MAT 531

Geometry/Topology II

Introduction to Smooth Manifolds

Claude LeBrun
Stony Brook University

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Corollary. *Any simply connected manifold M is orientable.*

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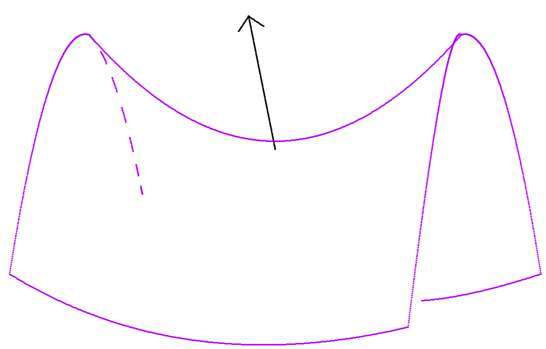

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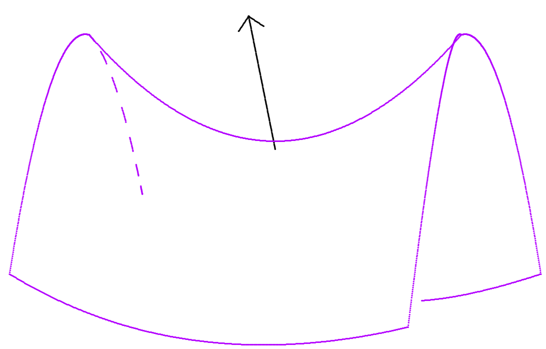

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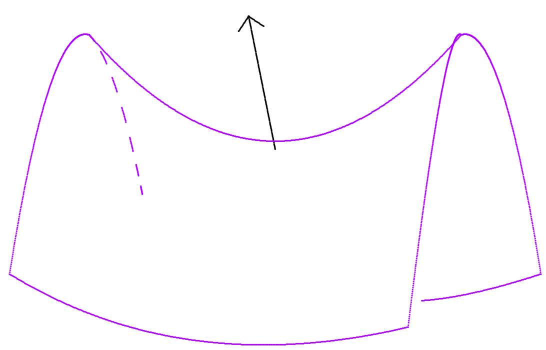

Orientation $\iff$ "Which side of X is V on?"

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(Diffeomorphism F sends a small cube of volume ε roughly to an parallepiped of volume $\varepsilon |\det dF|$.)

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Compatible with integration of n-forms because

$$dy^1 \wedge \cdots \wedge dy^n = \det\left(\frac{\partial y^j}{\partial x^k}\right) dx^1 \wedge \cdots \wedge dx^n$$

and $\det\left(\frac{\partial y^j}{\partial x^k}\right) > 0$ if orientation-preserving.

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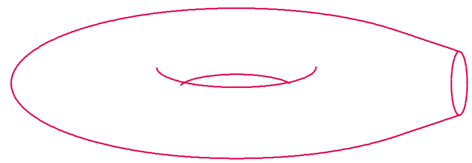

Let M be an oriented n-manifold

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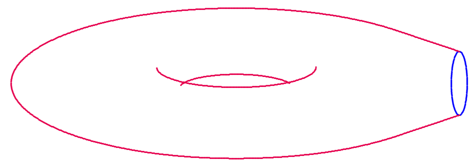

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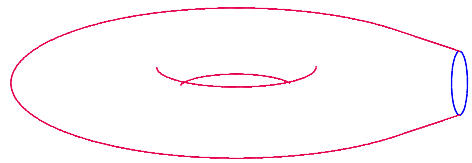

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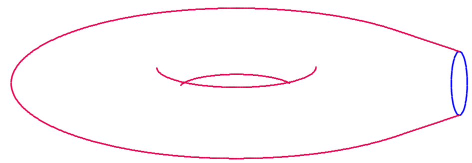

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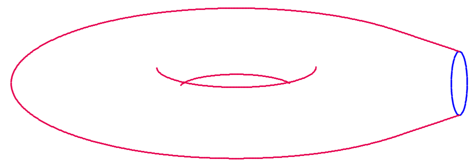

