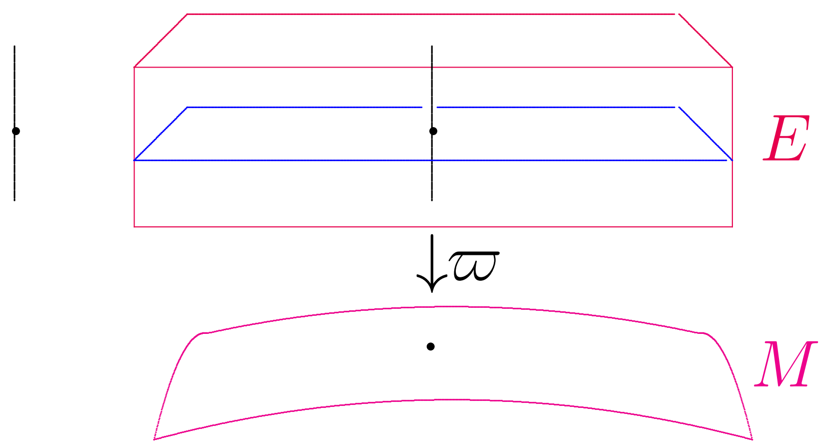
Let M be a smooth n-manifold.

A smooth rank-k vector bundle over M is

- a smooth $(n+k)$-manifold E;
- a smooth map $\varpi : E \to M$; and
- a real vector space structure on every "fiber"

$$E_p := \varpi^{-1}(p);$$



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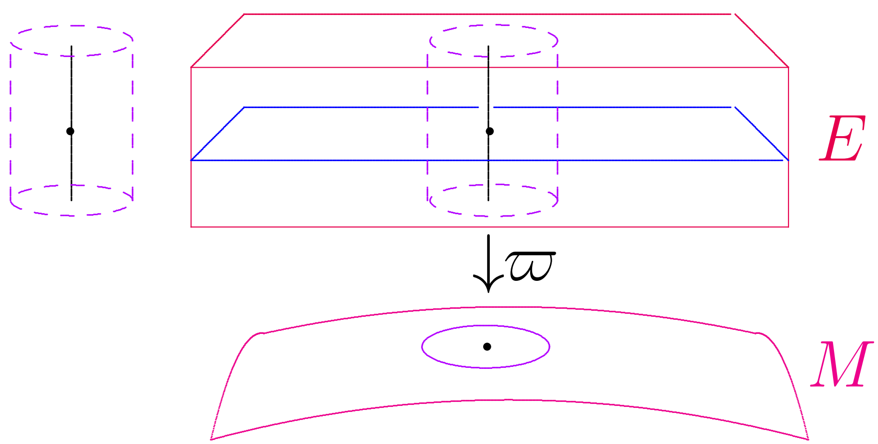

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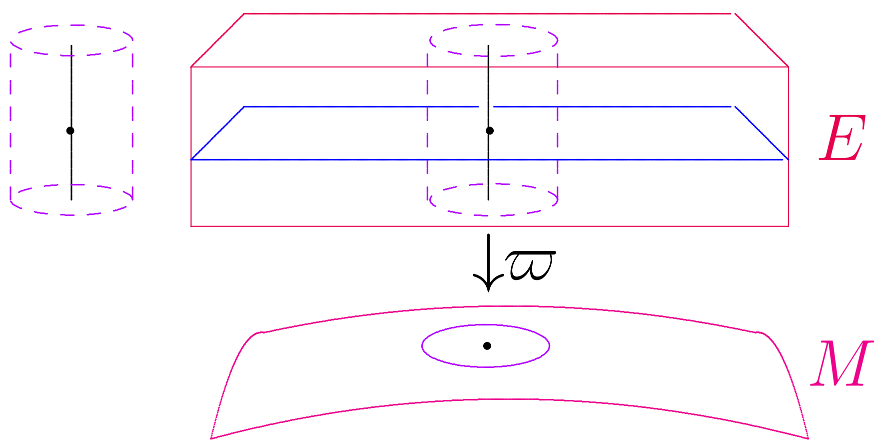

