MAT 531

Geometry/Topology II

Introduction to Smooth Manifolds

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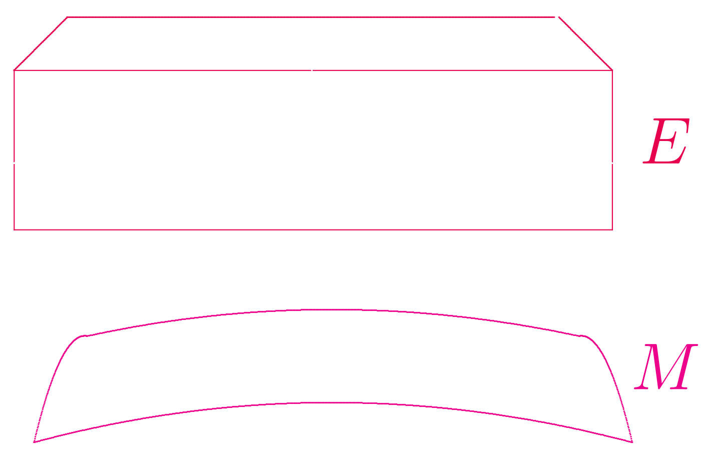

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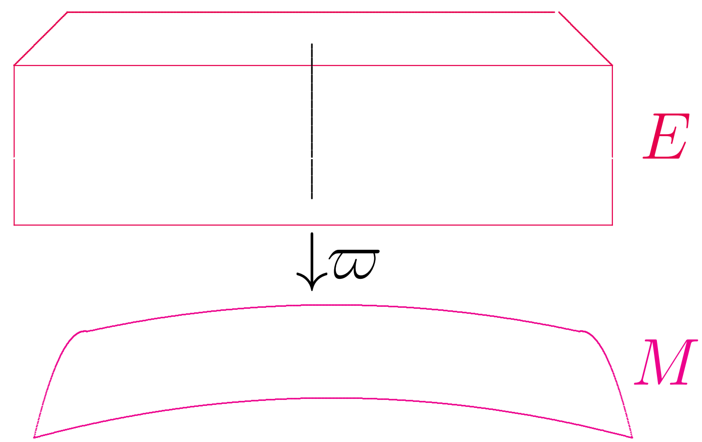

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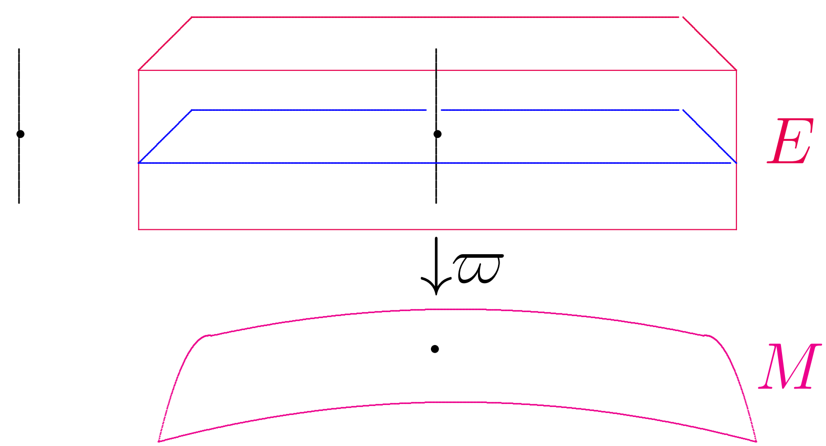