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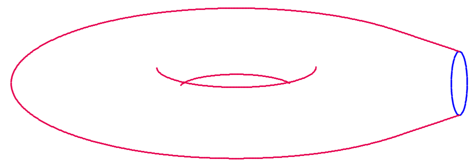

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One usually abbreviates this as:

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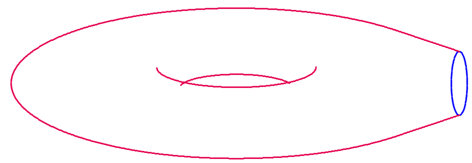

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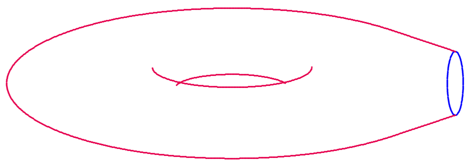

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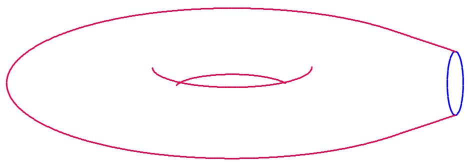

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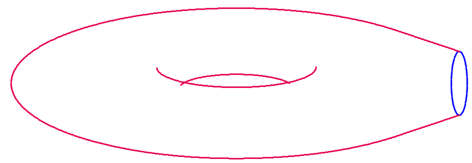

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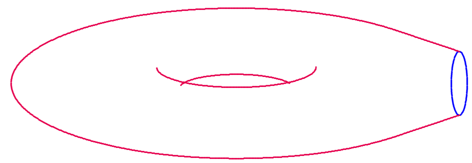

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Modern form due to Élie Cartan (1945).

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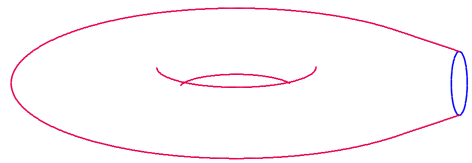

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Classical special case due to Lord Kelvin (1854).

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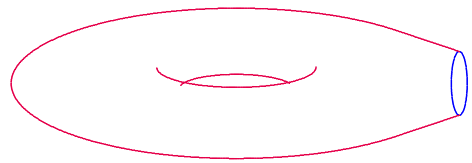
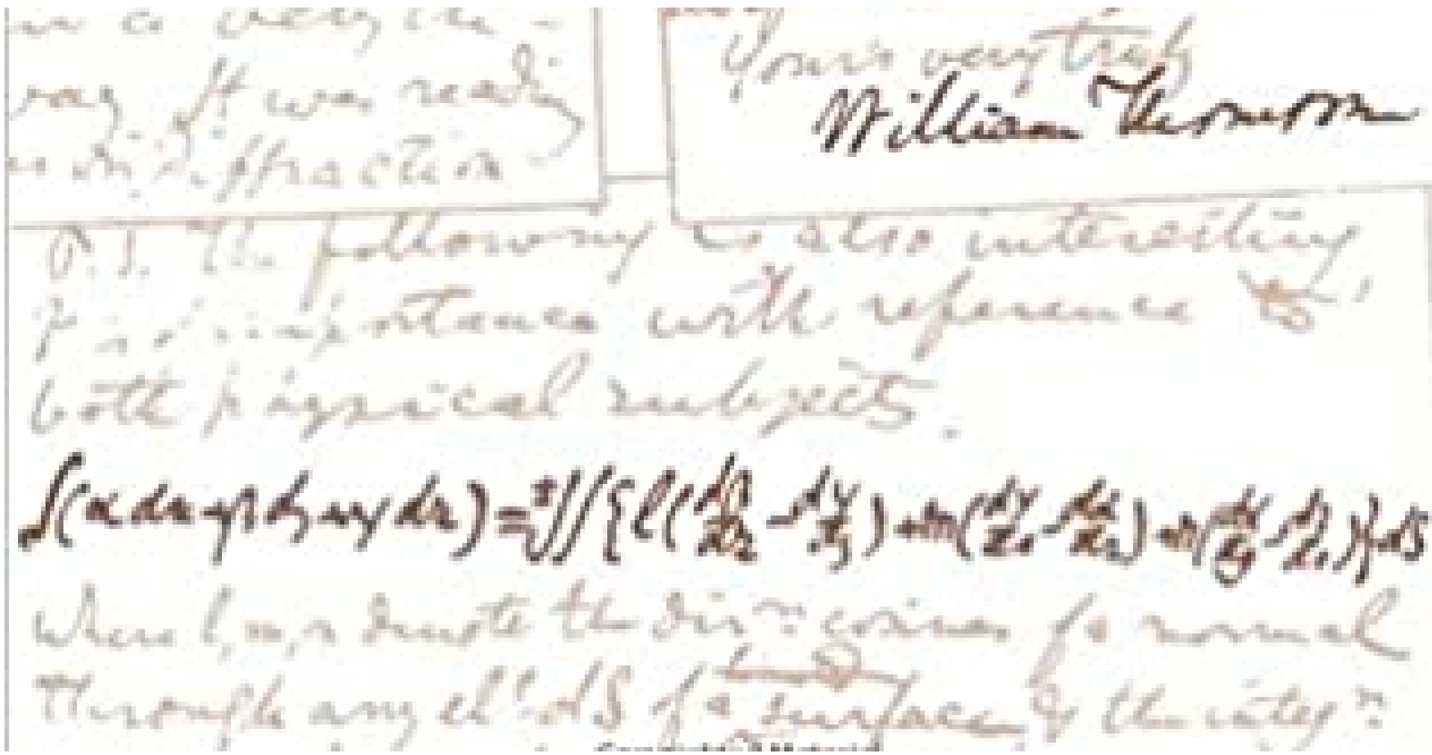