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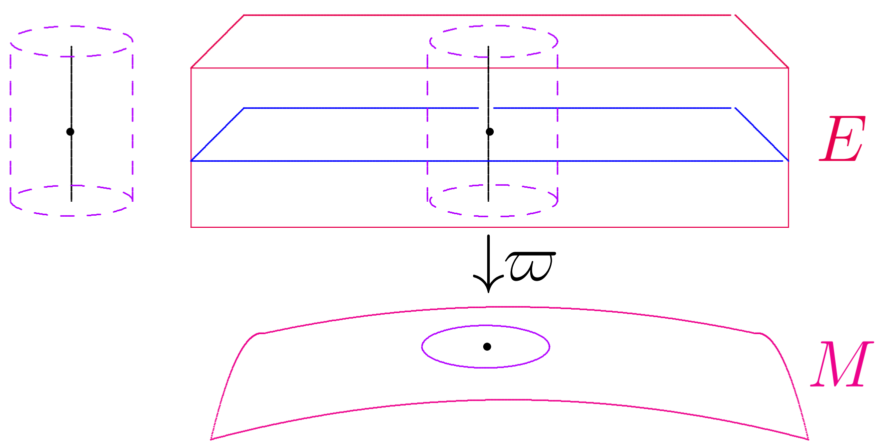

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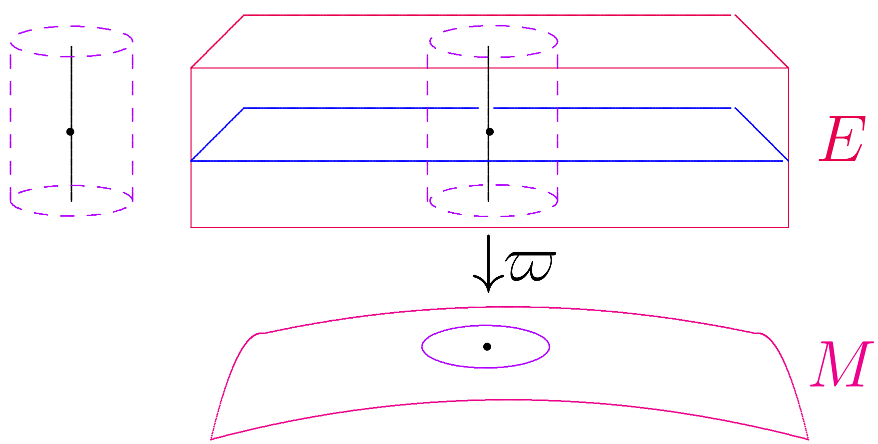

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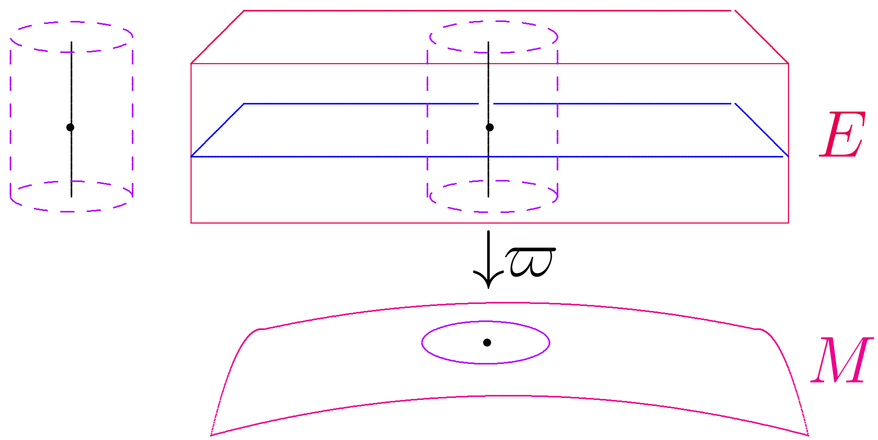

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Key observation. If $(x^1, \ldots, x^n)$ are coordinates on an open set $U \subset M$, then

$$T^*U \cong U \times \mathbb{R}^n$$

by using $dx^1, \ldots, dx^n$ as basis for T_p^*M, $\forall p \in U$.

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$$U_\beta \times \mathbb{R}^k \ni (p, \mathsf{v}) \sim (p, \tau_{\alpha\beta}(p)\mathsf{v}) \in U_\alpha \times \mathbb{R}^k \quad \forall p \in U_\alpha \cap U_\beta.$$

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$$\begin{bmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^1} \\ \vdots & & \vdots \\ \frac{\partial y^1}{\partial x^n} & \cdots & \frac{\partial y^n}{\partial x^n} \end{bmatrix} = \mathscr{J}^t$$

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Transition Functions for T_p^*M:

Thus, if

$$\tau_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{C^\infty} \mathbf{GL}(n, \mathbb{R})$$

is the system of transition functions used to build TM from a coordinate atlas for M^n, and if

$$\tilde{\tau}_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{C^\infty} \mathbf{GL}(n, \mathbb{R})$$

is the system of transition functions used to build T^*M, then one system is obtained from the other by taking the transpose-inverses:

$$\tilde{\tau}_{\alpha\beta} = \left[\left(\tau_{\alpha\beta} \right)^t \right]^{-1}.$$