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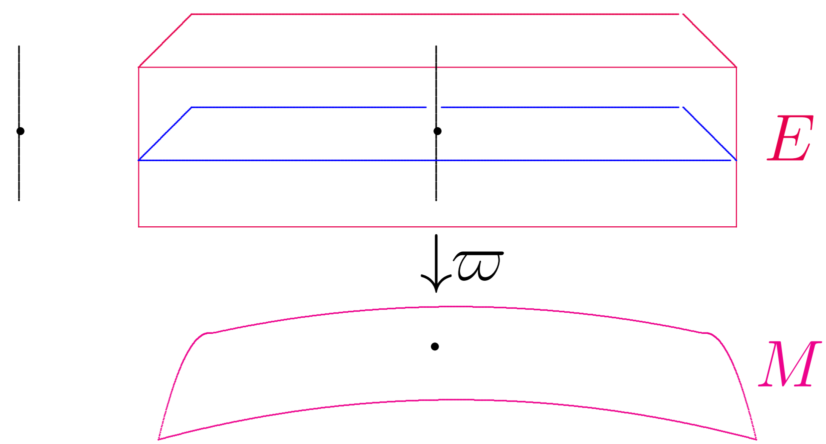

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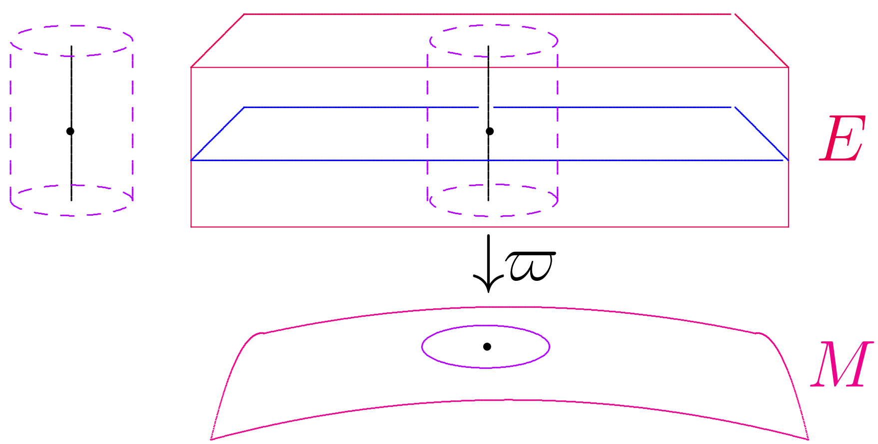

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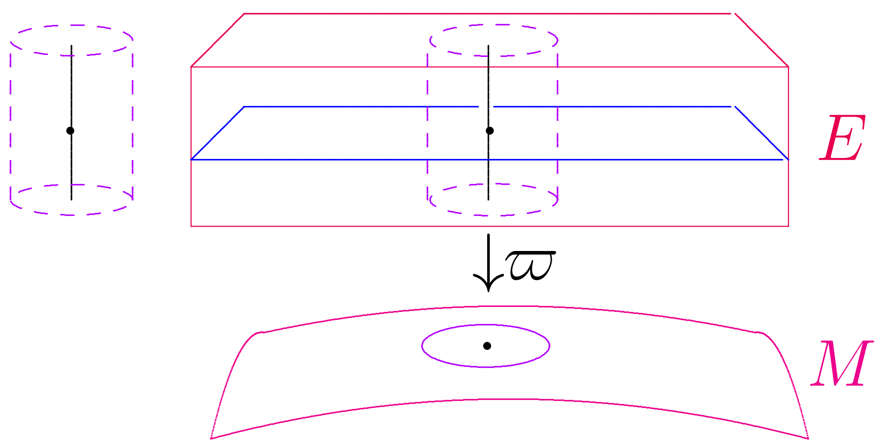