... it was ready ... fraction

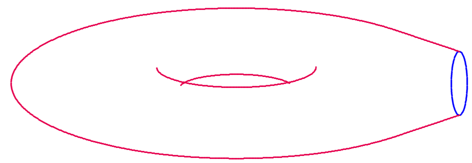
Yours very truly
William Thomson

P.S. The following is also interesting, & of importance with reference to both physical subjects.

$$\int(X\,dx + Y\,dy + Z\,dz) = \iint\left\{l\left(\frac{dZ}{dy} - \frac{dY}{dz}\right) + m\left(\frac{dX}{dz} - \frac{dZ}{dx}\right) + n\left(\frac{dY}{dx} - \frac{dX}{dy}\right)\right\}dS$$

where l, m, n denote the dir. cosines of normal through any el^t dS of a surface & the integ^n ...

Stokes' Theorem.



Do you know that the condition that $\alpha dx + \beta dy + \gamma dz$ may be the diff$^{\text{t}}$ of a function of two indep$^{\text{t}}$ variables for all points of a surface is

$$l\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)+m\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\right)+n\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right)=0?$$

I made this out some weeks ago with ref$^{\text{ce}}$ to electromagnetism. With ref$^{\text{ce}}$ to an elastic solid, the cond$^{\text{n}}$ may be expressed thus – the resultant axis of rotation at any point of the surface must be perp$^{\text{r}}$ to the normal.

