MAT 531

Geometry/Topology II

Introduction to Smooth Manifolds

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$$\begin{aligned} d\left(\sum_{|I|=k} \varphi_I dx^I\right) &= \sum_{|I|=k} (d\varphi_I) \wedge dx^I \\ &= \sum_{j,|I|=k} \frac{\partial \varphi_I}{\partial x^j} dx^j \wedge dx^I. \end{aligned}$$

Closed Forms.

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Definition.

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Definition. *If $\psi \in \Omega^k(M)$ can be written as*

$$\psi = d\varphi$$

for some $\varphi \in \Omega^{k-1}(M)$, we say that ψ in an exact form.

Thus

$$\text{exact} \Longrightarrow \text{closed}.$$

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Let's see this via some examples.

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Remember: ω is closed, because $\Omega^{n+1}(\mathbb{R}^n) = 0$.

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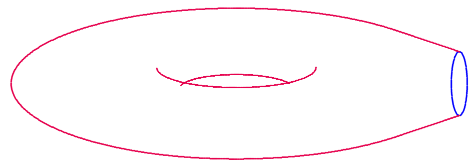
So any n-form on $\mathbb{R}^n$ is exact.

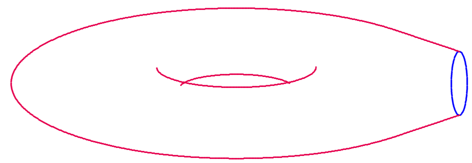
However, remember . . .

Stokes' Theorem.



Let M be an oriented n-manifold-with-boundary.

Stokes' Theorem.



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Let ω be compactly supported $(n-1)$-form on M.

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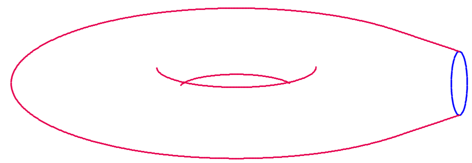

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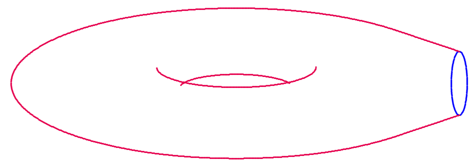

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$$\boxed{\int_M d\omega = \int_{\partial M} j^* \omega}$$

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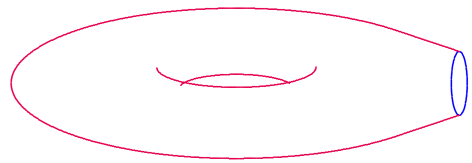

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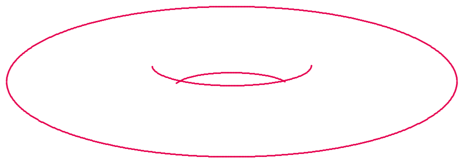
Corollary:

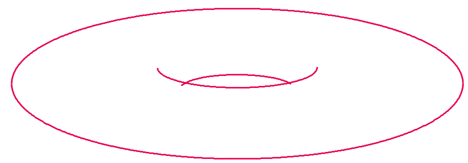


Let M be a compact oriented n-manifold, $\partial M = \emptyset$.

Let ω be any smooth $(n-1)$-form on M.

$$\boxed{\int_M d\omega = \int_\emptyset \omega}$$

Corollary:



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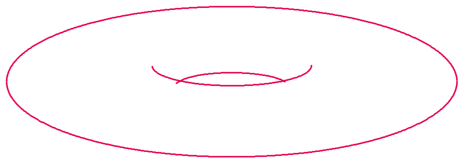

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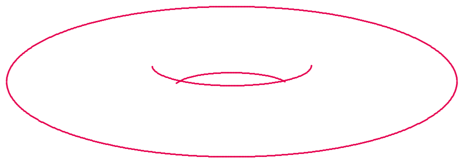

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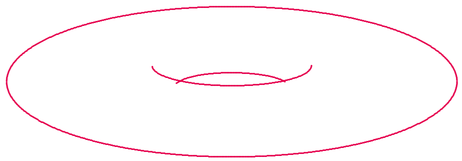

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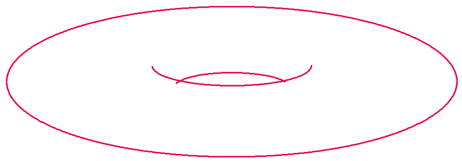

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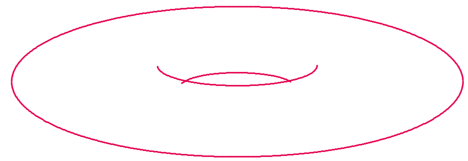

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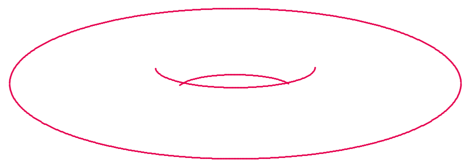

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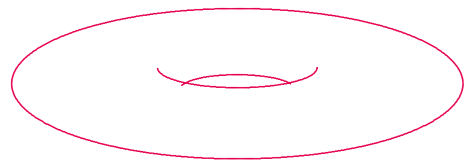

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Thus $\psi \neq d\omega$ for any $(n-1)$-form ω.

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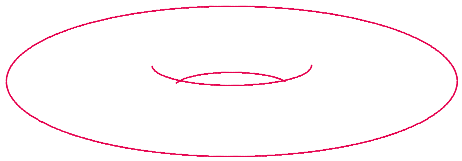

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$\therefore$ The closed form ψ is not exact.

$$\text{exact} \Longrightarrow \text{closed}.$$

What about converse?

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Answer: It depends!

It depends on the manifold M!

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This leads to interesting invariants of M...

De Rham Cohomology:

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(Named after Georges de Rham.)

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Sometimes called a “cohomology group” . . .

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Example. Or if M has ℓ connected components,

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That is, an n-form is exact iff its integral $= 0$.