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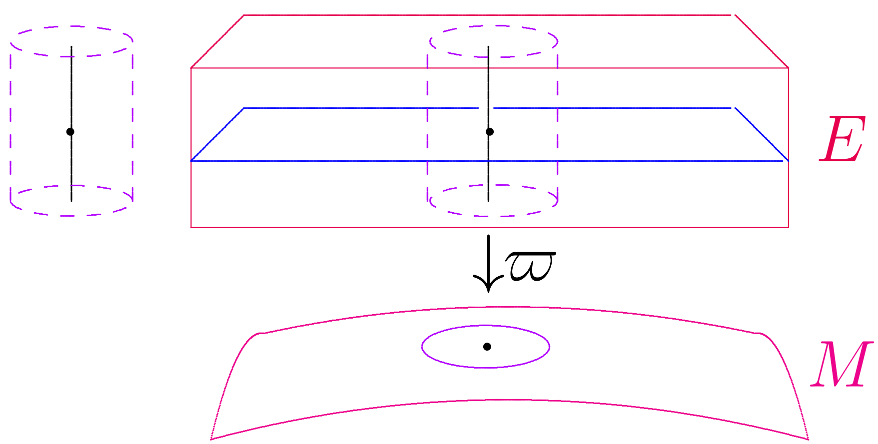

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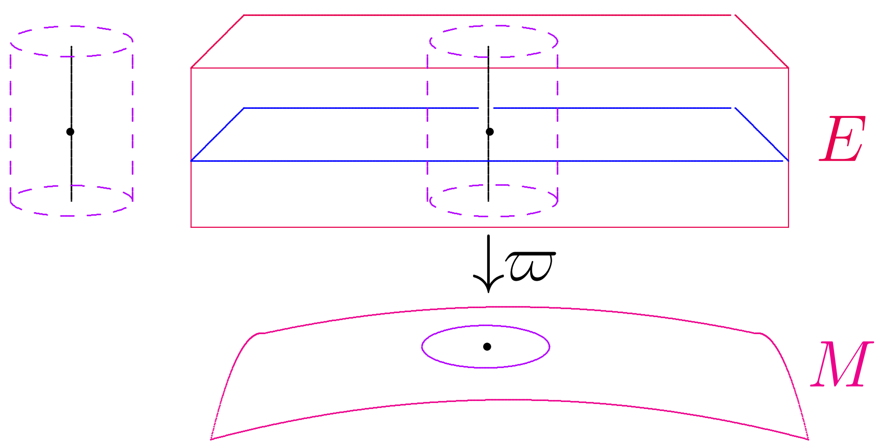

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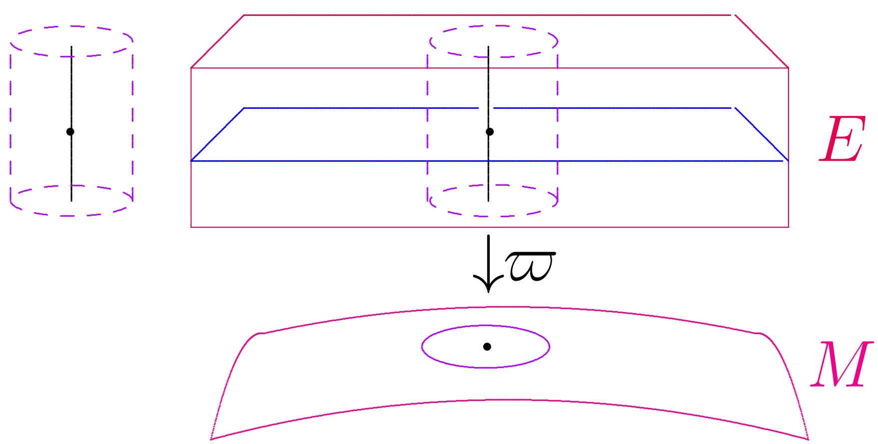