Your's very truly
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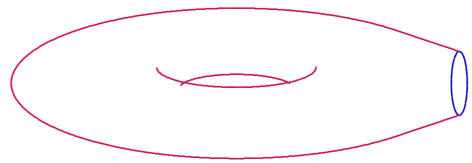
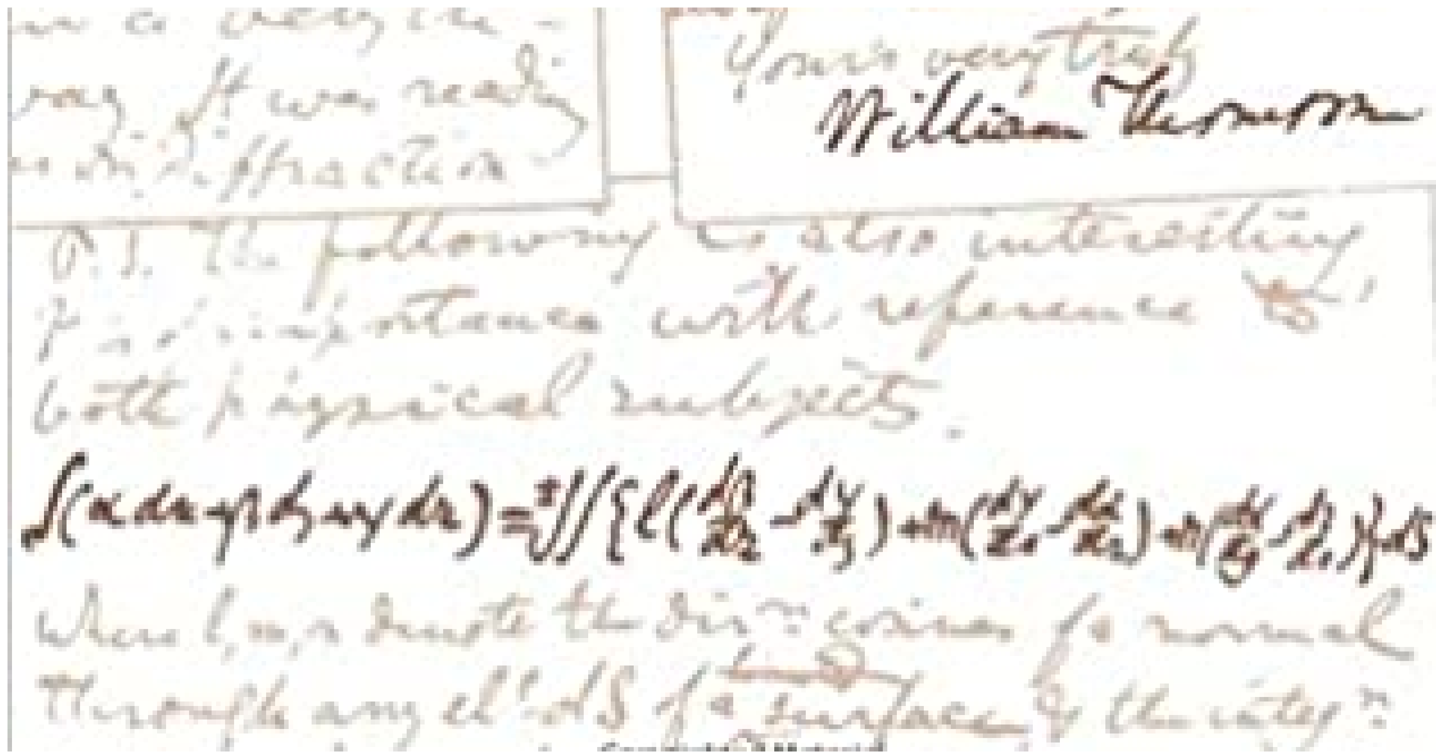
$$\int(\alpha dx+\beta dy+\gamma dz) = \pm\iint\left\{l\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)+m\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\right)+n\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right)\right\} dS$$

where l, m, n denote the dir$^{\text{n}}$ cosines of a normal through any el$^{\text{t}}$ dS of a ⟨limited⟩ surface; & the integ$^{\text{n}}$ in the sec$^{\text{d}}$ member is performed over a portion of this surface bounded by a curve round w$^{\text{h}}$ the int$^{\text{n}}$ in the 1$^{\text{st}}$ member is performed.[3]

1 Stokes (11).
2 Stokes (40).
3 Stokes included the equation in this postscript on the Smith's prize examination for 1854 (the year Maxwell took the examination), and it has become known as Stokes's Theorem. (See Larmor's footnote in Stokes's *MPP*, v, 320–1.)

Stokes' Theorem.





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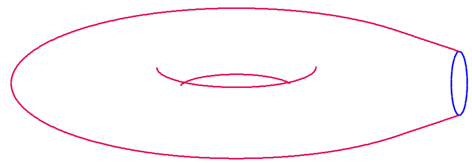

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