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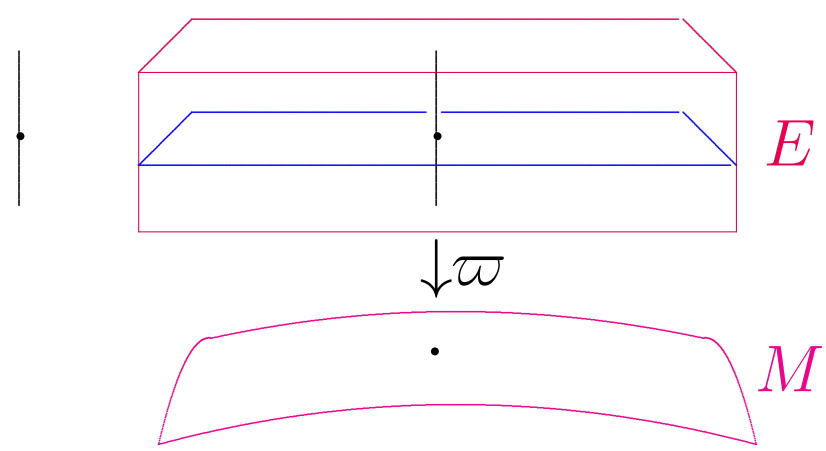

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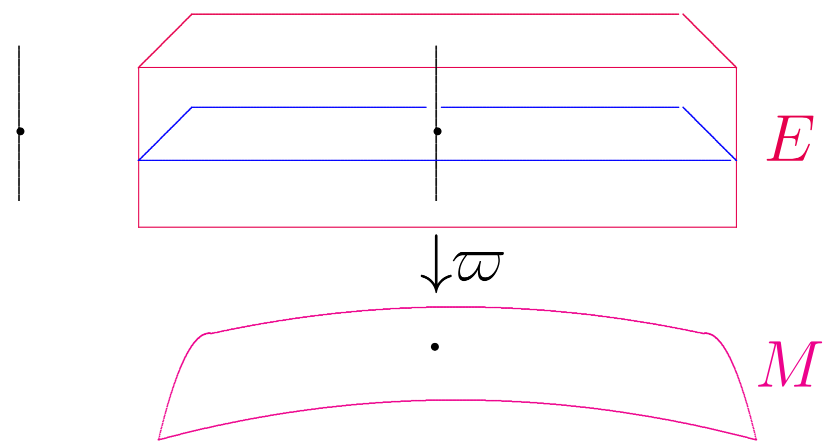

Example. Zero section: $\sigma(p) = \mathsf{0}_p \ \ \forall p$.

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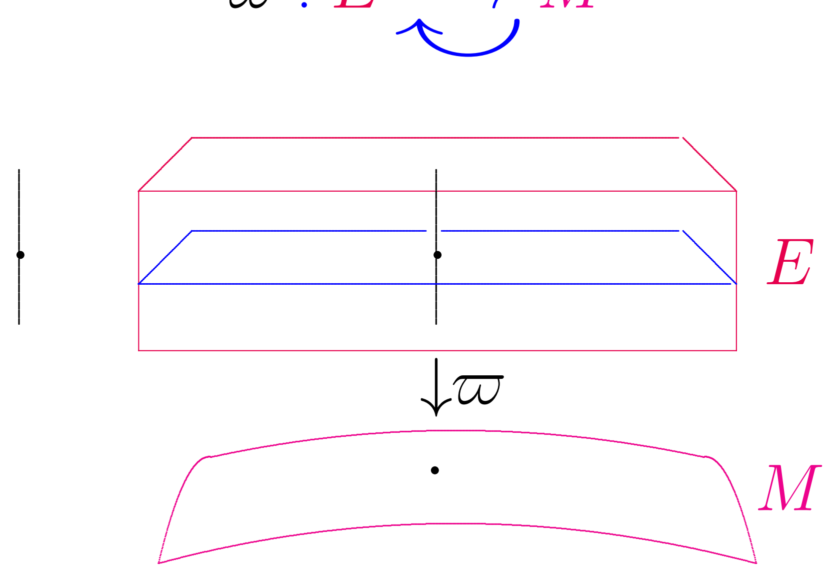

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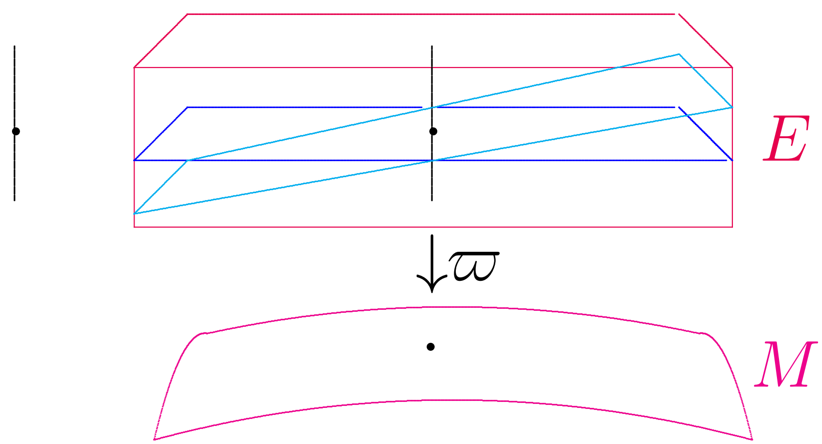

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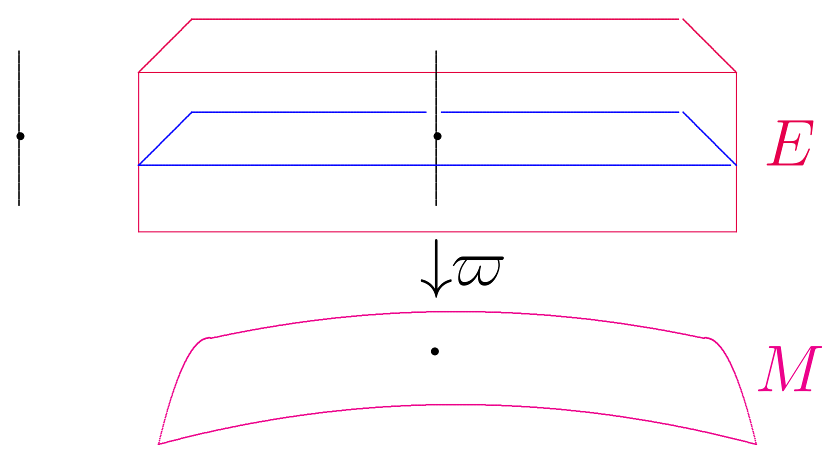

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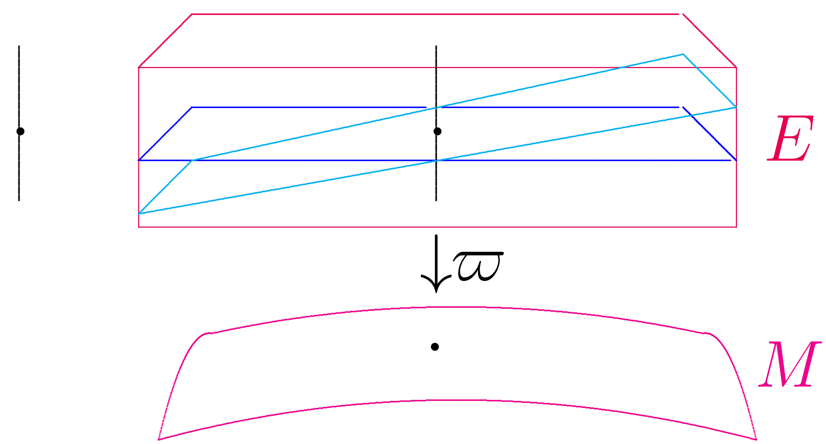

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Example. If $E = TM$, section = vector field.

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Local section = section of restriction $E|_U$ of bundle to some open subset $U \subset M$.

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Projection $\varpi : E \to \mathbb{R}/\mathbb{Z}$ given by $[(x, y)] \mapsto [x]$.

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$$U_{\beta} \times \mathbb{R}^{k} \ni (p, \mathsf{v}) \sim (p, \tau_{\alpha\beta}(p)\mathsf{v}) \in U_{\alpha} \times \mathbb{R}^{k} \quad \forall p \in U_{\alpha} \cap U_{\beta}.$$